

Definition

An *abstract cohomology theory* over a field k is a functor $E^\bullet: \mathrm{Sm}_k^{\mathrm{pair}} \rightarrow \mathrm{Ab}^{\mathrm{gr}}$ equipped with natural long exact sequences

$$\cdots \rightarrow E^*(X) \rightarrow E^*(X - Z) \xrightarrow{\partial} E_Z^{*+1}(X) \rightarrow E^{*+1}(X) \rightarrow \cdots, \quad (1)$$

where $E_Z^*(X) = E^*(X, X - Z)$ for $X \in \mathrm{Sm}_k$, and closed subschemes $Z \subset X$.

Define

$$\mathrm{Cous}(X, E) \cong [\cdots \rightarrow 0 \rightarrow E^0(X) \rightarrow \bigoplus_{x \in X^{(0)}} E^0(x) \rightarrow \cdots \rightarrow \bigoplus_{x \in X^{(c)}} E_x^c(X) \rightarrow \cdots].$$

Property

For any $X \in \mathrm{Sm}_k$, and a normal crossing divisor D , the complex $\mathrm{Cous}(X - D, E)$ is exact.

Examples

K-theory, Witt-groups, Hermitian K-theory, algebraic cobordisms.

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- 1) To use representability in $\mathbf{SH}(k)$ for $E^\bullet$, and the result for representable cohomology theories.

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- 1) To use representbality in $\mathbf{SH}(k)$ for $E^\bullet$, and the result for representbale cohomology theories.
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For any $X \in \mathrm{Sm}_k$, and a normal crossing divisor D , the complex $\mathrm{Cous}(X_x - D, E)$ is exact.

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- 1) To use representability in $\mathbf{SH}(k)$ for $E^\bullet$, and the result for representable cohomology theories.
- 2) To form a correspondences category or define framed transfers structure for $E^\bullet$, and use the results for presheaves with transfers.
- 3) To prove Nisnevich excision, and $\mathbb{A}^1_k$ -homotopy invariance for $E^\bullet$ and use the theorem below.

Theorem

The complex $\mathrm{Cous}(X_x - D, E)$ as above is exact for any $\mathbb{A}^1$ -invariant Nisnevich excisive abstract cohomology theory $E^\bullet$.

Let k be a field. Let $Y \in \mathrm{Sm}_k$ and V be an open subscheme. Recall regular Voevodsky's framed correspondences

$$\begin{aligned}\mathrm{Fr}(-, Y/V) &= \varinjlim_n \mathrm{Fr}_n(-, Y/V); \\ \mathrm{Fr}_n(-, Y/V) &= \mathrm{Shv}_{\bullet}(- \wedge \mathbb{P}^n / \mathbb{P}^{n-1}, Y/V \wedge \mathbb{A}^n / (\mathbb{A}^n - 0)).\end{aligned}$$

Use notation $\mathrm{Fr}(Y/V) = \mathrm{Fr}(-, Y/V)$.

Theorem (Garkusha & Panin + set of other authors)

There is a schemewise equivalence

$$\Omega_T^\infty \Sigma_T^\infty Y \simeq L_{\mathrm{zar}} L_{\mathbb{A}^1} \mathrm{Fr}(Y/V)^{\mathrm{gp}}.$$

Corollary

There is the Nisnevich local isomorphism of abelian presheaves

$$\pi_0(Y/V) \simeq_{\mathrm{nis}} \mathrm{coker}[\mathbb{Z}\mathrm{F}(- \times \mathbb{A}^1, Y/V) \rightarrow \mathbb{Z}\mathrm{F}(-, Y/V)],$$

and

$$\begin{aligned}[-, Y/V[n]]_{\mathbf{D}_{\mathbb{A}^1}(k)} &\simeq_{\mathrm{nis}} H^n(\mathbb{Z}\mathrm{F}(- \times \Delta_k^\bullet, Y/V)), \\ [- \wedge S^n, Y/V]_{\mathbf{SH}(k)} &\simeq_{\mathrm{nis}} \pi_n \mathrm{Fr}(- \times \Delta_k^\bullet, Y/V)^{\mathrm{gp}}.\end{aligned}$$

Let B be a separated noetherian B of finite Krull dimension and $Y \in \mathrm{Sm}_B$, V be an open subscheme. Let $\Omega_T^\infty \Sigma_T^\infty(Y/V) \in \mathrm{Shv}_\bullet(\mathrm{Sm}_B)$ denote the motivic ∞ -loop space of Y/V with respect to the category $\mathbf{SH}_{\mathbb{A}^1, \mathrm{nis}}(B)$.

Theorem

There is the Nisnevich locally equivalence

$$\Omega_T^\infty \Sigma_T^\infty(Y/V) \simeq_{\mathrm{nis}} \varinjlim_I \Omega_{\mathbb{G}_m^{\wedge 1} \wedge \Delta_S^1 / \partial \Delta_S^1}^I \mathcal{L}_{\mathrm{tf}} \mathrm{Fr}(Y/V \wedge T^{\wedge I}).$$

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$$\Omega_T^\infty \Sigma_T^\infty(Y/V) \simeq_{\mathrm{nis}} \varinjlim_I \Omega_{\mathbb{G}_m^{\wedge 1}}^I \mathcal{L}_{\mathbb{A}^1, \mathrm{tf}} \mathrm{Fr}(Y/V \wedge \mathbb{G}_m^{\wedge I})^{\mathrm{gp}} \cong \mathcal{L}_{\mathbb{A}^1} \mathrm{Fr}^{\mathrm{tf}}(Y/V)^{\mathrm{gp}},$$

and the schemewise isomorphisms

$$\mathcal{L}_{\mathbb{A}^1, \mathrm{tf}} \cong (\mathcal{L}_{\mathbb{A}^1} \mathcal{L}_{\mathrm{tf}})^d \mathcal{L}_{\mathbb{A}^1} \cong \mathcal{L}_{\mathbb{A}^1, d} \mathcal{L}_{\mathrm{tf}_d} \mathcal{L}_{\mathbb{A}^1, d-1} \cdots \mathcal{L}_{\mathrm{tf}_1} \mathcal{L}_{\mathbb{A}^1, 0}.$$

Here $\mathrm{Fr}(Y/U) = \mathrm{Fr}(-, Y/U)$ is the presheaf of framed correspondences.

Recall

$$\mathbb{Z}F(-, Y/U) = \varinjlim_n \mathbb{Z}F_n(-, Y/U).$$

Let B be one-dimensional local scheme, $z \in B$ be the closed point, Y be a smooth scheme over B , V be an open subscheme, and U be an essentially smooth local henselian scheme over B .

Corollary (the first theorem)

There is the isomorphism of Nisnevich stalks

$$\pi_{-1}(Y/V)(U) \simeq \varinjlim_I H_{\text{tf}}^1((\mathbb{G}_m^{\wedge 1} \wedge (\Delta_B^1 / \partial \Delta_B^1))^{\wedge I} \times U, \mathbb{Z}F(Y/V \wedge T^{\wedge I})_{\text{tf}}).$$

Note that

$$H_{\text{tf}}^1(U, F_{\text{tf}}) \cong \text{coker} \left(F(U \times_B (B - z)) \oplus F(U_z^h) \rightarrow F(U_z^h \times_B (B - z)) \right)$$

for any presheaf F .

Corollary (the second theorem)

Denote $\mathcal{Y}^n = Y/U \wedge \mathbb{G}_m^{\wedge n}$, and denote

$$E^{n,l} = U \wedge \mathbb{G}_m^n \times \mathbb{A}^{\times l}, F^{n,l} = (E^{n,l}) \times_B z,$$

and

$$V_0^{n,l} = (E^{n,l})_{F^{n,l}}^h, V_1^{n,l} = E^{n,l} - F^{n,l}, V_{01}^{n,l} = (E^h)_{F^{n,l}}^{n,l} - F^{n,l}.$$

The stalk $\pi_{-1}(Y)(U)$ is isomorphic to the cokernel of the group homomorphism

$$\varinjlim_n \mathbb{Z}F(V_0^{n,1}, \mathcal{Y}^n) \oplus \varinjlim_n \mathbb{Z}F(V_1^{n,1}, \mathcal{Y}^n) \oplus \varinjlim_n \mathbb{Z}F(V_{01}^{n,2}, \mathcal{Y}^n) \rightarrow \varinjlim_n \mathbb{Z}F(V_{0,1}^{n,1}, \mathcal{Y}^n).$$

Let B be a separated noetherian scheme of finite Krull dimension.

Definition

Define $\mathbf{SH}^{Cous}(B)$ as the subcategory in $\mathbf{SH}(B)$ spanned by objects E such that the Cousin complex $Cous(X_x, E)$ associated with the represented by the object E cohomology theory is exact for any $X \in \mathbf{Sm}_B$, $x \in X$.

Theorem

The subcategory $\mathbf{SH}^{Cous}(B)$ is a reflective subcategory of $\mathbf{SH}(B)$.

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Theorem

There are equivalences

$$\mathbf{SH}^{Cous}(B) \simeq \mathbf{SH}^{tfCous}(B) = \{E \in \mathbf{SH}(k) \mid \pi_I \text{Cous}_{\text{tf}}(E) \simeq \pi_I E \forall I\}.$$

Theorem

The subcategory $\{E \in \mathbf{SH}(k) \mid \pi_I \text{Cous}_{\text{tf}}(E) \simeq \pi_I E \forall I\}$ is reflective.

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Consider the functors

$$\text{Sm}_B \rightarrow \mathbf{SH}(B) \rightarrow \mathbf{SH}^{Cous}(B),$$

and the Nisnevich sheaves of stable motivic homotopy groups with respect to the right side.

Theorem

The following properties hold:

- a) the Nisnevich sheaves of homotopy groups of objects in $\mathbf{SH}^{Cous}(B)$ are strictly $\mathbb{A}^1$ -invariant;*
- b) the Nisnevich sheaves of negative homotopy groups of the motives of smooth schemes in $\mathbf{SH}^{Cous}(B)$ vanish.*

Definition

Let S be separated neotherian base scheme of dimension d . Consider the union of categories $\mathbf{Sm}_s$, for all points $s \in S$, consider the joint $\underline{\mathbf{Sm}}_S$ of the above categories by adding of arrows $U - U_{s_c} \rightarrow U_{s_{c+1}}$, $U_s = U \times_S s$, for all $U \in \mathbf{Sm}_S$, and pairs of points s_c and s_{c+1} of codimensions c and $c + 1$ such that the closure of s_c contains s_{c+1} .

Then for any additive presheaf $F \in \mathbf{Pre}^{\text{add}}(\underline{\mathbf{Sm}}_S)$ there is the sequence of morphisms

$$\bigoplus_{z \in S^{(0)}} F(U \times_S z) \rightarrow \bigoplus_{z \in S^{(l)}} F(U \times_S z) \rightarrow \cdots \rightarrow \bigoplus_{z \in S^{(d)}} F(U \times_S z) \rightarrow 0. \quad (2)$$

Definition

Consider the category of $\mathbb{A}^1$ -invariant presheaves of abelian groups $\mathbf{HI}^{\text{Ab}}(\underline{\mathbf{Sm}}_S)$. Let $\mathbf{HI}^{\text{Ab}}(\underline{\mathbf{Sm}}_S)[\mathbb{G}_m^{\wedge -1}]$ denote the category of homotopy modules of such presheaves i.e. sequences F_l , $l \in \mathbb{Z}$, $F_{l-1} \simeq \Omega_{\mathbb{G}_m^{\wedge 1}} F_l$.

We denote by $\mathbf{C}_{\text{tf}}\mathbf{HI}(S)$ the category of *tf-cycle modules of homotopy modules* given by such objects in $\mathbf{HI}^{\text{Ab}}(\underline{\mathbf{Sm}}_S)[\mathbb{G}_m^{\wedge -1}]$ that the sequences (2) are exact in all terms except the zero-th term $\bigoplus_{z \in S^0} F(U \times_S z)$.

Theorem

The category $\mathbf{SH}^{\text{Cous}}(B)^{\heartsuit}$ is equivalent to $\mathbf{C}_{\text{tf}}\mathbf{HI}(S)$

Theorem

Let B be a spectrum of an excellent discrete valuation ring, with perfect residue fields, then $\mathbf{SH}^{\text{Cous}}(B)^{\heartsuit}$ is equivalent to the category of Milnor-With cyclemodules that satisfy additional axiom (R5) in sense of the arXiv:2203.07801 .

Let B a local scheme and p be the characteristic of the residue field of the closed point.

Proposition

The subcategory $\langle \mathrm{SmProj} \rangle[\frac{1}{p}]$ generated by smooth projective schemes in $\mathbf{SH}(B)[\frac{1}{p}]$ is a reflective subcategory.

Note that $\langle \mathrm{SmProj} \rangle \simeq \langle \mathrm{Sm}^{\mathrm{good}} \rangle$, where $\mathrm{Sm}^{\mathrm{good}}$ is the category of good schemes in the following sense.

Definition

A scheme X is *good* if there is an open immersion $X \rightarrow \overline{X}$ and filtration of closed subschemes $X_{\infty}^{(0)} \supset \dots \supset X_{\infty}^{(l)} \supset \dots$ such that $\overline{X} - X = X_{\infty}^{(0)}$, and the stratas $X_{\infty}^{(l)} - X_{\infty}^{(l+1)}$ for $l > 0$ are good.

Definition

Denote by $\underline{\mathrm{Sm}}$ the category that objects are pairs $(X, \overline{X})$. Define

$$\mathbf{SH}_{\underline{\mathbb{A}^1}}(B) = \mathrm{Pre}(\underline{\mathrm{Sm}})[w_{\mathrm{nis}}^{-1}, w_{\mathbb{A}^1}^{-1}][\mathbb{P}^{\wedge -1}],$$

where w_{nis} is generated by squares that induces Nisnevich square in view of the functor $(X, \overline{X}) \rightarrow X$.

Conjecture

There are equivalence of categories

$$\mathbf{SH}^{Cous}(B) \simeq \langle \mathbf{SmProj} \rangle \simeq \mathbf{SH}_{\underline{\mathbb{A}^1}}(B).$$

Conjecture

For any $X \in \mathbf{Sm}^{\text{good}}$,

a) there is the levelwise schemewise equivalence of prespectra of presheaves on $\mathbf{Sm}_B$

$$L_{\text{smot}}(\Sigma_T^\infty X) \cong L_{\text{nis}} L_{\mathbb{A}^1}(h^{\text{fr}}(\Sigma_T^\infty X))^{\text{gp}},$$

b) the Cousin complex associated with the cohomology theory represented by X in $\mathbf{SH}_{\mathbb{A}^1, \text{nis}}(B)$ for any essentially smooth local scheme is exact.

Theorem

Let F be an $\mathbb{A}^1$ -invariant quasi-stable framed additive presheaf of abelian groups over a field k , and F_{nis} denote the associated Nisnevich sheaf, and similarly F_{zar} for Zariski topology. Then for any $X \in \text{Sm}_k$, $n \in \mathbb{Z}$, the canonical projection induces the isomorphism of Nisnevich cohomologies

$$H_{\text{nis}}^n(\mathbb{A}_k^1 \times X, F_{\text{nis}}) \cong H_{\text{nis}}^n(X, F_{\text{nis}}), \quad (3)$$

and similarly for Zariski cohomologies

$$H_{\text{zar}}^n(\mathbb{A}_k^1 \times X, F_{\text{zar}}) \cong H_{\text{zar}}^n(X, F_{\text{zar}}), \quad (4)$$

and even more, there is the canonical isomorphism

$$H_{\text{zar}}^n(X, F_{\text{zar}}) \cong H_{\text{nis}}^n(X, F_{\text{nis}}). \quad (5)$$

In particular, the claim holds for presheaves with classical Voevodsky's transfers with respect to the finite correspondences $\text{Cor}(k)$.

Theorem

Let U be an essentially smooth local scheme, and D be a normal-crossing divisor.

Denote $V = U - D$, and let $\eta \in V$ be the generic point.

Then for a quasi-stable additive framed presheaf of abelian groups F over k , and $E_Z^l(X) = H_Z^l(X, F_{\text{nis}})$, where F_{nis} is the Nisnevich sheafification, the Cousin complex

$$0 \rightarrow E^l(V) \rightarrow E^l(\eta) \rightarrow \bigoplus_{x \in V^{(1)}} E_x^{l+1}(V) \rightarrow \cdots \bigoplus_{x \in V^{(r)}} E_x^{l+r}(V) \rightarrow \cdots \bigoplus_{x \in V^{(d)}} E_x^{l+d}(V) \rightarrow 0$$

is exact, $d = \dim V$.

Let k be a field.

Theorem

Given an abstract cohomology theory $E^\bullet$, there is a functor

$$E^\bullet: \langle K/k, K \text{ is finitely generated} \rangle \rightarrow \text{Ab}^{\text{gr}},$$

and boundary homomorphisms $E_*(\eta) \rightarrow E_{*-1}(z)$, where η and z are the generic and closed points of a scheme S , $\dim S = 1$, that

a) form the Gersten complex

$$\cdots \rightarrow 0 \rightarrow E^0(X) \rightarrow \bigoplus_{x \in X^{(0)}} E_0(x) \rightarrow \cdots \rightarrow \bigoplus_{x \in X^{(c)}} E_{-c}(x) \rightarrow \cdots$$

for any $U \in \text{Sm}_k$,

(b) that is exact on schemes of the form $U = X_x - D$ for a normal-crossing D in $X \in \text{Sm}_k$.

Remark: There are isomorphisms $E_c(x) \cong E_x^c(X)$.

Theorem

There is the equivalence of the heart of the homotopy t -structure $\mathbf{DM}^{\heartsuit}(k)$ and the category of Rost cyclemodules over k .

Theorem

Let V be a local henselian scheme, and F be an $\mathbb{A}^1$ -invariant quasi-stable framed additive presheaf of abelian groups over V . Let X be an open subscheme of $V \times \mathbb{A}^1$. Then

$$\underline{H}_{\mathrm{Nis}_Z^X}^*(X, F) = 0$$

for any closed subscheme Z in X finite over V .

Theorem

Let X be a smooth scheme over a one-dimensional irreducible base scheme B . Let $x \in X$ be a point over a closed point $z \in B$. Let F be an $\mathbb{A}^1$ -invariant quasi-stable framed presheaf of abelian groups over the generic point $\eta \in B$. Then

$$\underline{H}^i(X_x \times_B \eta, F) \cong 0$$

Proposition

(Zariski excision with finite support) Let V be a local scheme. Let $Z_0, Z_1 \rightarrow \mathbb{A}_V^1$ be closed immersions, $Z_0 \cap Z_1 = \emptyset$. Assume that Z_0 is finite over V .

Then for any quasi-stable linear framed presheaf

$$G: \mathbb{Z}F_*^{\text{pair}}(V) \rightarrow \text{Ab},$$

the morphism

$$j^*: G(\mathbb{A}_V^1, \mathbb{A}_V^1 - Z_0) \rightarrow G(\mathbb{A}_V^1 - Z_1, \mathbb{A}_V^1 - Z_1 - Z_0), \quad (6)$$

induced by the canonical open immersion of pairs

$$j: (\mathbb{A}_V^1 - Z_1, \mathbb{A}_V^1 - Z_1 - Z_0) \rightarrow (\mathbb{A}_V^1, \mathbb{A}_V^1 - Z_0)$$

is an isomorphism.

Proposition

Let B be one-dimensional irreducible local scheme, with the closed point $z \in B$ and the generic point $\eta \in B$. Let $X \in \text{Sm}_B$, $x \in X \times_B z$.

Denote $\mathcal{V} = X_{(x)} \times_B \eta$. Then for any closed subscheme Z in $X \times_B \eta$, there are framed correspondences $c \in \mathbb{Z}F_N(\mathcal{V} \times \mathbb{A}^1, X_\eta)$, $c' \in \mathbb{Z}F_N(\mathcal{V} \times \mathbb{A}^1, X_\eta - Z)$ such that

- (1) $c \circ i_0 = \sigma_{\mathcal{V}}^N \text{can}_\eta$, $c \circ i_0 = j \circ c'$, where $i_0, i_1: \mathcal{V} \rightarrow \mathcal{V} \times \mathbb{A}^1$ are the zero and unit sections, $\text{can}_\eta: \mathcal{V} \rightarrow X_\eta$ is the canonical morphism, $j: X_\eta - Z \rightarrow X_\eta$ is the open immersion,
- (2) $c^{-1}(Z)$ is finite over $\mathcal{V}$.

Lemma

Let B be a one-dimensional local scheme, $z \in B$ be the closed point, $\eta \in B$ be the generic point, $X \in \text{Sm}_B$, Z be a closed subscheme in $X_\eta = X \times_B \eta$.

Assume there is a one-dimensional compactified $X_{(x)}$ -focused framed correspondence $(\overline{C}, C_\infty, \Psi)$ from the η -scheme $\mathcal{V} = (X_{(x)})_\eta = X_{(x)} \times_B \eta$ to the η -scheme X_η such that

- (1) $(\overline{C}, C_\infty, \Psi)$ smoothly contains the canonical map $\text{can}_\eta: (X_{(x)})_\eta \rightarrow X_\eta$,
- (2) the closure $\overline{\mathcal{Z}}$ of $\mathcal{Z} = \text{val}^{-1}(Z)$ in $\overline{C}$ is finite over $X_{(x)}$, and $(\overline{\mathcal{Z}} \cap C_\infty) \times_B \eta = \emptyset$,
where $\Psi = (C_\eta, \varphi, \text{val})$, $C_\eta = (\overline{C} - C_\infty) \times_B \eta$.

Then there are framed correspondences $c \in \text{Fr}_N(\mathcal{V} \times \mathbb{A}^1, X_\eta)$,

$c_0^+, c_1 \in \text{Fr}_N(\mathcal{V} \times \mathbb{A}^1, X_\eta - Z)$, $N \in \mathbb{Z}$, such that

- (1) $c \circ i_0 = \sigma_{\mathcal{V}}^N \text{can}_\eta + c_0^+$, $\tilde{c} \circ i_1 = j \circ c_1$, in $\mathbb{Z}\text{F}_N(\mathcal{V}, X_\eta)$, where $i_0, i_1: \mathcal{V} \rightarrow \mathcal{V} \times \mathbb{A}^1$ are the zero and unit sections, $j: X_\eta - Z \rightarrow X_\eta$ is the open immersion,
- (2) $c^{-1}(Z)$ is finite over $\mathcal{V}$.