

$$d \log_{\mathbb{P}' \times \mathbb{P}'}^{A'}(T) = \frac{d}{dt} \log \frac{1}{(1-T)(1-q_\epsilon^{-1}T)} + \langle -1 \rangle \frac{d}{dt} \log \frac{1}{(1-q_\epsilon T)^2}$$

$$d \log_{\text{Res}_{\mathbb{F}_q^x/\mathbb{F}_q}}^{A'} \mathbb{P}'(T) = \frac{d}{dt} \log \frac{1}{(1-T)(1-q_\epsilon^{-1}T)} + \langle -u \rangle \frac{d}{dt} \log \frac{1}{1-q_\epsilon \omega T} + \langle -1 \rangle \frac{d}{dt} \log \frac{1}{1+q_\epsilon T}$$

$u \in \mathbb{F}_q^x / (\mathbb{F}_q^x)^2$ non-trivial

$$d \log_{\mathbb{P}^n}^{A'}(T) = \frac{d}{dt} \log \prod_{r \text{ even}} \frac{1}{(1-q_\epsilon^r T)} + \langle -1 \rangle \frac{d}{dt} \log \prod_{r \text{ odd}} \frac{1}{(1-q_\epsilon^r T)}$$

$$d \log_{\mathbb{P}^n}^{A'}(T) = d \log_{\mathbb{P}^n}^{A'}(T) = d \log_{\mathbb{P}^n}^{A'}(T) = d \log_{\mathbb{P}^n}^{A'}(T)$$

A Quadratically Enriched A' -Zeta Function

joint with M Bilu, W. Ho, P. Sin, I. Vogt

Weil Conjectures $X_{S_n, pr} / \mathbb{F}_q$

$$N_m = |X(\mathbb{F}_{q^m})|$$

connection with $X(\mathbb{C})$

Thm. (Dwork, Grothendieck, Deligne)

$$(\star) \sum_{m=1}^{\infty} N_m t^{m-1} = \frac{d}{dt} \log Z_X(T)$$

$$Z_X(T) = \prod_{i=0}^{2d} \det(1 - T F | H_{\text{ét}}^i(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell^{\text{cor}}))$$

$d = \text{Kull dim } X$

F Frobenius

$$\text{rk } H_{\text{ét}}^i = \text{rk } H^i(X(\mathbb{C}))$$

Algebraic Lemma

$$\Rightarrow \oplus \Leftarrow$$

Traces (

X dual

$$\text{Tr}(f) = 1$$

Vect_k

$D(R)$

SH^{top}

$\text{SH}(k)$

Notation $q \in K^*$, $\langle a \rangle \in \text{GW}(K)$ is the class of $R \otimes K \rightarrow R$
 $(x, y) \mapsto axy$

$$\text{GW}(\mathbb{F}_q) \cong \frac{\mathbb{Z}[u]}{\langle u^2-1, 2(u-1) \rangle}$$

Given $R \subseteq L$ finite separable

$$\text{Tr}_{L/K} : \text{GW}(L) \rightarrow \text{GW}(K)$$

$$(V \otimes L \rightarrow L) \mapsto (V \otimes L \rightarrow L \xrightarrow{\text{Tr}_{L/K}} K)$$

$$\text{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q} \langle 1 \rangle = \begin{cases} m & \text{for } m \text{ odd} \\ m-1 + \langle u \rangle & \text{for } m \text{ even} \end{cases} \quad \text{where } u \in \mathbb{F}_q^* \text{ is not a square}$$

Lefschetz FPT $f: X \rightarrow X$

$$\sum_{x: f(x)=x} \omega_x f = \text{Tr} f = \sum (-1)^i \text{Tr} f|H^i$$

$x: f(x)=x$

For $X/\mathbb{F}_q$ $\hookrightarrow F = \text{Frobenius}$ $\text{ind}_* f = [X/\mathbb{F}_q, K]$

$$\Rightarrow |N_m| = \sum (-1)^i \text{Tr} f|H^i_{\mathbb{Z}} \Rightarrow \text{rationality } \zeta_X$$

"Grothendieck-
with gap"

Group
completion

$d(u) = \text{of symm}$

Def (Bilu-Ho-Srinivas-Vogt-W) For $X \in \text{Sm}_k$, X dualizable (e.g. X sm and proper) and $f: X \rightarrow X$

$$\text{dlog}^{A'} \mathbb{Z}_{X,f}(T) = \sum_{m=1}^{\infty} \text{Tr}(f^m) T^{m-1} \in \text{GW}(k)[\langle T \rangle]$$

$k \subset \mathbb{R}$ $\text{End}_{\text{SH}^{\text{tr}}}(1) \xrightarrow{\mathbb{Q}\text{-points}} \text{End}_{\text{SH}(k)}(1) \xrightarrow{\mathbb{R}\text{-points}} \text{End}_{\text{SH}^{\text{tr}}}(1)$ Prop. (i) For $f: X \rightarrow X$ in $\text{Sm}_{\mathbb{R}}$
 $\downarrow \text{deg}^{\text{top}}$ $\downarrow \text{deg}^{A'}$ $\downarrow \text{deg}^{\text{top}}$ $\text{Sign } \text{dlog}^{A'} \mathbb{Z}_{X,f} \stackrel{(*)}{=} \frac{d}{dt} \log \prod (1 - T f | H^i(X, \mathbb{R}))$
 $\mathbb{Z} \xleftarrow{\text{rank}} \text{GW}(k) \xrightarrow{\text{Sign}} \mathbb{Z}$ (ii) For $X/\mathbb{F}_q$ with a lift of Frobenius to $\mathbb{Z}$,
 then $\text{dlog}^{A'} \mathbb{Z}_{X,f} \in \mathbb{Z}[\langle 1 \rangle, \langle -1 \rangle][\langle t \rangle]$ and $\bigoplus$

Ex Toric Varieties (Borger)

Prop. $\text{dlog} \mathbb{Z}_X^{A'}: K_0(\text{Var}_{\mathbb{F}_q}) \rightarrow \text{GW}(\mathbb{F}_q)[\langle T \rangle]$ motivic measure.

Rmk. We know of no way to recover $\text{dlog} \mathbb{Z}^{A'}$ from Kapranov's motivic Zeta function

$$d \log \mathbb{A}'(T) = \frac{d}{dt} \log \prod_{r \text{ even}} \frac{1}{(1 - q_\epsilon^r T)} + \langle -1 \rangle \frac{d}{dt} \log \prod_{r \text{ odd}} \frac{1}{(1 - q_\epsilon^r T)}$$

$u \in \mathbb{F}_q^* / (\mathbb{F}_q^*)^2$ non-trivial

Thm (Hoyois LFT) X $\text{Sm, proj} / K$ $f: X \rightarrow X$

Suppose $X^f = \{\text{fixed points}\}$ are étale over K

and $1 - df_x: T_x X \rightarrow T_x X$ is an isomorphism,

then $\text{Tr}(f) = \sum_{x \in X^f} \text{Tr}_{K(X)/K} \langle \det(1 - df) \rangle$

$$\Rightarrow \text{Tr}(\text{Frob}_{X/\mathbb{F}_q}^m) = \sum_{i|m} \alpha(i) \text{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q} \langle 1 \rangle$$

$\alpha(i) = \# \text{ points of } X \text{ with residue field } \mathbb{F}_{q^i}$

Ex: (SAGE) $y^2 = x^3 + x + 1$ over $\mathbb{F}_{13}$

$$d \log Z_X^{\mathbb{A}'}(T) = 18 + (179 + \langle u \rangle) T + 2106 T^2 + 28,800 T^3 + 371,538 T^4 + (4822739 + \langle u \rangle) T^5 + \dots$$