MOMENTS OF THE MAXIMUM OF TWO CORRELATED NORMALS

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Abstract. Formulaes for the expectation and variance of the maximum of two correlated normal distributed random variables are given. The variance requires the evaluation of an expectation using numerical integration. Certain special cases are considered in detail and several illustrative examples are provided.

1. Expectation and variance

Consider a bi-normally distributed pair (X, Y) with means μ_1 and μ_2, standard deviations σ_1 and σ_2, and correlation ρ. The two first moments for the maximum of these are sought.

A useful algebraic identity is

$$\max(X, Y) = \tfrac{1}{2}(X + Y + |X - Y|),$$

where $|\cdot|$ is the absolute value. Moreover for $U \sim N(\mu, \sigma^2)$ one has

$$\mathrm{E}\big\{|U|\big\} = \sigma A(\mu/\sigma),$$

where

$$A(u) = 2\phi(u) + u\big(2\Phi(u) - 1\big),$$

and $\phi(u)$ is the standard normal density and $\Phi(u)$ is the standard normal cumulative distribution function. The function $A(u)$ is shown in Figure 1.1. It is seen that $A(u)$ is symmetric and approaches $|u|$ rapidly away from $u = 0$. Let now

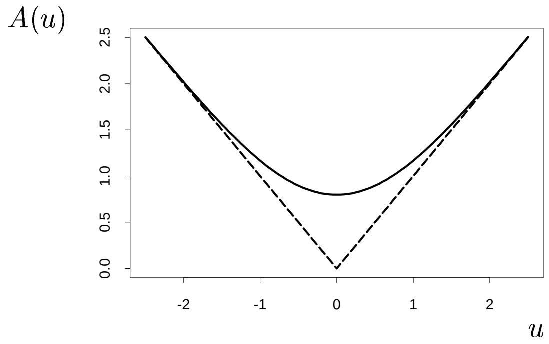

Figure 1.1: The function $A(u)$. The dashed lines are $|u|$.

$$v_+^2 = \mathrm{Var}\big\{X + Y\big\} = \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2$$
$$v_-^2 = \mathrm{Var}\big\{X - Y\big\} = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2.$$

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Then the expectation of the maximum is

$$\mathrm{E}\big\{\max(X,Y)\big\} = \tfrac{1}{2}\big[\mu_1 + \mu_2 + v_- A\big(\tfrac{\mu_1-\mu_2}{v_-}\big)\big], \tag{1}$$

and the variance is

$$\mathrm{Var}\big\{\max(X,Y)\big\} = \tfrac{1}{4}(v_+^2 + v_{|-|}^2 + 2\mathrm{cov}), \tag{2}$$

where v_+ is as above and where

$$v_{|-|}^2 = \mathrm{Var}\big\{|X-Y|\big\} = (\mu_1-\mu_2)^2 + v_-^2 - v_-^2 A\big(\tfrac{\mu_1-\mu_2}{v_-}\big)^2.$$

It remains to find an expression for $\mathrm{cov} = \mathrm{Cov}\big\{X+Y, |X-Y|\big\}$.

For a general bi-normal pair (U,V) with means μ_u and μ_v, standard deviations σ_u and σ_v, and correlation ρ_{uv}, then

$$\begin{aligned}\mu_{v|u} &= \mathrm{E}\big\{V\big|U\big\} = \mu_v + \rho_{uv}\frac{\sigma_v}{\sigma_u}(U-\mu_u)\\ \sigma_{v|u}^2 &= \mathrm{Var}\big\{V\big|U\big\} = \sigma_v^2(1-\rho_{uv}^2).\end{aligned}$$

Therefore

$$\mathrm{E}\big\{U|V|\big\} = \mathrm{E}\big\{\mathrm{E}\{U|V|\big|U\}\big\} = \sigma_{v|u}\,\mathrm{E}\big\{UA(\mu_{v|u}/\sigma_{v|u})\big\}, \tag{3}$$

and the covariance is

$$C(\mu_u,\sigma_u,\mu_v,\sigma_v,\rho_{uv}) = \mathrm{Cov}\big\{U,|V|\big\} = \mathrm{E}\big\{U|V|\big\} - \mu_u\sigma_v A(\mu_v/\sigma_v).$$

The expectation in (3) must be evaluated by numerical integration, but note that this is only a one-dimensional integral. The cov term needed in (2) is now given as

$$\mathrm{cov} = \mathrm{Cov}\big\{X+Y,|X-Y|\big\} = C\big(\mu_1+\mu_2, v_+, \mu_1-\mu_2, v_-, \tfrac{\sigma_1^2-\sigma_2^2}{v_+v_-}\big).$$

1.1. **Some special cases.** If $\sigma_1 = \sigma_2$ the last argument of C vanishes. This corresponds to V and U in (3) being independent. In this case $\mu_{v|u} = \mu_v$ is independent of U and the expectation in (3) becomes $\mu_u A(\mu_v/\sigma_v)$ so that C and cov vanishes. Thus numerical integration is not needed.

If $\sigma_1 = 0$ or $\sigma_2 = 0$ then ρ must also be zero. Then $v_+ = v_- = \sigma^2$ where σ is the non-zero standard deviation. Further it follows that $\rho_{uv} = \pm 1$ so that $\mu_{v|u} = \mu_v \pm (U-\mu_u)$ and $\sigma_{v|u} = 0$. For small $\sigma_{v|u} \to 0$ and non-zero $\mu_{v|u}$ then $A\big(\mu_{v|u}/\sigma_{v|u}\big) \to |\mu_{v|u}/\sigma_{v|u}|$. Therefore $\mathrm{E}\big\{U|V|\big\} \to \mathrm{E}\big\{U|\mu_{v|u}|\big\}$. The limit fails at $\mu_{v|u} = 0$ but the contribution to the integral is zero since a point has zero measure. Thus

$$C = \begin{cases}\mathrm{E}\big\{U|\mu_v+\mu_u-U|\big\} - \mu_u\sigma_2 A(\mu_v/\sigma_2) & \text{for } \sigma_1 = 0\\ \mathrm{E}\big\{U|\mu_v-\mu_u+U|\big\} - \mu_u\sigma_1 A(\mu_v/\sigma_1) & \text{for } \sigma_2 = 0.\end{cases} \tag{4}$$

If $\sigma_1 = 0$ or $\sigma_2 = 0$ and $\mu_1 = \mu_2 = 0$ then $\mu_v = \mu_v = 0$ and the expectation in (4) becomes zero due to symmetry. Moreover, the last term in (4) also vanishes so that C is zero.

2. Examples

To illustrate the properties of the results in the previous section two random variables X and Y are considered having moments

$$\mu_1 = 0 \qquad \sigma_1 = \tfrac{9}{10}$$
$$\mu_2 = x \qquad \sigma_2 = 1.$$

Results are given for different values of μ_2.

2.1. Varying ρ.

Figure 2.1 shows the expectation of $\max(X, Y)$ for different correlations. The corresponding standard deviations are given in Figure 2.2.

Figure 2.1 shows that the difference between $\max(\mu_1, \mu_2)$ and the expectation of $\max(X, Y)$ almost vanish for $\rho = 1$. However, the differences are significant as ρ approaches -1.

Figure 2.2 shows that the standard deviation of $\max(X, Y)$ makes a "jump" from σ_1 to σ_2 for $\rho = 1$. For zero correlation however the standard deviation is smaller than μ_1 around the intersection point and this reduction is almost 50% as ρ approaches -1. Also note the remarkably large area influenced.

2.2. Varying σ_1.

The example above is modified by letting $\rho = 0$ and varying σ_1 from 0 to 1. Figure 2.3 illustrates expectations of $\max(X, Y)$ while Figure 2.4 shows the corresponding standard deviations. The expectation is almost insensitive to variations in σ_1.

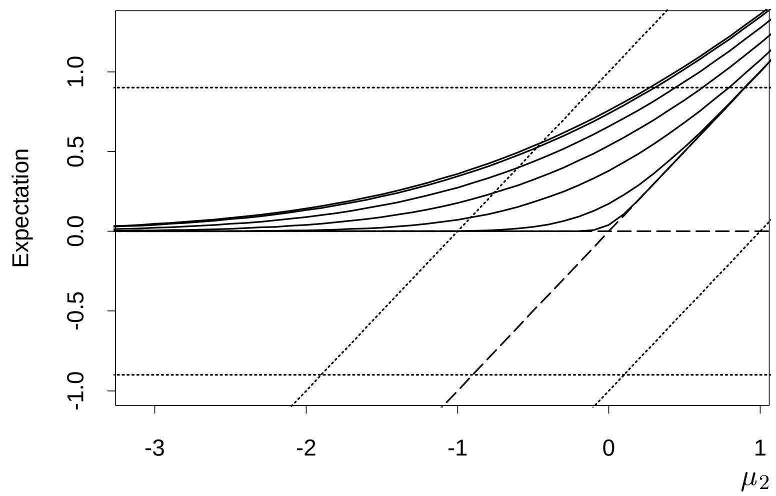

FIGURE 2.1: Expectations as a function of μ_2. The two dashed lines are μ_1 and μ_2 with surrounding dotted lines; $\mu_1 \pm \sigma_1$ and $\mu_2 \pm \sigma_2$. The solid curves are the expectation of $\max(X, Y)$ for $\rho = -1$, $-9/10$, $-1/2$, 0, 1/2, 9/10, and 1. The solid curve closest to the dashed lines corresponds to $\rho = 1$.

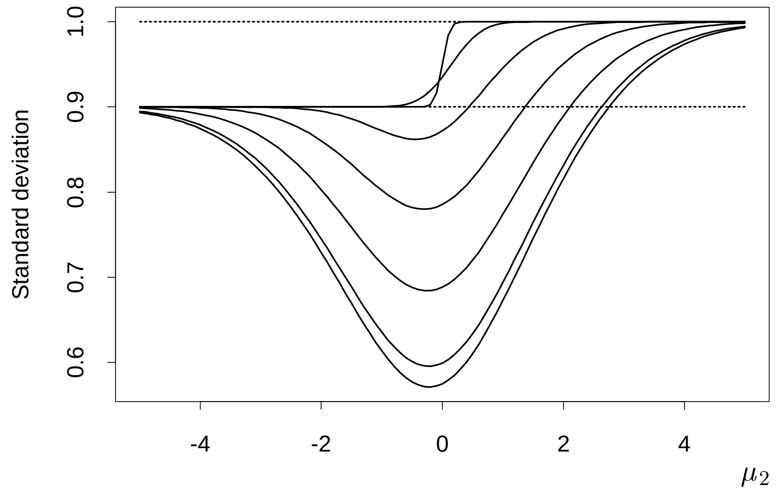

FIGURE 2.2: Standard deviations as a function of μ_2. The dotted lines are $\sigma_1 = \frac{9}{10}$ and $\sigma_2 = 1$. The solid curves are standard deviations of $\max(X, Y)$ for $\rho = -1$, $-9/10$, $-1/2$, 0, 1/2, 9/10, and 1. The lowest solid curve is for $\rho = -1$.

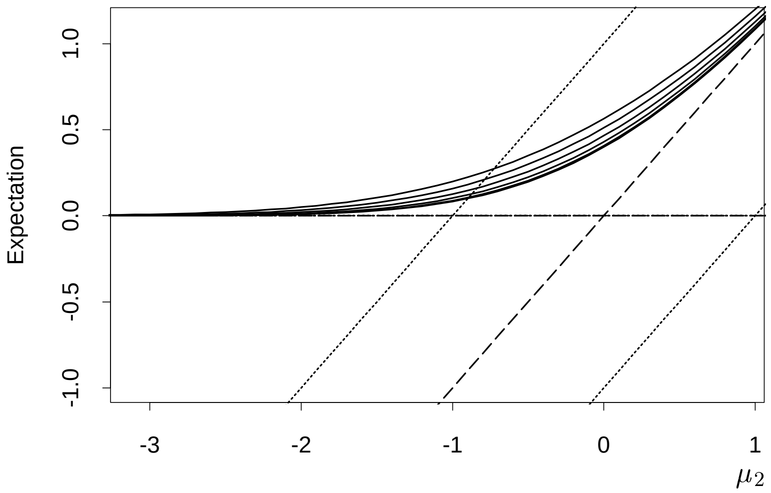

FIGURE 2.3: Expectations as a function of μ_2. The two dashed lines are μ_1 and μ_2. The solid curves are the expectation of $\max(X, Y)$ for $\sigma_1 = 0$, 0.2, 0.4, 0.6, 0.8, and 1. The deepest curve corresponds to $\rho = 1$.

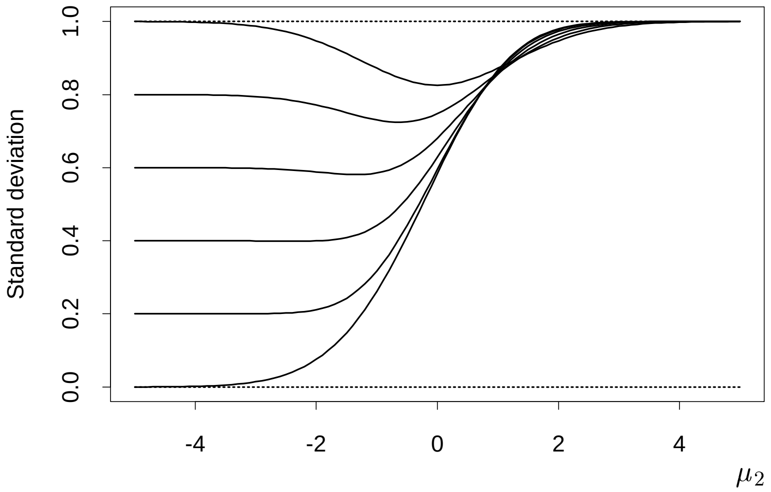

FIGURE 2.4: Standard deviations as a function of μ_2. The dotted lines are $\sigma_1 = 0$ and $\sigma_2 = 1$. The solid curves are standard deviations of $\max(X, Y)$ for $\sigma_1 = 0$, 0.2, 0.4, 0.6, 0.8, and 1. The lowest curve corresponds to $\sigma_1 = 0$.

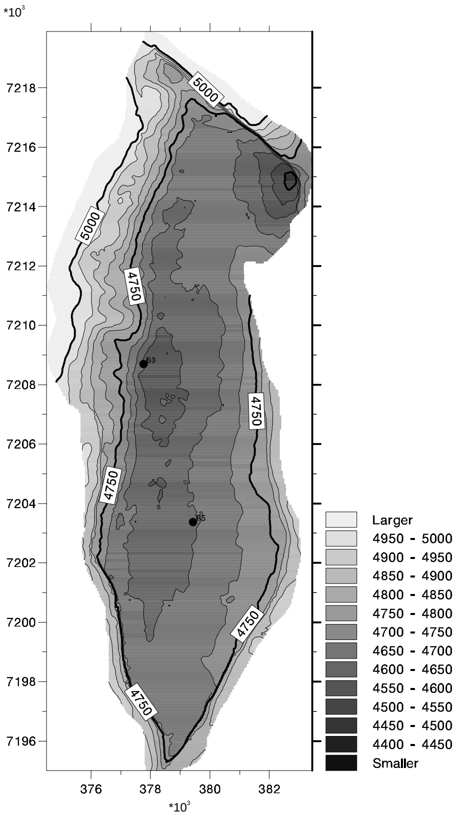

FIGURE 3.1: Predicted depth to eroding surface.

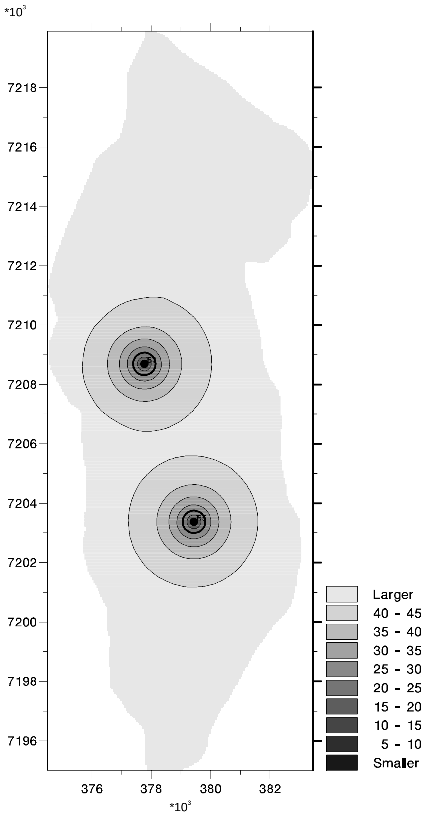

FIGURE 3.2: Prediction error to eroding surface.

3. Geological surfaces

In this example an eroding surface truncates the cap-rock of a petroleum reservoir. The two subsurfaces are predicted by Bayesian kriging using seismic data as trends and information from two wells. One of the wells penetrates the lower surface in the erosion zone so that no direct observation of the depth to the lower surface at this location is available

FIGURE 3.1 shows a contoured map of the predicted depth to an eroding surface called Base Cretaceous. The corresponding prediction error is shown in FIGURE 3.2. FIGURE 3.3 shows a contoured map of the predicted depth to a petroleum reservoir. This surface is extrapolated to the east and must be eroded by Base Cretaceous to obtain the correct depth to the reservoir. In FIGURE 3.5 the top is eroded using (1). The corresponding prediction error from (2) is shown in FIGURE 3.6. We see how the prediction error is reduced along the truncation line. The correlation between Base Cretaceous and the uneroded top of the reservoir is assumed to be zero.

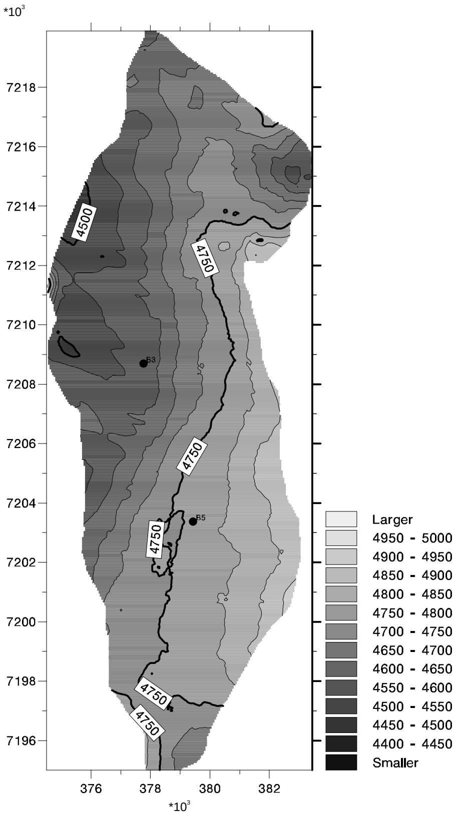

FIGURE 3.3: Predicted depth to uneroded surface.

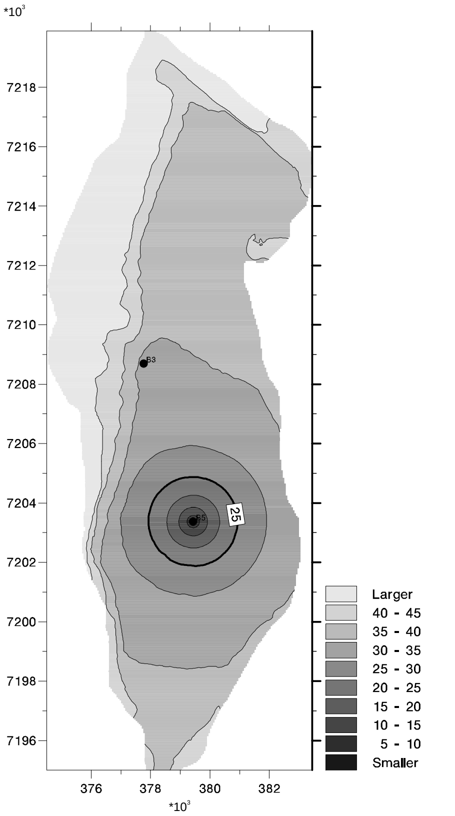

FIGURE 3.4: Prediction error to uneroded surface.

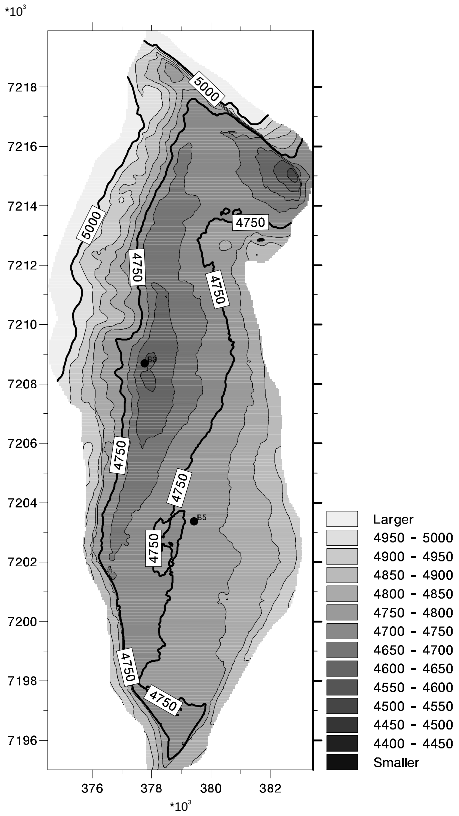

FIGURE 3.5: Predicted depth to eroded surface.

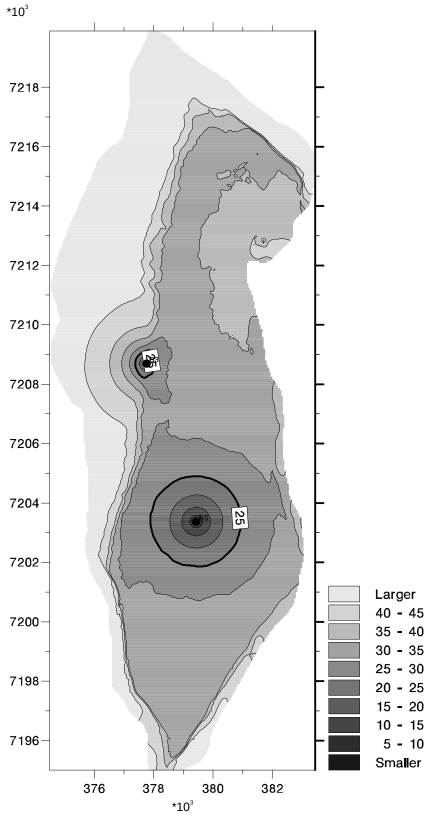

FIGURE 3.6: Prediction error to eroded surface.

Appendix A. S-plus code

```
muMax_function(m1,s1,m2,s2,r)
{
  v1_sqrt(s1^2 + s2^2 + 2*r*s1*s2)
  v2_sqrt(s1^2 + s2^2 - 2*r*s1*s2)
  mu_ifelse(m1>m2,m1,m2)         # Handle s1=s2=0 => v2=0
  mu_ifelse(v2>0,(m1 + m2 + v2*A((m1-m2)/v2))/2,mu)
  mu
}

sigmaMax_function(m1,s1,m2,s2,r)
{
  v1_sqrt(s1^2 + s2^2 + 2*r*s1*s2)
  v2_sqrt(s1^2 + s2^2 - 2*r*s1*s2)
  rr_ifelse(v1*v2>0,(s1^2-s2^2)/(v1*v2),0)
  cov_Cc(m1+m2,v1,m1-m2,v2,rr)
  a_ifelse(v2>0,A((m1-m2)/v2),0)
  tau2_(m1-m2)^2 + v2^2 * (1 - a^2)
  sigma_ifelse(v2>0,sqrt(v1^2 + tau2 + 2 * cov)/2,0)
  sigma
}

A_function(u)
{
   tmp_2*dnorm(u)+u*(2*pnorm(u)-1)
   tmp
}

Cc_function(mu,su,mv,sv,rr)
{
  t2_sv*sqrt(1-rr^2)
  cc_mu-mu
  for (i in 1:length(mu))
  {
     if (rr[i] != 0.0 && su[i]>0.0)
     {
        lower_mu[i]-7*su[i]
        upper_mu[i]+7*su[i]
        step_(upper-lower)/5000
        u_seq(from=lower,to=upper,by=step) #Integration steps.
        if (t2[i]>0.0) {
           t1_mv[i] + rr[i]*sv[i]*(u-mu[i])/su[i]
           cc[i]_t2[i]*sum(u*A(t1/t2[i])*dnorm(u,mean=mu[i],sd=su[i]))*step
        }
        else
        {
           t1_mv[i] + rr[i]*(u-mu[i])
           cc[i]_sum(u*abs(t1)*dnorm(u,mean=mu[i],sd=su[i]))*step
        }
     }
     else if (sv[i]>0)
        cc[i]_t2[i]*mu[i]*A(mv[i]/sv[i])
  }
  out_ifelse(sv>0,cc - mu*sv*A(mv/sv),0)
  out
}
```