Static continuum theory of nematic liquid crystals.

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[1] Some images are from Google Earth and http://www.personal.kent.edu/ bisenyuk/liquidcrystals

Nematic Liquid Crystals

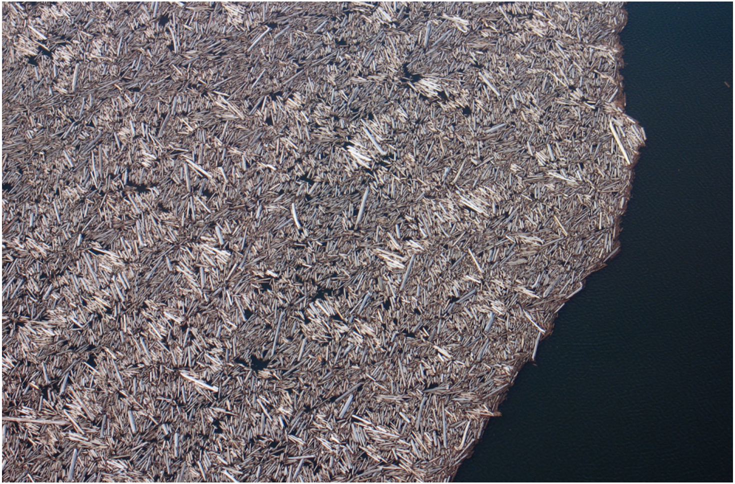


Figure: Logs in the Spirit Lake, Mt. St. Helens.

Nematic Liquid Crystals

Figure: Logs in the Spirit Lake, Mt. St. Helens.

Director-Based Theory

Suppose that a nematic occupies a domain $\Omega \subset \mathbb{R}^3$ and $\mathbf{n} : \Omega \to \mathbb{S}^2$. The *director* field $\mathbf{n}(\mathbf{x})$ represents local orientation of nematic molecules near $\mathbf{x} \in \Omega$.

To formulate a continuum variational theory, need a functional space and an energy functional that take into account

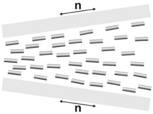
- Elastic distortions of the director field $\mathbf{n}$ in Ω
- Interactions of the nematic with the walls of the container, i.e. the boundary or *anchoring* conditions satisfied by the director field $\mathbf{n}$ on $\partial\Omega$.

Note: Additional effects (magnetic field, etc.) can be taken into account—beyond the scope of this talk.

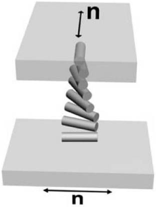
Oseen-Frank Model

Oseen-Frank elastic energy density (Frank, 1958):

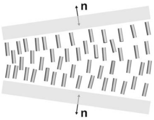
$$f_{OF}(\mathbf{n},\nabla\mathbf{n}) := \frac{K_1}{2}(\operatorname{div}\mathbf{n})^2 + \frac{K_2}{2}(\operatorname{curl}\mathbf{n}\cdot\mathbf{n})^2 + \frac{K_3}{2}|\operatorname{curl}\mathbf{n}\times\mathbf{n}|^2$$
$$+ \frac{K_2+K_4}{2}\left(\operatorname{tr}(\nabla\mathbf{n})^2 - (\operatorname{div}\mathbf{n})^2\right)$$



Splay



Twist



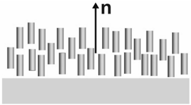
Bend

Saddle Splay

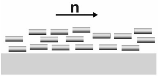
Anchoring Conditions

Controlled, e.g., by mechanical treatment or use of surfactants. Two possible types of boundary conditions:

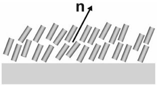
- Strong anchoring:



Homeotropic (Dirichlet)



Planar



Tilted

- Weak anchoring via a surface energy density term, e.g.:

$$f_{OF}^s(\mathbf{n},\nu) = \gamma(\mathbf{n}\cdot\nu)^2 \text{ or } f_{OF}^s(\mathbf{n},\nu) = \gamma\left((\mathbf{n}\cdot\nu)^2 - \cos^2\alpha\right)^2$$

where ν is an outward unit normal to $\partial\Omega$. The first expression is a *Rapini-Papoular* surface energy density.

Variational problem (Strong anchoring):

Minimize

$$F_{OF}[\mathbf{n}] := \int_\Omega \left\{ \frac{K_1}{2}(\operatorname{div}\mathbf{n})^2 + \frac{K_2}{2}(\operatorname{curl}\mathbf{n}\cdot\mathbf{n})^2 + \frac{K_3}{2}|\operatorname{curl}\mathbf{n}\times\mathbf{n}|^2 \right.$$
$$\left. + \frac{K_2+K_4}{2}\left(\operatorname{tr}(\nabla\mathbf{n})^2 - (\operatorname{div}\mathbf{n})^2\right) \right\}$$

in $H^1(\Omega, \mathbb{S}^2)$ subject to the appropriate boundary data, i.e., $\mathbf{n}|_{\partial\Omega} = \nu$ for the homeotropic anchoring.

(Hardt, Kinderlehrer and Lin, 1986) *For the positive K_1, K_2, K_3 global minimizers of F_{OF} exist among all maps in $H^1(\Omega, \mathbb{S}^2)$ subject to Lipschitz, $\mathbb{S}^2$-valued Dirichlet boundary data. Any minimizer is smooth except for a closed set of Hausdorff dimension strictly less than* 1.

Facts:

- When $K_1 = K_2 = K_3 = K$ and $K_4 = 0$, the Oseen-Frank energy reduces to the Dirichlet integral

$$F_{OF}[\mathbf{n}] = K \int_{\Omega} |\nabla \mathbf{n}|^2.$$

- The saddle-splay term is a null Lagrangian, i.e., its integral over Ω depends only on the boundary data $\rightarrow$ this term reduces to a constant for Dirichlet boundary conditions on $\mathbf{n}$.
- Any configuration with a line singularity (observed experimentally), e.g.,

$$\mathbf{n}(\mathbf{x}) = (x/\sqrt{x^2 + y^2}, y/\sqrt{x^2 + y^2}, 0)$$

has an infinite energy.

Inability to model line defects with a finite energy within OF theory can be addressed by

- Moving to higher-dimensional order parameters (Ericksen's theory for nematics with variable degree of orientations, Landau-de Gennes theory)
- Making suitable modifications to the energy functional and the class of admissible maps (Ball and Bedford, 2014)—e.g., replacing $|\nabla \mathbf{n}|^2$ with $|\nabla \mathbf{n}|^p$, where $1 < p < 2$ and/or allowing $\mathbf{n}$ to jump across surfaces by assuming that $\mathbf{n} \in SBV\left(\Omega, \mathbb{S}^2\right)$. The second modification addresses another shortcoming of director-based theories, the issue of orientability.

Note: Any version of a continuum theory based on a single vector field only works for uniaxial nematics and does not allow to model biaxiality.

Nematics with variable degree of orientation

To allow for line defects, Ericksen (1991) proposed to supplement **n** with a scalar field $s : \Omega \to \left(-\frac{1}{2}, 1\right)$ to describe the degree of local orientational order.

Simplified version of the energy functional:

$$F_E[s, \mathbf{n}] := \int_\Omega \left\{ K_s |\nabla s|^2 + K_n s^2 |\nabla \mathbf{n}|^2 + W(s, T) \right\}$$

Here

$$\min_{s \in \left(-\frac{1}{2}, 1\right)} W(s, T) = W(s_0(T), T) = 0$$

and

$$\lim_{s \to -1/2} W(s, T) = \lim_{s \to 1} W(s, T) = \infty.$$

The model allows for a phase transition between nematic and isotropic states:

$$s(T_0) = \begin{cases} s_0 \neq 0, & T < T_c, \quad \text{(nematic state = order)}, \\ 0, & T > T_c, \quad \text{(isotropic state = disorder)}, \end{cases}$$

where $T_c \in \mathbb{R}$ is a critical temperature.

If $K_s = K_n = K$, set $\mathbf{u} = s\mathbf{n}$ then

$$F_E[\mathbf{u}] = \int_\Omega K|\nabla \mathbf{u}|^2 + W(|\mathbf{u}|, T)$$

- Ginzburg-Landau model.

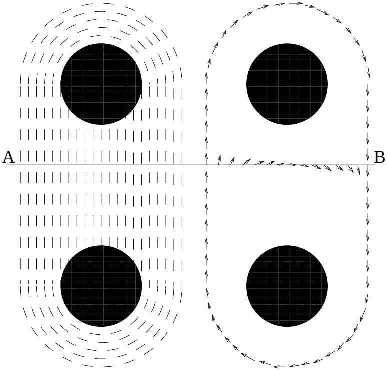
Note: As formulated, Ericksen's model does not resolve orientability issue and it cannot be used to model biaxiality.

Orientability

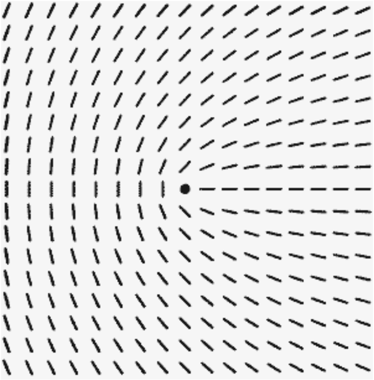
Experimental fact: Probability of finding the head of a molecule pointing in a given directions is equal to the probability of finding the tail of a molecule pointing in the same direction.

Consequence: $\mathbf{n}(\mathbf{x}) = \mathbf{n}_0$ and $\mathbf{n}(\mathbf{x}) = -\mathbf{n}_0$ for some $\mathbf{n}_0 \in \mathbb{S}^2$ correspond to the same nematic state at $\mathbf{x}$. The tensor field $\mathbf{n} \otimes \mathbf{n}$ (possibly, translated and/or dilated) is, however, invariant under inversion $\mathbf{n} \to -\mathbf{n}$.

Conclusion: Local orientation of nematic molecules is described by a line field with values in $\mathbb{RP}^2$ and not a vector field with values in $\mathbb{S}^2$. The classical OF theory will give incorrect predictions when a minimizing line field is not orientable (Ball and Zarnescu, 2010).



"Stadium" configuration (Ball and Zarnescu, 2010)



Nematic disclination

Q-tensor theory

Let $\Omega \subset \mathbb{R}^3$ and $\rho(\mathbf{n}, x)$ be a pdf of molecular orientations at $x \in \Omega$, where $\mathbf{n} \in \mathbb{S}^2$.

Since head and tail are equiprobable $\Longrightarrow \rho(-\mathbf{n}, x) = \rho(\mathbf{n}, x)$ and the first moment of ρ vanishes.

Nontrivial information about LC configuration at x is given by the second moment

$$M(x) = \int_{\mathbb{S}^2} (\mathbf{n} \otimes \mathbf{n}) \rho(\mathbf{n}, x) d\mathbf{n}$$

<u>Note:</u> $M^T(x) = M(x)$ and $\operatorname{tr} M(x) = 1$ for all $x \in \Omega$.

LC is isotropic at x if $\rho(\mathbf{n}, x) \equiv \frac{1}{4\pi} \Longrightarrow M(x) = M_{iso} = \frac{1}{3}\mathrm{I}$.

Q−tensor: $Q(x) = M(x) - M_{iso}$ so that Q vanishes in the isotropic state.

Nematic Q-tensor

$Q \in M^{3\times 3}_{sym}$ is a traceless tensor $\Rightarrow$ eigenvalues satisfy $\lambda_1 + \lambda_2 + \lambda_3 = 0$ with a mutually orthonormal eigenframe $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

Uniaxial nematic: repeated nonzero eigenvalues $\lambda_1 = \lambda_2 \Rightarrow$ $Q = S\left(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3}\mathbf{I}\right)$, where $S := \frac{3\lambda_3}{2}$ is the uniaxial nematic order parameter and $\mathbf{n} \in \mathbf{S}^2$ is the nematic director.

Biaxial nematic: no repeated eigenvalues $\Rightarrow$ $Q = S_1\left(\mathbf{e}_1 \otimes \mathbf{e}_1 - \frac{1}{3}\mathbf{I}\right) + S_3\left(\mathbf{e}_3 \otimes \mathbf{e}_3 - \frac{1}{3}\mathbf{I}\right)$, where $S_1 := 2\lambda_1 + \lambda_3$ and $S_3 = \lambda_1 + 2\lambda_3$ are biaxial order parameters.

Isotropic: all eigenvalues are equal zero $\Rightarrow$ $Q = 0$.

By construction, $\lambda_i \in \left[-\frac{1}{3}, \frac{2}{3}\right]$, where $i = 1, 2, 3$.

Bulk elastic energy density:

$$f_e(Q,\nabla Q) := \frac{L_1}{2}|\nabla Q|^2 + \frac{L_2}{2}Q_{ik,j}Q_{ij,k} + \frac{L_3}{2}Q_{ij,j}Q_{ik,k} + \frac{L_4}{2}Q_{lk}Q_{ij,k}Q_{ij,l}$$

Bulk Landau-de Gennes energy density:

$$f_{LdG}(Q) := a\,\mathrm{tr}\left(Q^2\right) + \frac{2b}{3}\mathrm{tr}\left(Q^3\right) + \frac{c}{2}\left(\mathrm{tr}\left(Q^2\right)\right)^2$$

Here $a(T)$ is temperature-dependent, $c > 0$, and $f_{LdG} \geq 0$ by adding an appropriate constant. Function of eigenvalues of Q only. Depending on T, minimum is either isotropic or nematic w/specific s.

Surface energy density (Either strong or weak anchoring):

$$f_s(Q) := f(Q,\nu)$$

on the boundary of the container and $\nu \in \mathbb{S}^2$ is a normal to the surface of the liquid crystal.

Remarks:

- When $L_4 \neq 0$, the energy

$$F_{LdG}[Q] = \int_\Omega f_e(Q, \nabla Q) + f_{LdG}(Q)$$

  is unbounded from below. The existence of the global minimizer can be established, subject to constraints on $L_i,\ i = 1, \ldots, 4$ if the potential term is modified according to the Ericksen's idea (Ball and Majumdar (2009)).
- When $L_4 = 0$, the functional is coercive subject to constraints on $L_i,\ i = 1, 2, 3$ (Gartland and Davis (1998), Longa et al (1987)). However, in this case, two of the three elastic constants in a Oseen-Frank reduction (when $Q = s\left(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3} I\right)$ with a constant s and $|\mathbf{n}| = 1$) must be equal.

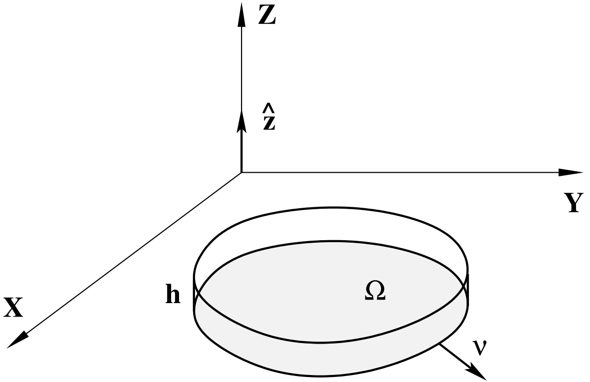


FIGURE: Geometry of the problem.

Here $\Omega \subset \mathbf{R}^2$ and $h > 0$ is small.

Nematic energy functional:

$$E[Q] := \int_{\Omega\times[0,h]} \{f_e(Q,\nabla Q) + f_{LdG}(Q)\}\ dV + \int_{\Omega\times\{0,h\}} f_s(Q,\hat{z})\, dA$$

Uniaxial data on the lateral boundary of the film:

$$Q|_{\partial\Omega\times[0,h]} = g \in H^{1/2}(\partial\Omega;\mathcal{A}).$$

Admissible class:

$$\mathcal{C}_h^g := \left\{ Q \in H^1\left(\Omega\times[0,h];\mathcal{A}\right) : Q|_{\partial\Omega\times[0,h]} = g \right\},$$

where $\mathcal{A}$ is the set of three-by-three symmetric traceless matrices.

Osipov-Hess surface energy

"Bare" surface energy (Osipov-Hess):

$$f_s(Q,\nu) := c_1(Q\nu\cdot\nu) + c_2 Q\cdot Q + c_3(Q\nu\cdot\nu)^2 + c_4|Q\nu|^2$$

where c_i, $i = 1,\ldots,4$ are constants.

Observe that:

$$Q\cdot Q = 2|Q\nu|^2 - (Q\nu\cdot\nu)^2 + Q_2\cdot Q_2,$$

where $Q_2 \in M^{2\times 2}_{sym}$ is a nonzero square block of $(\mathbf{I}-\nu\otimes\nu)\,Q\,(\mathbf{I}-\nu\otimes\nu)$.

The traceless condition for Q:

$$\operatorname{tr} Q_2 + x\cdot\nu = 0$$

where $x := Q\nu \in \mathbf{R}^3$.

In terms of x and Q_2:

$$f_s(Q,\nu) = c_1(x\cdot\nu) + c_2 Q_2\cdot Q_2 + (c_3 - c_2)\,(x\cdot\nu)^2 + (2c_2 + c_4)\,|x|^2$$

This expression has a family of surface-energy-minimizing tensors that is

1. parameterized by at least one free eigenvalue
2. normal to the surface of the liquid crystal is an eigenvector

as long as $c_2 = 0$, $\alpha = c_3 + c_4 > 0$, and $\gamma = c_4 > 0$. Then the surface energy has the form

$$f_s(Q,\nu) = \alpha\left[(Q\nu\cdot\nu) - \beta\right]^2 + \gamma|(\mathbf{I} - \nu\otimes\nu)\,Q\nu|^2$$

where $\beta = -\frac{c_1}{2(c_3+c_4)}$.

Let $L_4 = 0$ and

$$\tilde{x} = \frac{x}{D},\ \tilde{y} = \frac{y}{D},\ \tilde{z} = \frac{z}{h},\ F_\epsilon = \frac{2}{L_1 h} E,$$

where $D := \mathrm{diam}(\Omega)$. Set

$$\xi = \frac{L_1}{2D^2},\ \epsilon = \frac{h}{D},\ \delta = \sqrt{\frac{2\xi}{c}}$$

$$K_2 = \frac{L_2}{L_1},\ K_3 = \frac{L_3}{L_1}$$

$$A = \frac{a}{c},\ B = \frac{b}{c}$$

$$\tilde{\alpha} = \frac{\alpha}{\xi},\ \tilde{\gamma} = \frac{\gamma}{\xi}$$

Nondimensional energy

$$F_\epsilon[Q] = \int_{\Omega\times[0,1]} \left(f_e(\nabla Q) + \frac{1}{\delta^2} f_{LdG}(Q) \right) dV + \frac{1}{\epsilon}\int_{\Omega\times\{0,1\}} f_s(Q,\hat{z})\, dA,$$

where

$$\begin{aligned} f_e(\nabla Q) := & \left[|\nabla_{xy} Q|^2 + K_2 Q_{ik,j} Q_{ij,k} + K_3 Q_{ij,j} Q_{ik,k} \right] \\ & + \frac{2}{\epsilon} \left[K_2 Q_{i3,j} Q_{ij,3} + K_3 Q_{ij,j} Q_{i3,3} \right] \\ & + \frac{1}{\epsilon^2} \left[|Q_z|^2 + (K_2+K_3) Q_{i3,3}^2 \right], \end{aligned}$$

$$f_{LdG}(Q) = 2A\,\mathrm{tr}\left(Q^2\right) + \frac{4}{3} B\,\mathrm{tr}\left(Q^3\right) + \left(\mathrm{tr}\left(Q^2\right)\right)^2,$$

$$f_s(Q,\hat{z}) = \alpha\left[(Q\nu\cdot\nu) - \beta\right]^2 + \gamma\left|(\mathbf{I} - \nu\otimes\nu)\,Q\nu\right|^2.$$

Assumptions

Suppose for simplicity that $K_2 = K_3 = 0$ then for every $Q \in \mathcal{C}_1^g$

$$F_\epsilon[Q] = \int_{\Omega\times[0,1]} \left\{ |Q_x|^2 + |Q_y|^2 + \frac{1}{\epsilon^2}|Q_z|^2 \right. \\ \left. + \frac{1}{\delta^2}\left(2A\,\mathrm{tr}\left(Q^2\right) + \frac{4}{3}B\,\mathrm{tr}\left(Q^3\right) + \left(\mathrm{tr}\left(Q^2\right)\right)^2\right)\right\} dV \\ + \frac{1}{\epsilon}\int_{\Omega\times\{0,1\}} \left(\alpha\left[\left(Q\hat{z}\cdot\hat{z}\right) - \beta\right]^2 + \gamma\left|\left(\mathbf{I} - \hat{z}\otimes\hat{z}\right)Q\hat{z}\right|^2\right) dA,$$

and set

$$f_s(Q,\hat{z}) =: f_s^{(0)}(Q,\hat{z}) + \epsilon f_s^{(1)}(Q,\hat{z})$$

—this allows for different asymptotic regimes for α and γ.

Let

$$F_0[Q] := \begin{cases} 2\int_\Omega \left\{ |\nabla_{xy} Q|^2 + \frac{1}{\delta^2} f_{LdG}(Q) + f_s^{(1)}(Q, \hat{z}) \right\} dA & \text{if } Q \in H^1_g, \\ +\infty & \text{otherwise.} \end{cases}$$

Here

$$H^1_g := \left\{ Q \in H^1(\Omega; \mathcal{D}) : Q|_{\partial\Omega} = g \right\}$$

and

$$\mathcal{D} := \left\{ Q \in \mathcal{A} : Q \in \operatorname{argmin}_{Q \in \mathcal{A}} f_s^{(0)}(Q) \right\},$$

for some boundary data $g : \partial\Omega \to \mathcal{D}$.

Theorem (G, Montero, Sternberg (2015))

Fix $g : \partial\Omega \to \mathcal{D}$ such that H^1_g is nonempty. Then Γ-$\lim_\epsilon F_\epsilon = F_0$ weakly in $\mathcal{C}^g_1$. Furthermore, if a sequence $\{Q_\epsilon\}_{\epsilon>0} \subset \mathcal{C}^g_1$ satisfies a uniform energy bound $F_\epsilon[Q_\epsilon] < C_0$ then there is a subsequence weakly convergent in $\mathcal{C}^g_1$ to a map in H^1_g.

Proof.

Idea: can use a trivial recovery sequence. Indeed, if $Q_\epsilon \equiv Q \in \mathcal{C}^g_1 \backslash H^1_g$ then $\lim_{\epsilon\to 0} F_\epsilon[Q_\epsilon] = +\infty = F_0[Q]$ and when $Q_\epsilon \equiv Q \in H^1_g$ then $F_\epsilon[Q_\epsilon] = F_0[Q_\epsilon] = F_0[Q]$ for all ϵ. □

Let

$$f_s^{(0)} = \alpha \left[(Q\hat{z} \cdot \hat{z}) - \beta \right]^2 + \gamma \left| (\mathbf{I} - \hat{z} \otimes \hat{z})\, Q\hat{z} \right|^2 \text{ and } f_s^{(1)} \equiv 0$$

$\Rightarrow$ two types of $\mathcal{D}$-valued uniaxial Dirichlet data on $\partial\Omega$:

- $Q = -3\beta \left(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3}\mathbf{I} \right)$, where $\mathbf{n} \perp \hat{z}$ is **any** $\mathbb{S}^1$-valued field on $\partial\Omega$.
- $Q = \frac{3\beta}{2} \left(\hat{z} \otimes \hat{z} - \frac{1}{3}\mathbf{I} \right)$.

Note: In the first case the boundary data can have any degree, while in the second case the Dirichlet condition is completely rigid as Q is equal to a constant.

Can represent $Q \in H^1_g$ as

$$Q = \begin{pmatrix} p_1 - \frac{\beta}{2} & p_2 & 0 \\ p_2 & -p_1 - \frac{\beta}{2} & 0 \\ 0 & 0 & \beta \end{pmatrix}.$$

Then

$$F_0[Q] = \tilde{F}_0[\mathbf{p}] := \int_\Omega \left\{ 2|\nabla \mathbf{p}|^2 + \frac{1}{\delta^2} W(|\mathbf{p}|) \right\} dV,$$

where $\mathbf{p} = (p_1, p_2)$ and

$$W(t) = 4t^4 + \tilde{C}t^2 + \tilde{D},$$

with $\tilde{C} = 6\beta^2 - 4B\beta + 4A$ and $\tilde{D} \in \mathbb{R}$.

If

$$Q|_{\partial\Omega\times[0,1]} = \frac{3}{2}\beta\left(\hat{z}\otimes\hat{z} - \frac{1}{3}\mathbf{I}\right),$$

admissible functions satisfy the boundary condition

$$\mathbf{p}|_{\partial\Omega} = \mathbf{0}.$$

The minimizer of

$$\tilde{F}_0[\mathbf{p}] = \int_\Omega \left\{2|\nabla\mathbf{p}|^2 + \frac{1}{\delta^2}W(|\mathbf{p}|)\right\} dV$$

then has a constant angular component $\Rightarrow$ scalar minimization problem for $p := |\mathbf{p}|$ and

1. If $\tilde{C} \geq 0$ then the minimizer $p \equiv 0$.
2. If $\tilde{C} < 0$ then the minimizer p solves the problem

$$-\Delta p + \frac{1}{\delta^2}W'(p) = 0 \text{ in } \Omega, \quad p = 0 \text{ on } \partial\Omega.$$

Now suppose

$$Q|_{\partial\Omega\times[0,1]} = -3\beta\left(\mathbf{n}\otimes\mathbf{n} - \frac{1}{3}\mathbf{I}\right),$$

where $\mathbf{n} : \partial\Omega \to \mathbb{S}^1$.

We have

$$\mathbf{p} = -3\beta\left(n_1^2 - \frac{1}{2}, n_1 n_2\right),$$

on $\partial\Omega$ where $|\mathbf{p}| = \frac{3\beta}{2}$. If $\mathbf{p}$ is smooth and nonvanishing, it has a well-defined winding number $d \in \mathbb{Z}$. We set the degree of g to be equal to $d/2$. Then $\mathbf{p}_0$ must vanish somewhere within a vortex core structure of a characteristic size of δ in Ω.

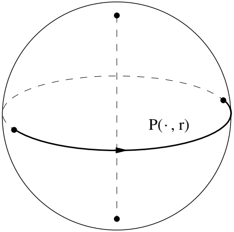


FIGURE: Geometry of the target space.

Topologically nontrivial boundary data will cause the director to "escape" from the xy-plane to the z-direction. The requirement that Q_0 takes values in $\mathcal{D}$ forces the escape to happen through biaxial states that are heavily penalized by the Landau-de Gennes energy.

Degree of biaxiality:

$$\xi(Q)^2 := 1 - 6\frac{\left(\operatorname{tr} Q^3\right)^2}{\left(\operatorname{tr} Q^2\right)^3} = 1 - 27\,\frac{\beta^2\left(4p^2 - \beta^2\right)^2}{(4p^2 + 3\beta^2)^3}$$

where $\xi(Q) = 0$ implies that Q is uniaxial.

Asymtotic regimes - Regime II

Let $\alpha = \epsilon a$ for some $a > 0$ and $\gamma = O(1)$. Then

$$f_s^{(0)}(Q, \hat{z}) = \gamma \left| (\mathbf{I} - \hat{z} \otimes \hat{z})\, Q\hat{z} \right|^2$$

and

$$f_s^{(1)}(Q\hat{z}) = a \left[(Q\hat{z} \cdot \hat{z}) - \beta \right]^2$$

and the target set $\mathcal{D}$ consists of traceless symmetric tensors having $\hat{z}$ as one of its eigenvectors. The limiting functional for $Q \in H^1_g$ is

$$F_0[Q] = 2 \int_\Omega \left\{ |\nabla Q|^2 + \frac{1}{\delta^2} f_{LdG}(Q) + a \left[(Q\hat{z} \cdot \hat{z}) - \beta \right]^2 \right\} dA$$

(cf. Bauman-Park-Phillips when $\delta \to 0$ and $a = 0$).