

The motivic Hopf invariant one problem

Monday, August 8, 2022 7:09 AM

joint work with Balderama and Culver

§1. Classical story. ($n \geq 2$)

Def. Let $f: S^{2n-1} \rightarrow S^n$. $H^*(\text{cofib}(f); \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & * = 0, n, 2n, \\ 0 & \text{otherwise.} \end{cases}$

Let β gen H^{2n} , α gen H^n .

The Hopf invariant of f , $H(f)$, is the integer s.t.
 $\alpha^2 = H(f) \cdot \beta$.

Ex. n odd $\Rightarrow H(f) = 0$ ($\alpha^2 = -\alpha^2 = 0$)

Ex. n even $\Rightarrow H([1_n, 1_n]) = 2$.

Question: For which n does there exist $f \in \pi_{2n-1}(S^n)$ with $H(f) = 1$?

Prop. If S^{n-1} is an H-space, then there exists $f \in \pi_{2n-1}(S^n)$ with $H(f) = 1$.
(Idea: Hopf construction on product.)

Cor. Hopf invariant one elements exist for $n = 2, 4, 8$.

Pf. $S^1 = S(\mathbb{C})$, $S^3 = S(\mathbb{H})$, $S^7 = S(\mathbb{O})$. //

Thm. (Adams, 1960) There exists a HI 1 element in $\pi_{2n-1}(S^n)$ iff $n = 2, 4, 8$.

Cor. S^n admits H-space structure iff $n = 0, 1, 3, 7$.

§2. Motivic story, I.

Question (Asok-Doran-Fasel, 2014) Which motivic spheres, $S^{m,n} := (S^{n-m}_*) \wedge G_m^{\wedge n}$

$S^{0,0}, S^{1,1}, S^{3,2}, S^{7,4}$.

* The following admit H-space structures:

$S^{1,0}, S^{3,0}, S^{7,0}$.

* The following might admit H-space structures:

$S^{3,1}, S^{7,3}, S^{7,2}, S^{7,1}$.

Cor. (BCQ) If $F = \mathbb{R}$, only

$S^{0,0}, S^{1,1}, S^{3,2}, S^{7,4}, S^{1,0}, S^{3,0}, S^{7,0}$

admit H-space structures.

Prop. If $\mu: S^{a-1,b} \times S^{a-1,b} \rightarrow S^{a,b}$ is an H-space product, then its Hopf construction in $\pi_{2a-1,2b}(S^{a,b})$ has Hopf invariant 1.

Prop. Let $f: S^{2a-1,2b} \rightarrow S^{a,b}$ have $H(f) = 1$. Then the associated stable class $\alpha \in \pi_{a-1,b}^F$ is detected by a permanent cycle in $E_{\text{Ext}}^{a-1,b}_F$.

E_2 -term of motivic Adams SS

$$E_2^{s,f,w} := E_{\text{Ext}}^{s,f,w}_F \Rightarrow (\pi_{s,w}^F)_{(2,1)}.$$

Pf of Thm/Cor: * $E_{\text{Ext}}^{s,1,w}_F = 0$ for $(s,w) \in \{(3,3), (7,5), (7,6), (7,7)\}$.

* $E_{\text{Ext}}^{s,1,w}_{\mathbb{R}} = 0$ for $(s,w) \in \{(3,1), (7,3), (7,2), (7,1)\}$. //

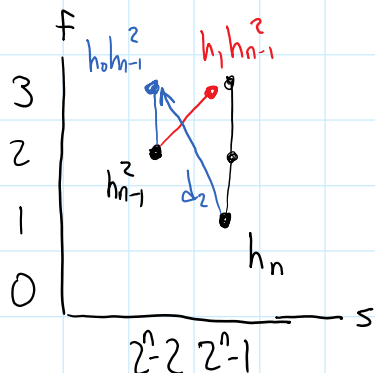
Cor. There is an $\mathbb{R}$ -motivic $f: S^{2a-1,2b} \rightarrow S^{a,b}$ with $H(f) = 1$ iff $(a,b) \in \{(1,0), (2,1), (4,2), (8,4), (2,0), (4,0), (8,0)\}$. //

§3. Motivic story, II.

Classically, solving HI 1 is equivalent to completely understanding

$i=1$ part of Adams SS.

$$E_2^{f=1} = Ext_{cl}^{f=1} = \mathbb{F}_2 \{h_0, h_1, h_2, \dots\} \quad |h_i| = 2^i - 1.$$



$n \geq 2$

$n \geq 4$

$n \geq 5$

Thm (Adams) $d_2(h_n) = h_0 h_{n-1}^2$
for $n \geq 4$.

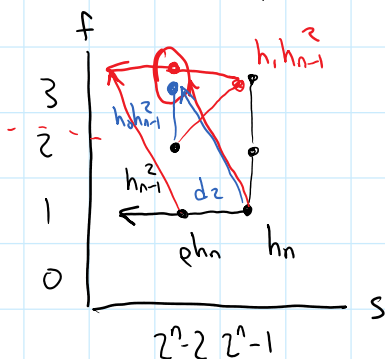
$$F: \bullet = \mathbb{F}_2[\tau] \quad d_2(\tau^k h_n) = \tau^k h_0 h_{n-1}^2.$$

$$Ext_{\overline{F}}^{f=1} = \mathbb{F}_2[\tau] \{h_0, h_1, h_2, \dots\} \quad |h_i| = (2^i - 1, 2^{i-1}), \quad |\tau| = (0, -1)$$

$$Ext_{\mathbb{R}}^{f=1} = \mathbb{F}_2[\rho] \{h_1, h_2, h_3, \dots\} \quad |\rho| = (-1, -1)$$

$$\oplus \mathbb{F}_2 \{ \tau^{2k} h_0 : k \geq 0 \}$$

$$\oplus \bigoplus_{n \geq 1} \mathbb{F}_2[\rho] / (\rho^{2^n}) \{ \tau^{2^{n-1}(4k+1)} h_n : k \geq 0 \}$$



$n \geq 2$

$n \geq 4$

$n \geq 5$

Thm (BCG) Over any base
field F of $\text{char} \neq 2$,
 $d_2(h_n) = (h_0 + \rho h_1) h_{n-1}^2$
for $n \geq 4$.

Thm (BCG) The subgroup of permanent cycles in $Ext_{\mathbb{R}}^{f=1}$ is
 $\mathbb{F}_2[\rho] \{h_1, h_2, h_3, \rho h_4\} \oplus \mathbb{F}_2 \{ \tau^{2k} h_0 : k \geq 0 \}$

$$\oplus \bigoplus_{n \geq 1} \mathbb{F}_2[\rho] / (\rho^{\psi(n)}) \{ \rho^{2^{n-1}(4k+1)} h_n : k \geq 0 \}$$

where if $n = c + 4d$, $0 \leq c \leq 3$,
 $\psi(n) = 2^c + 8d$.