

Volatility and intensity*

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Abstract

When studying models and estimators in the setting of high-frequency data, simplifying assumptions are typically imposed on the relation between the observation times and the observable process. In this paper we study a certain form of endogeneity of the observation times, namely that they depend on non-observable spot processes. The prototypical example is that of a stochastic volatility model observed at times which are in part determined by the stochastic volatility process. We introduce an estimator of the quadratic covariation between two spot process semimartingales, and apply this estimator to the problem of assessing the relation between the spot volatility and the intensity process governing the observations times. Consistency of this estimator is proved, and its convergence rate is derived. In an empirical study of the Apple stock over 21 trading days we find indications of a correlation between the spot volatility and the observation times.

KEYWORDS: Asynchronous times; consistency; convergence rates; counting processes; endogenous observation times; high-frequency; intensity; irregular times; microstructure; observed asymptotic variance; two-scales estimation.

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1 Introduction

When estimating parameters associated with a continuous time process that is only observed at discrete times, simplifying assumptions are often imposed on the relation between the observation times and the underlying process. The observation times are typically either taken as fixed and equidistant, or they are governed by a stochastic process postulated to be independent of the observable process (see e.g., Aït-Sahalia and Jacod (2014, Ch. 9) for a discussion). We refer to both cases as ‘exogenous times’. It is easy to think of scientific settings where the assumption of exogenous times makes perfect sense (think of experiments in biology, physics and neuroscience, where a measurement instrument is rigged to measure some continuous phenomenon at regular intervals), but it is just as easy to think of situations where it is clearly violated (e.g., in longitudinal studies where measurements are taken when an individual shows up for a medical test). The case of high-frequency data is a pertinent example of the latter: That decisions to buy or sell a given security is, in part, determined by features of that security, seems obvious. And since it is only at the times at which transactions are conducted that we get a glimpse of the continuous processes ticking in the background (modulo microstructure noise), one would; (i) expect the observation times to be correlated with the transaction-igniting features of the underlying process; (ii) ask in what ways ones sample may be biased; and consequently, (iii) empirically assess the direction and magnitude of the correlation; and (iv) study the implications of (i)-(iii) for estimators of quantities of interest.

In recent years, much progress has been made when the assumption of exogenous times is relaxed. In Li et al. (2013, 2014) the realised volatility estimator is studied in the presence of endogenous observation times, and it is shown that a bias term appears in the limiting distribution of this estimator, i.e., a high-level example of point (ii) above. Jacod et al. (2019) construct an estimator of the integrated volatility in the presence of microstructure noise, jumps and endogenous times. Other papers have dealt with consistency and central limit theorems under irregular and random times (Renault and Werker 2011; Hayashi et al. 2011; Fukasawa and Rosenbaum 2012). Common for all the above papers is that the endogeneity of the observation times, to the extent that it is present, comes about because the times depend on the efficient price process itself.

In this paper we study forms of observation time endogeneity where the observation times depend on underlying non-observable features of the efficient price process, such as its volatility, the associated volatility-of-volatility, the leverage effect, and so on. The crucial assumption is that these features are semimartingales, that they can be adequately estimated from the data, and that the intensity process of the observation times is itself a semimartingale. Under these assumptions we introduce an estimator of the quadratic covariation (realised covariance) between such spot process semimartingales and the intensity process of the observation times. As will become clear, the quadratic covariation between spot processes of the underlying process and the intensity process of the observation times is just one example of an estimand for which our estimator may be applied. Under some regularity conditions, the estimator we introduce in this paper works for any two spot

process semimartingales, say θ_t and λ_t , as long as their integrated versions, that is $\int_0^t \theta_s ds$ and $\int_0^t \lambda_s ds$, can be consistently estimated from the data.

The paper proceeds as follows. In Section 2 we provide a framework for the study of dependence between the observations times and non-observable spot processes associated with the observable process. To fix ideas, a parametric model that exhibits the phenomenon we have in mind is presented. In Section 3 we present an estimator of the quadratic covariation between two non-observable spot process semimartingales, and prove that this estimator is consistent. Subsequently, we specialise this estimator to the endogenous time case that motivated the present paper. To further assess the accuracy of these estimators, Section 3.3 contains a general theorem on the convergence rate of the approximation of certain rolling and overlapping sums of squared semimartingale increments to the quadratic variation of these semimartingales. Section 4 contains a simulation study providing insight into the appropriate choice of tuning parameters, as well as an empirical analysis of the Apple stock over one month. The empirical analysis indicates that the observation times and the volatility process of the Apple stock are correlated. In Section 5 we conclude, and discuss some problems for future research along with complementing remarks. Most of the proofs are deferred to the appendix.

2 The problem and a model

We observe data at high frequency over a time period from 0 to T . The observations are samples from a semimartingale X_t , typically contaminated by microstructure noise (see e.g., Zhang et al. (2005)). All the random quantities introduced in the following are defined on a fixed filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{0 \leq t \leq T}, P)$, $\mathcal{F}_T \subseteq \mathcal{F}$. When we say that a process is a semimartingale, we mean that it is a semimartingale relative to $\{\mathcal{F}\}_{0 \leq t \leq T}$. The filtered probability space is taken to satisfy the so-called usual conditions (Jacod and Shiryaev 2013, Definitions I.1.2-I.1.3, p. 2). For a given frequency of observations, indexed by $n \geq 1$, the successive observations occur at times $0 = T_{n,0} < T_{n,1} < \dots$, where $\{T_{n,i}\}_{n \geq 1}$ is a sequence of finite stopping times. Set $\Delta_{n,i} = T_{n,i} - T_{n,i-1}$ and define the sequence of counting processes $N_{n,t} = \sum_{i \geq 1} I\{T_{n,i} \leq t\}$. We are going to assume (in Condition 1) that, for observation frequency n , the inter-observational lags $\Delta_{n,i}$ are of the same order of magnitude as $1/n$ (see Li et al. (2014) and Jacod et al. (2017, 2019) for similar constructions), and moreover, that $n^{-1}N_{n,t}$ has a possibly random probability limit when n goes to infinity.

Based on the $N_{n,T}$ observations of X_t , we form an estimator $\hat{\Theta}_t^n$ of $\Theta_t = \int_0^t \theta_u du$, where the spot process θ_t is itself assumed to be a semimartingale, and assume that $\hat{\Theta}_t^n$ is a consistent estimator of Θ_t . By the Doob–Meyer decomposition (see e.g., Jacod and Shiryaev (2013, Theorem I.3.15, p. 32)), the counting process $N_{n,t}$ can be decomposed as $N_{n,t} = M_{n,t} + \Lambda_{n,t}$, in terms of a martingale $M_{n,t}$ and an increasing and predictable process $\Lambda_{n,t}$. We assume that the latter process is absolutely continuous, so that $\Lambda_{n,t} = \int_0^t \lambda_{n,s} ds$, and that $\lambda_{n,t}$, called the intensity process, is itself

a semimartingale. As stated in the introduction, we are concerned with nonparametric inference in situations where the observation times are thought to be dependent on an underlying spot process θ_t . With the notation now introduced, this means that we seek to estimate covariance matrices Γ_t of the type

$$\Gamma_t = \begin{pmatrix} [\theta, \theta]_t & [\theta, \lambda]_t \\ [\theta, \lambda]_t & [\lambda, \lambda]_t \end{pmatrix}. \quad (1)$$

REMARK 1. Even though the primary motivation for this study was the problem of endogenous times, i.e., the situation where $\lambda_{n,t}$ is the intensity of observation times, the estimator we introduce in Section 3.1 applies more generally. As an example, the λ_t in (1) can be a spot process associated with a second observable process, and our estimator can be used to estimate the covolatility of two spot processes. Also note that the adaption of our theory to covariance matrices larger than the 2×2 -matrix above is straightforward. $\square$

All the processes are observed over the finite interval $[0, T]$, where T is fixed, and our arguments will be based on asymptotics as the observation frequency gets higher, that is $\max_{i \geq 1} \Delta_{i,n} \rightarrow 0$. To let the number of observations $N_{n,T}$ tend to infinity, and at the same time get a stable expression for the limiting intensity of the observation times, we impose the following condition.¹

CONDITION 1. *There is a non-negative semimartingale λ_t such that $n^{-1}\Lambda_{n,t} \xrightarrow{p} \Lambda_t := \int_0^t \lambda_s ds$, for all $t \in [0, T]$.*

One may think of $1/n$ as proportional to the expected distance between two observation times, or n as being proportional to the expected number of observations per period (see Li et al. (2014) and Jacod et al. (2019) for similar discussions). The point is that Condition 1 allows us to develop asymptotic theory in terms for $N_{n,T}$ for the estimators we construct. In applications it might be easy to show pointwise convergence in probability of $n^{-1}\lambda_{n,t}$ to a limiting process λ_t . If also $n^{-1}\lambda_{n,t}$ is uniformly integrable, and there exists a function $k(t)$, with $\int_0^T k(t) dt < \infty$, such that $n^{-1}\mathbb{E}|\lambda_{n,t}| \leq k(t)$ for all n and t , then Condition 1 holds (Andersen et al. 1993, Proposition II.5.2, p. 85). For the (finite sample) empirical applications of our estimator, the index n will turn out to be immaterial.

Before proceeding to the construction of the estimator, we provide an example of a simple model satisfying the above assumptions.

EXAMPLE 1. Suppose that we observe samples from the process $X_t = X_0 + \int_0^t \sigma_s dW_s$, where the spot volatility and the intensity follow CIR processes (Cox et al. 1985) given by,

$$\begin{aligned} d\sigma_t^2 &= \kappa(\alpha - \sigma_t^2) dt + \gamma\sigma_t dZ_t, \quad \sigma_0 = \alpha, \\ d\lambda_{n,t} &= \beta_n(\xi_n - \lambda_{n,t}) dt + \nu_n\lambda_{n,t}^{1/2} dB_t, \quad \lambda_{n,0} = \xi^n, \end{aligned} \quad (2)$$

¹Compare to Assumption (O- ρ , ρ') in Jacod et al. (2019, p. 82).

where W_t, Z_t and B_t are Wiener processes such that $\text{corr}(W_t, Z_t) = \text{corr}(W_t, B_t) = 0$ and $\text{corr}(Z_t, B_t) = \rho$. The parameters κ, α and σ as well as β_n, ξ_n and ν_n are positive and we assume that the Feller condition (Feller 1951) holds for both the volatility and the intensity, that is $2\kappa\alpha \geq \sigma^2$, and $2\beta_n\xi_n \geq \nu_n^2$ for all $n \geq 1$. In this model, the dependency between σ_t^2 and $\lambda_{n,t}$ is introduced by the correlation between Z_t and B_t . Suppose that $\xi^n = n\xi$, $\nu_n = \sqrt{n}\nu$ and that $0 < \beta \leq \beta_n \rightarrow \infty$ as $n \rightarrow \infty$. Then, for each $t \in [0, T]$, we have that $n^{-1}\Lambda_{n,t} \xrightarrow{P} \xi t$, and that

$$n^{-1}[\sigma^2, \lambda_n]_t \xrightarrow{P} \rho\gamma\nu\xi^{1/2} \int_0^t \sigma_s ds = [\sigma^2, \lambda]_t, \quad (3)$$

as $n \rightarrow \infty$. See Appendix B for details. $\square$

By $[X, Y]_t$ we mean the continuous-time quadratic covariation (quadratic variation when X equals Y) of two semimartingales X and Y from time zero to time t (see e.g., Jacod and Shiryaev (2013, pp. 51-52) or Protter (2004, p. 66)).

In Section 4 the model of Example 1 is used as the basis for a simulation study.

3 The estimator

In this section we first present a general result for our estimator of covariance matrices of the type given in (1). We then specialise to the case of estimating the covariance between the volatility of a continuous semimartingale and the intensity of the observations times.

3.1 A general result

Divide the time interval $[0, \tau] \subset [0, T]$ into B blocks $(t_{i-1}^n, t_i^n]$, of equal length, with $t_0^n = 0$ and $t_{B_n}^n = \tau$. Set $\Delta_n = \tau/B_n$, and for convenience, assume that $t_i^n = i\Delta_n$. Since we shall permit rolling and overlapping intervals, let k_n be an integer no greater than $B_n/2$. From now on we drop the index n from the t_i^n , B_n and k_n when it does not cause confusion. For functions Θ_t and Λ_t , define

$$\text{QV}_{B,k}(\Theta, \Lambda)_\tau = \frac{1}{k} \sum_{i=k}^{B-k} (\Theta_{(t_i, t_{i+k}]} - \Theta_{(t_{i-k}, t_i]}) (\Lambda_{(t_i, t_{i+k}]} - \Lambda_{(t_{i-k}, t_i]}), \quad (4)$$

where $\Theta_{(s,t]} = \Theta_t - \Theta_s$, and write $\text{QV}_{B,k}(\Theta)_\tau = \text{QV}_{B,K}(\Theta, \Theta)_\tau$. For $l = 1, \dots, 2k_n$, the notation $i \equiv l[2k_n]$ means that

$$i = 2k_n j + l, \quad \text{for } k_n \leq i \leq B_n - k_n,$$

with j an increasing sequence of integers. As an example of this notation, notice that

$$\sum_{i=k}^{B-k} (\Theta_{t_{i+k}} - \Theta_{t_{i-k}})^2 = \sum_{l=1}^{2k} \sum_{i \equiv l[2k]} (\Theta_{t_{i+k}} - \Theta_{t_{i-k}})^2,$$

where, importantly, the inner sum on the right contains adjacent and non-overlapping intervals, e.g., for $k_n = 2$ and $l = 1$, the first terms on the right hand side are

$$(\Theta_{t_7} - \Theta_{t_3})^2 + (\Theta_{t_{11}} - \Theta_{t_7})^2 + (\Theta_{t_{15}} - \Theta_{t_{11}})^2 + \dots$$

Now, suppose that $\Theta_t = \int_0^t \theta_s ds$ and $\Lambda_t = \int_0^t \lambda_s ds$ are two integrated spot processes, and that θ_t and λ_t are semimartingales. These two spot processes might be associated with the same underlying semimartingale (in which case we can have $\theta = \lambda$), or with two different semimartingales concurrently observed. In the latter case, the sampling times can be asynchronous, and the total number of observations may differ. To not overburden the notation, however, we assume that the number of samples are the same for both processes, and equals n . As will become clear, the results generalise to situations with unequal sample sizes. We are given the estimators $\hat{\Theta}_t^n$ and $\hat{\Lambda}_t^n$ of Θ_t and Λ_t , respectively. Both $\hat{\Theta}_t^n$ and $\hat{\Lambda}_t^n$ are consistent and admit representations of the type $\hat{\Theta}_t^n = M_{n,t} + e_{n,t}^\theta - \tilde{e}_{n,0}^\theta$, in terms of a semimartingale $M_{n,t}^\lambda$ and edge effects $e_{n,t}^\theta$ and $\tilde{e}_{n,0}^\theta$. For $s < t$ we write $\hat{\Theta}_{(s,t]}^n = \hat{\Theta}_t^n - \hat{\Theta}_s^n$. This means that for $s < t$ the estimators can be represented as

$$\begin{aligned} \hat{\Theta}_{(s,t]} - \Theta_{(s,t]} &= M_{n,t}^\theta - M_{n,s}^\theta + e_{n,t}^\theta - e_{n,s}^\theta, \\ \hat{\Lambda}_{(s,t]} - \Lambda_{(s,t]} &= M_{n,t}^\lambda - M_{n,s}^\lambda + e_{n,t}^\lambda - e_{n,s}^\lambda. \end{aligned} \tag{5}$$

The assumption, implicit in (5), that the edge effect of phasing in an estimator at $s < t$ is the same as the edge effect associated with phasing out an estimator at t , is also chosen for notational convenience; the results extend with little effort to situations where the edge effects in the two ends of the interval behave differently.

CONDITION 2. Assume that (5) holds, and that there are $\alpha > 0$ and $\beta > 0$ such that, as $n \rightarrow \infty$,

$$n^\alpha M_{n,t}^\theta \xrightarrow{\mathcal{L}} L_t^\theta \quad \text{and} \quad n^\beta M_{n,t}^\lambda \xrightarrow{\mathcal{L}} L_t^\lambda \quad \text{stably,}$$

with respect to a σ -algebra $\mathcal{G}$. Both $n^\alpha M_{n,t}^\theta$ and $n^\beta M_{n,t}^\lambda$ are P-UT, and the quadratic variations $[L^\theta, L^\theta]_\tau$ and $[L^\lambda, L^\lambda]_\tau$ are measurable with respect to $\mathcal{G}$.

Define

$$\widetilde{\text{QV}}_{B,k}(\Theta, \Lambda)_\tau = \begin{pmatrix} \text{QV}_{B,k}(\Theta)_\tau & \text{QV}_{B,k}(\Theta, \Lambda)_\tau \\ \text{QV}_{B,k}(\Theta, \Lambda)_\tau & \text{QV}_{B,k}(\Lambda)_\tau \end{pmatrix}.$$

We can now state our main theorem.

THEOREM 1. (CONSISTENCY OF THE COVARIANCE ESTIMATOR) Assume that $\hat{\Theta}_t^n$ and $\hat{\Lambda}_t^n$ satisfy (5) and Condition 2. Let $k = k_n$ be positive integers, assume that $k_n \Delta_n \rightarrow 0$, and that the edge

effects e_t^θ and e_t^λ are $o_p((k_n \Delta_n)^{1/2} n^{-\alpha})$ and $o_p((k_n \Delta_n)^{1/2} n^{-\beta})$, respectively. Then

$$\begin{aligned} \widetilde{\text{QV}}_{B,k_n}(\hat{\Theta}^n, \hat{\Lambda}^n)_\tau &= \frac{2}{3}(k_n \Delta_n)^2 \begin{pmatrix} [\theta, \theta]_{\tau-} & [\theta, \lambda]_{\tau-} \\ \cdot & [\lambda, \lambda]_{\tau-} \end{pmatrix} \\ &\quad + 2 \begin{pmatrix} [M_n^\theta, M_n^\theta]_{\tau-} & [M_n^\theta, M_n^\lambda]_{\tau-} \\ \cdot & [M_n^\lambda, M_n^\lambda]_{\tau-} \end{pmatrix} + o_p((k_n \Delta_n)^2) + o_p(n^{-2\alpha} \vee n^{-2\beta}). \end{aligned}$$

In particular,

$$\text{QV}_{B,k}(\hat{\Theta}, \hat{\Lambda}) = 2[M_n^\theta, M_n^\lambda]_\tau + \frac{2}{3}(k_n \Delta_n)^2[\theta, \lambda]_{\tau-} + o_p((k_n \Delta_n)^2) + o_p(n^{-(\alpha+\beta)}).$$

Proof. The proof follows with trivial adjustments from Mykland and Zhang (2017a, Theorem 3, p. 208). A brief sketch of the proof along with some remarks on the edge effects are given in Appendix C. $\square$

In Appendix C we also provide the conclusion of the above theorem with slightly more stringent restrictions on the edge effects. Corresponding results for all combinations of assumptions on the edge effects can be deduced from the results in Appendix C.

As an estimator of the matrix Γ_t in (1) we propose the *Two Scales Quadratic Covariation* (TSQC) estimator. The TSQC-estimator of the full matrix $\Gamma_{\tau-}$ is defined by

$$\widetilde{\text{TSQC}}_{B,k_1,k_2}(\hat{\Theta}^n, \hat{\Lambda}^n)_{\tau-} = \frac{3}{2} \frac{\widetilde{\text{QV}}_{B,k_2}(\hat{\Theta}^n, \hat{\Lambda}^n) - \widetilde{\text{QV}}_{B,k_1}(\hat{\Theta}^n, \hat{\Lambda}^n)}{(k_{n,2}^2 - k_{n,1}^2)(\Delta_n)^2}, \quad (6)$$

where $k_{n,2} > k_{n,1}$ are user specified sequences of integers (tuning parameters) tending to infinity. The element wise TSQC-estimators, denoted $\text{TSQC}_{B,k_1,k_2}(\cdot, \cdot)$, are obtained by replacing the $\widetilde{\text{QV}}_{B,k}(\cdot, \cdot)$ in (6) with $\text{QV}_{B,k}(\cdot, \cdot)$. For the TSQC-estimators of diagonal elements we write $\text{TSQC}_{B,k_1,k_2}(\hat{\Theta}^n, \hat{\Theta}^n) = \text{TSQC}_{B,k_1,k_2}(\hat{\Theta}^n)$, and so on.

COROLLARY 1. (CONSISTENCY OF THE TSQC-ESTIMATOR) *Assume that the conditions of Theorem 1 are in force, and that $\Delta_n = o(n^{-\alpha} \vee n^{-\beta})$. Let $k_{n,2} > k_{n,1}$ be positive integers such that $k_{j,n} \Delta_n \rightarrow 0$ for $j = 1, 2$, with both $k_{j,n} \Delta_n$, $j = 1, 2$ being of the same order as $n^{-\alpha} \vee n^{-\beta}$, and that $\liminf_{n \rightarrow \infty} k_{n,2}/k_{n,1} > 1$. Then,*

$$\widetilde{\text{TSQC}}_{B,k_1,k_2}(\hat{\Theta}^n, \hat{\Lambda}^n)_{\tau-} = \Gamma_{\tau-} + o_p(1).$$

Proof. We have that $\widetilde{\text{TSQC}}_{B,k_1,k_2}(\hat{\Theta}^n, \hat{\Lambda}^n)_{\tau-} = \Gamma_{\tau-} + (k_{n,1}^2 - k_{n,2}^2)^{-1} \Delta_n^{-2} o_p(n^{-2\alpha} \vee n^{-2\beta}) + o_p(1)$. Now, use that $\liminf_{n \rightarrow \infty} k_{n,2}/k_{n,1} > 1$ and that $\Delta_n = o(n^{-\alpha} \vee n^{-\beta})$ to obtain the result. $\square$

REMARK 2. The above corollary treats the estimation of the entire matrix Γ_τ in one go, so to say. In applications, however, it can be advantageous to estimate the matrix element by element. That is, the best choice of tuning parameters $k_{n,1}$ and $k_{n,2}$ might not be the same for each of the elements of the matrix. $\square$

3.2 Volatility and intensity

We now specialise to the case that motivated the present paper, namely the estimation of the quadratic covariation between the spot volatility of a process and the intensity of the observation times. Suppose that we have at hand an estimator $\hat{\Theta}_t^n$ of the integrated volatility $\int_0^t \sigma_s^2 ds$, assume that this estimator satisfies the decomposition in (5), and that its error process martingale $M_{n,t}^\theta$ obeys Condition 2. We return to the assumptions on the edge effects in due time. Define $\tilde{\Lambda}_t^n = n^{-1}N_{n,t}$. We observe the counting process $N_{n,t}$, whereas n is a non-observable abstraction introduced so that the asymptotic theory developed Section 3.1 generalises to volatility-intensity estimation. This means that $\tilde{\Lambda}_t^n$ is not an estimator, however, for the time being we will treat it as it were.

Notice that there are no edge effects associated with $\tilde{\Lambda}_t^n$, so (5) becomes $\tilde{\Lambda}_t^n = n^{-1}\Lambda_{t,n} + M_{n,t}^\lambda$, where $M_{n,t}^\lambda = n^{-1}(N_{n,t} - \Lambda_{t,n})$ is a martingale sequence. Moreover, as $n \rightarrow \infty$,

$$[M_n^\lambda, M_n^\lambda]_t = n^{-1}N_{n,t} \xrightarrow{p} \Lambda_t, \quad (7)$$

by Condition 1 and because $M_{n,t}^\lambda$ is $o_p(1/n)$ by Lenglart's inequality (see e.g., Andersen et al. (1993, p. 86)). The convergence in (7) combined with the fact that Λ_t is increasing and continuous, yield

$$n^{1/2}M_{n,t}^\lambda \xrightarrow{\mathcal{L}} \int_0^t \lambda_s^{1/2} dW'_s \quad \text{stably,}$$

where W'_s is a Wiener process defined on an extension of the original probability space (Mykland and Zhang 2012, Theorem 2.28, p. 152). Set $L_t^\lambda = \int_0^t \lambda_s^{1/2} dW'_s$, and we have the first part of Condition 2. For Theorem 1 to be applicable, the sequence of martingales $n^{1/2}M_{n,t}^\lambda$ must also be P-UT.

LEMMA 1. *Assume Condition 1. Then $n^{1/2}M_{n,t}^\lambda$ is P-UT.*

Proof. The (predictable) quadratic variation of $n^{1/2}M_{n,t}^\lambda$ is $n\langle M_n^\lambda, M_n^\lambda \rangle_t = n^{-1} \int_0^t \lambda_{n,s} ds$, which converges in probability to $\Lambda_t = \int_0^t \lambda_s ds$ by Condition 1. Since Λ_t is non-decreasing and continuous, the convergence $n\langle M_n^\lambda, M_n^\lambda \rangle_t \xrightarrow{p} \Lambda_t$ is also in law on the space of càdlàg functions on $[0, T]$ (Jacod and Shiryaev 2013, Theorem VI.3.37, p. 354), from which it follows that $n\langle M_n^\lambda, M_n^\lambda \rangle_t$ is tight (Jacod and Shiryaev 2013, Proposition VI.3.26, p. 351). Being a counting process martingale and by continuity of $\Lambda_{n,t}$, we have that $|n^{1/2}(M_{n,t}^\lambda - M_{n,t-}^\lambda)| \leq 1$ for all n . This bound on the jumps

in combination with the tightness of the angle bracket process give the result (Jacod and Shiryaev 2013, Proposition VI.6.13, p. 379). $\square$

In the absence of edge effects on the part of $\tilde{\Lambda}_t^n$, $\text{QV}(\hat{\Theta}^n, \tilde{\Lambda}^n)$ can be decomposed as (cf. (C.3)),

$$\text{QV}(\hat{\Theta}^n, \tilde{\Lambda}^n) = \overline{\text{QV}}(\hat{\Theta}^n, \tilde{\Lambda}^n) + O_p(\text{QV}(\tilde{\Lambda}^n)^{1/2} R_{n,k}(\Theta)^{1/2}), \quad (8)$$

by the Cauchy–Schwarz inequality, where

$$\overline{\text{QV}}(\hat{\Theta}^n, \tilde{\Lambda}^n) = \text{QV}(\Theta, \Lambda_n/n) + \text{QV}(M^\theta, \Lambda_n/n) + \text{QV}(\Theta, M_n^\lambda) + \text{QV}(M^\theta, M_n^\lambda),$$

and $R_{n,k}(\Theta) = k^{-1} \sum_{i=k}^{B-k} (e_{t_{i+k}}^\theta - e_{t_i}^\theta - (e_{t_i}^\theta - e_{t_{i-k}}^\theta))^2$. For Theorem 1 to generalise to the volatility-intensity estimation of this section we need the following technical condition.

CONDITION 3. *The sequence $\{\lambda_{n,t}/n\}_{n \geq 1}$ is $O_p(1)$ in the sense of Definition 1.*

COROLLARY 2. *Suppose that $\hat{\Theta}_t^n$ satisfies Condition 2, that Condition 3 holds, and that e_t^θ are $o_p((k_n \Delta_n)^{1/2} n^{-\alpha})$. Then, as $k_n \Delta_n \rightarrow 0$*

$$\overline{\text{QV}}(\hat{\Theta}^n, \tilde{\Lambda}^n) = 2[M_n^\theta, M_n^\lambda] + \frac{2}{3}(k_n \Delta_n)^2 [\sigma^2, \lambda_n/n]_{\tau-} + o_p((k_n \Delta_n)^2) + o_p(n^{-\alpha} n^{-1/2}),$$

Proof. By Lemma 1, the sequence $\tilde{\Lambda}_t^n = n^{-1} \Lambda_{n,t} + M_{n,t}^\lambda$ satisfies Condition 2, and Condition 3 assures that Theorem 7 in Mykland and Zhang (2017b) is applicable. The second part of Theorem 1 then gives the result. $\square$

We have that $\text{QV}(\tilde{\Lambda}^n) = O_p(k_n \Delta_n + n^{-1/2})$, which via (8) shows how differing restrictions on the edge effects associated with the integrated volatility estimator give differing conclusions about $\text{QV}(\hat{\Theta}^n, \tilde{\Lambda}^n)$ (see the discussion in Appendix C). If we assume that the edge effects associated with $\hat{\Theta}_t^n$ are $o_p((k_n \Delta_n)^{3/4} n^{-\alpha})$, which is not unrealistic when working with two-scales estimators and pre-averaged observations (see e.g., Zhang et al. (2005) or Mykland et al. (2019)), then the conclusion of Corollary 2 is

$$\text{QV}(\hat{\Theta}^n, \tilde{\Lambda}^n) = 2[M_n^\theta, M_n^\lambda] + \frac{2}{3}(k_n \Delta_n)^2 [\sigma^2, \lambda_n/n]_{\tau-} + O_p((k_n \Delta_n)^{5/2}) + O_p((k_n \Delta_n)^{1/2} n^{-\alpha} n^{-1/2}).$$

Under Conditions 1 and 3 and Corollary 1, $\text{TSQC}_{B,k_1,k_2}(\hat{\Theta}^n, \tilde{\Lambda}^n)$ is consistent because $[\sigma^2, \lambda_n/n]_{\tau-}$ converges in probability to $[\sigma^2, \lambda]_{\tau-}$. It is, however, unattainable since n is unknown. Consider therefore the process $\rho_t(\cdot, \cdot)$, given by

$$\rho(\sigma^2, \lambda)_t = \frac{[\sigma^2, \lambda]_t}{([\sigma^2, \sigma^2]_t [\lambda, \lambda]_t)^{1/2}}.$$

For each t we see that $\rho(\sigma^2, \lambda_n)_t = \rho(\sigma^2, \lambda_n/n)_t \xrightarrow{p} \rho(\sigma^2, \lambda)_t$ by the continuous mapping theorem, which means that the coefficient $\rho(\sigma^2, \lambda)_t$ can be consistently estimated using the estimators $\hat{\Theta}_t^n$

and $\hat{\Lambda}_t^n$, the latter simply defined as $\hat{\Lambda}_t^n = N_{n,t}$. In particular, define

$$\hat{\rho}(\hat{\Theta}^n, \hat{\Lambda}^n)_{\tau-} = \frac{\text{TSQC}(\hat{\Theta}^n, \hat{\Lambda}^n)_{\tau-}}{(\text{TSQC}(\hat{\Theta}^n)_{\tau-} \text{TSQC}(\hat{\Lambda}^n)_{\tau-})^{1/2}},$$

and note that $\hat{\rho}(\hat{\Theta}^n, \hat{\Lambda}^n)_{\tau-} = \hat{\rho}(\hat{\Theta}^n, \tilde{\Lambda}^n)_{\tau-}$, from which consistency of this estimator follows. The estimator $\hat{\rho}(\hat{\Theta}^n, \hat{\Lambda}^n)_{\tau-}$ has a similar flavour to it, but is different from, the first-order correlation estimator introduced in Barndorff-Nielsen and Shephard (2004, Sections 3.1-3.2, pp. 899–903).

In Section 4.1 we study the performance of $\hat{\rho}(\hat{\Theta}^n, \hat{\Lambda}^n)_{\tau-}$ on simulated data, and investigate its sensitivity to the choice of tuning parameters k_1 and k_2 .

3.3 Convergence rates for rolling quadratic variation

The convergence rate of the TSQC-estimator depends in part on the accuracy of the approximation of rolling and overlapping sums of the type $k^{-1} \sum_{l=1}^{2k} \sum_{i \equiv l[2k]} (\alpha_{t_{i+k}}^{(l,n)} - \alpha_{t_{i-k}}^{(l,n)}) (\beta_{t_{i+k}}^{(l,n)} - \beta_{t_{i-k}}^{(l,n)})$ to the average quadratic variation $k^{-1} \sum_{l=1}^{2k} [\alpha^{(l,n)}, \beta^{(l,n)}]_{\tau-}$. Here

$$\alpha_t^{(l,n)} = \int_0^t f_{s-}^{(l,n)} d\alpha_s^{(n)}, \text{ and } \beta_t^{(l,n)} = \int_0^t g_{s-}^{(l,n)} d\beta_s^{(n)}, \quad \text{for } l = 1, \dots, 2k_n, \quad (9)$$

where $\alpha_t^{(n)}$ and $\beta_t^{(n)}$ are sequences of semimartingales, and $f_t^{(l,n)}$ and $g_t^{(l,n)}$ are deterministic càdlàg functions bounded by 1. We denote by $\mathbb{F}$ the collection $f^{(l,n)}$ $l = 1, \dots, 2k_n$, $n = 1, 2, \dots$, of such functions, and similarly $g^{(l,n)}$ belongs to the collection $\mathbb{G}$ (see Appendix A for details). In Mykland and Zhang (2017b, Theorem 7, p. 1) it was shown that

$$\frac{1}{k} \sum_{l=1}^{2k} \sum_{i \equiv l[2k]} (\alpha_{t_{i+k}}^{(l,n)} - \alpha_{t_{i-k}}^{(l,n)}) (\beta_{t_{i+k}}^{(l,n)} - \beta_{t_{i-k}}^{(l,n)}) = \frac{1}{k} \sum_{l=1}^{2k} [\alpha^{(l,n)}, \beta^{(l,n)}]_{\tau-} + o_p(1). \quad (10)$$

In this section we turn to the rate of convergence for the approximation in (10). Such statements will help with accuracy and with optimal calibration of the TSQC-estimators, as well as other rolling intervals estimators that depend on approximations such as the one in (10). In other words, the theorem presented shortly is quite general and has ramifications well beyond the estimation of quadratic variations of the type studied in this paper, for example for the Observed Asymptotic Variance (AVAR) estimator of (Mykland and Zhang 2017a). The main restrictions are that $\alpha^{(n)}$ and $\beta^{(n)}$ are semimartingales whose predictable quadratic variations $\langle \alpha^{(n)}, \alpha^{(n)} \rangle_t$ and $\langle \beta^{(n)}, \beta^{(n)} \rangle_t$ are absolutely continuous and locally bounded, uniformly n .

THEOREM 2. *Suppose that $\alpha_t^{(n)}$ and $\beta_t^{(n)}$ satisfy Conditions 4-5, that $f_t^{(l,n)} \in \mathbb{F}$ and $g_t^{(l,n)} \in \mathbb{G}$, and*

that $\alpha_t^{(l,n)}$ and $\beta_t^{(l,n)}$ are as defined in (9). Then

$$\frac{1}{k_n} \sum_{l=1}^{2k_n} \sum_{i \equiv l[2k_n]} (\alpha_{t_{i+k}}^{(l,n)} - \alpha_{t_{i-k}}^{(l,n)}) (\beta_{t_{i+k}}^{(l,n)} - \beta_{t_{i-k}}^{(l,n)}) = \frac{1}{k_n} \sum_{l=1}^{2k_n} [\alpha^{(l,n)}, \beta^{(l,n)}]_{\tau-} + O_p((k_n \Delta_n)^{1/2}).$$

Proof. See Appendix D. □

4 Simulations and data analysis

In this section we apply the TSQC-estimators to simulated data to assess its applicability in realistic settings. The choice of tuning parameters $k_{n,1}$ and $k_{n,2}$ is graphically investigated by plotting the estimates against the choice of k_1 , with $k_2/k_1 = 2$ held constant. In the empirical analysis, TSQC-estimators are applied to the Apple stock as observed over 21 trading days in January 2018.

4.1 A simulation study

The data are simulated from the model presented in Example 1, with the first observations for the volatility and intensity processes were sampled from a Gamma $(2\kappa\alpha/\gamma^2, 2\kappa/\gamma^2)$ and a Gamma $(2\beta_n\xi_n/\nu_n^2, 2\beta_n/\nu_n^2)$ distribution, respectively. The parameter values were $\alpha = 2.172, \kappa = 2.345, \gamma = 1.000$ (volatility model), $\xi_n = n8.912, \beta_n = n^{1/4}0.169, \nu = \sqrt{n}1.000$, with $n = 40\,000$. The microstructure noise was taken as additive on the efficient price and independent of the three underlying Brownian motions, that is, we observe $Y_{t_i} = X_{t_i} + \varepsilon_{t_i}$, where the ε_{t_i} were independent mean zero normals with standard deviation 0.0005, independent of W, Z and B . These three process were all Brownian motions, W was independent of Z and B , while Z and B were jointly Brownian with correlation $\rho = 0.912$. The data were simulated to mimic features of the actual Apple stock data that we analyse in Section 4.2. With $[0, T]$ one trading day (6.5 hours) the intensity function $\lambda_{n,t}$ is such that we have about 275 000 observations of Y_t per day. This is a common number of daily trades of a liquid stock such as that of Apple. Figure 1 provides a summary of the five estimators.

For each simulation we estimated the matrix Γ_{T-} , the coefficient $\rho(\sigma^2, \lambda)_{T-}$ and β_{T-} , the latter defined as $\beta_t = [\sigma^2, \lambda]_t / [\lambda, \lambda]_t$. As an estimator of the integrated volatility we used the Two-Scales Realised Volatility (TSRV) of Zhang et al. (2005), while $\tilde{\Lambda}^n$ was used to estimate the cumulative intensity of the observation times. Recall that n is an abstraction, so that $\tilde{\Lambda}^n$ as used here is not an estimator. In practice, however, not knowing n is just a question of scaling for four of the five estimators, while for $\rho(\sigma^2, \lambda)_T$ it is irrelevant. In Figure 1 we have plotted the deviance of the estimates from the (random) estimands for various values of k_1 , with $k_2 = 2k_1$ throughout.

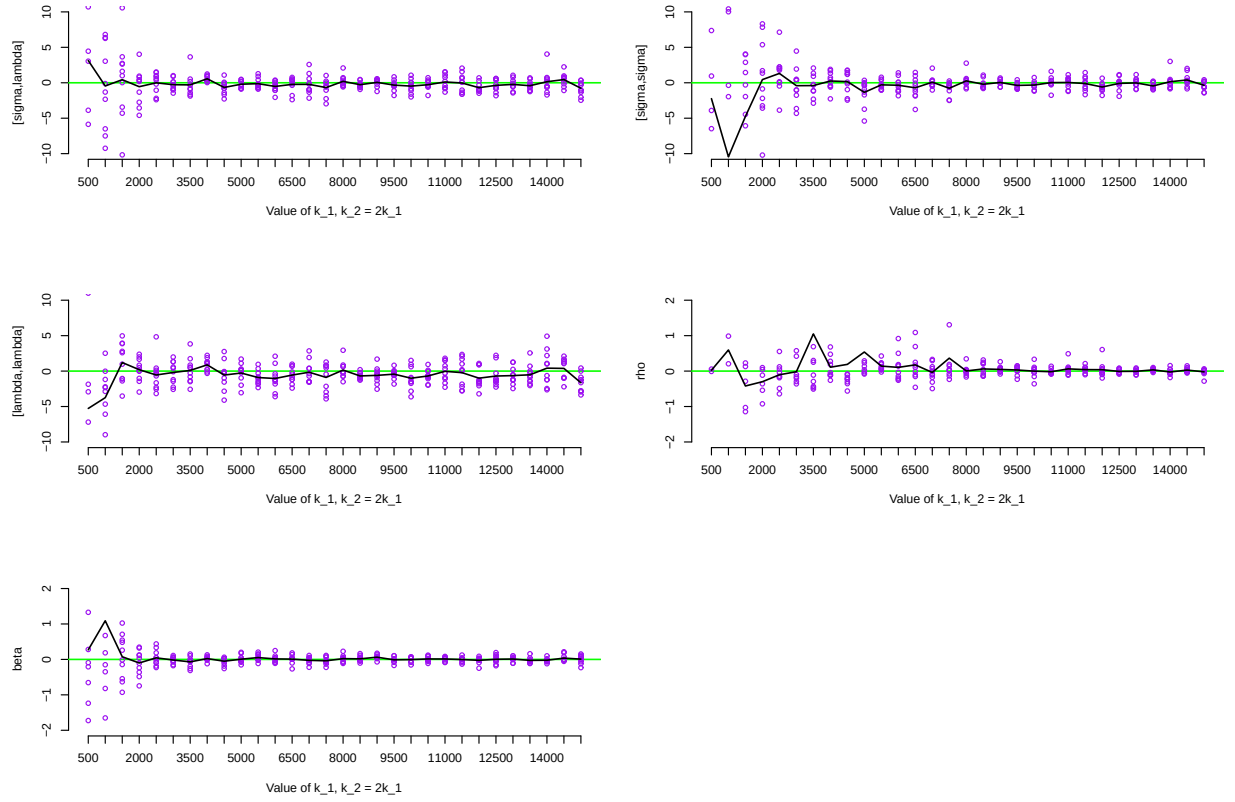


Figure 1: Values of k_1 on the x -axis ($k_2 = 2k_1$). Deviance of the estimate from the random truth, i.e., $\hat{\theta} - \theta$, on the y -axis. The wiggly lines are the means of the 10 simulations performed for each value of k_2 ; the dots are the actual deviances; the straight lines indicate zero deviance. The TSRV-estimator (with $K = 2$ and $J = 1$, see Mykland et al. (2019, Eq. (17), p. 106)) was used to estimate the integrated volatility.

4.2 An empirical analysis

In the empirical study we analyse features of the Apple stock as traded over a period of 21 trading days in January, 2018. All transactions registered in the U.S. National Market System conducted between 9:45 am - 3:45 pm Eastern Standard Time are included. The reason for choosing this window is to avoid abnormal trading activity during the opening and closing of the New York Stock Exchange, and to avoid those hours of the day during which the trading frequency is low (Wang and Mykland 2014, p. 205). The Apple stock data is recorded down to the nanosecond (10^{-9} seconds), and for the period under study the mean number of transactions over a trading day the time window we use was 203 924, which is about nine transactions per second. After some data cleaning, the data was pre-averaged and the TSRV estimator of Zhang et al. (2005) was used to estimate the integrated volatility. The cumulative intensity of the observation times was estimated by $10^{-6}N_t$, where N_t counts the number of transactions conducted from 9:45 am to 9:45 am plus t . Besides making the plots more aesthetically pleasing, the number 10^{-6} plays no role.

We used the TSQC-estimator for daily estimation of the volatility-intensity covariance matrix and the two transformations thereof, $\rho(\sigma^2, \lambda)_t$ and β_t . The estimates of $\rho(\sigma^2, \lambda)_t$ lies between 0.5 and 0.8 for most of the days under study, indicating that the two processes are indeed correlated. To estimate the (pointwise) confidence bands of our TSQC-estimators we employed the Observed Asymptotic Variance (AVAR) of Mykland and Zhang (2017a). This needs some comment. The applicability of the AVAR requires (among other technical details, some of whom relate to the edge effects) that the estimand, say Ξ_t , can be written as $\Xi_t = \int_0^t \xi_s ds$, where ξ_t is a semimartingale, and that the error process $\hat{\Xi}_t^n - \Xi_t$, after proper normalisation, satisfy Condition 2 (i.e., satisfy a central limit theorem). The soundness of the confidence bands in Figure 2 thus rests on the assumption that the TSQC-estimators satisfy these requirements. Whether this is indeed the case is beyond the scope of this paper.

5 Conclusion

This paper introduces a consistent estimator of the quadratic covariation between two non-observable spot process semimartingales, and derives the convergence rates of this estimator. As recognised in much recent literature on estimation in high-frequency data, the assumption of exogenous observation times is often untenable, and one typically allows for dependency between the observation times and the price process. In this paper we have considered possible dependencies between the observation times and non-observable spot processes associated with the price process, of which the spot volatility is a prime example. A simulation study shows that the estimators perform well with decent amounts of data. The empirical study of the Apple stock indicates that the observation times and the volatility process of this stock are positively correlated.

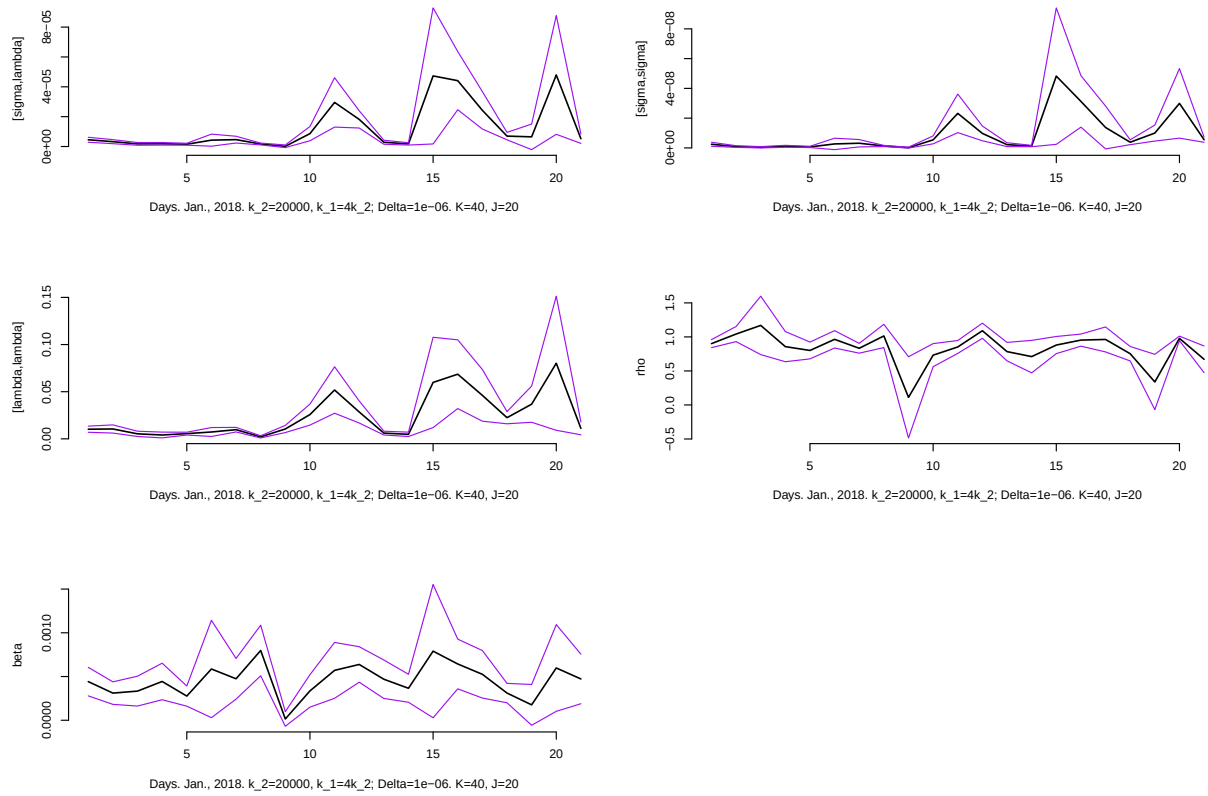


Figure 2: The Apple stock January 2.-31., 2018. Daily estimates of $[\sigma^2, \lambda]_T$, $[\sigma^2, \lambda]_T$ and $[\lambda, \lambda]_T$, as well as the parameters β_T and ρ_T . The TSRV was used as the estimator of the integrated volatility. The purple lines are pointwise 95 percent confidence bands computed using the Observed asymptotic variance (AVAR) of Mykland and Zhang (2017a), along with the delta method.

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APPENDIX: PROOFS AND TECHNICAL ISSUES

A Notation and conditions

We start by recalling some definitions from Mykland and Zhang (2017a).

DEFINITION 1. (ORDERS IN PROBABILITY.) *For a sequence $\alpha_t^{(n)}$ of semimartingales, we say that $(\alpha_t^{(n)}) = O_p(1)$ if the sequence is tight, with respect to convergence in law relative to the Skorokhod topology on $\mathbb{D}$ (Jacod and Shiryaev (2013, Theorem VI.3.21, p. 350)), and also P-UT (ibid., Chapter VI.3.b, and Definition VI.6.1, p. 377). For scalar random quantities, $O_p(\cdot)$ and $o_p(\cdot)$ are defined as usual, see, e.g., Pollard (1984, Appendix A).*

CONDITION 4. *Let $\alpha_t^{(n)}$ and $\beta_t^{(n)}$ be sequences (in n) of semimartingales. Each of these sequences are (separately) assumed to be $O_p(1)$.*

DEFINITION 2. (NOTATION.) *The symbol $\mathbb{F}$ will refer to a collection of nonrandom functions $f^{(l,n)}$ càdlàg on $[0, T]$, with $n \in \mathbb{N}$, and $l = 1, \dots, 2K_n$, satisfying*

$$|f_t^{(l,n)}| \leq 1 \text{ for all } t, l, \text{ and } n.$$

Similarly, $\mathbb{G}$ will refer to a collection $g_t^{(l,n)}$ with the same size and properties.

Given $\mathbb{F}$ and $\mathbb{G}$, set

$$\alpha_t^{(l,n)} = \int_0^t f_{s-}^{(l,n)} d\alpha_s^{(n)} \text{ and } \beta_t^{(l,n)} = \int_0^t g_{s-}^{(l,n)} d\beta_s^{(n)} \text{ for } l = 1, \dots, 2k_n.$$

CONDITION 5. (CONDITIONS FOR RATE-OF-CONVERGENCE STATEMENTS.) *The sequence of semimartingales $\alpha_t^{(n)}$, defined on a fixed filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$, is said to satisfy this condition if it can be written as $\alpha_t^{(n)} = \alpha_0^{(n)} + \alpha_t^{(n, \text{MG})} + \int_0^t b_s^{(n)} ds$, where for each n , $\alpha_t^{(n, \text{MG})}$ is a square integrable martingale with predictable quadratic variation $\langle \alpha^{(n)}, \alpha^{(n)} \rangle_t$ that is absolutely continuous. We assume that $b_t^{(n)}$ and $d\langle \alpha^{(n)}, \alpha^{(n)} \rangle_t/dt$ are locally bounded uniformly in n , in other words, there is a sequence of stopping times τ_m , so that $b_t^{(n)}$ and $d\langle \alpha^{(n)}, \alpha^{(n)} \rangle_t/dt$ are bounded by a constant for $t \in [0, \tau_m]$, while $P(\tau_m = T) \rightarrow 1$ as $m \rightarrow \infty$.*

A single semimartingale α_t is said to satisfy this condition if the above is satisfied for the constant sequence $\alpha_t = \alpha_t^{(n)}$.

B Proof of claims in Example 1

Assume that $\xi^n = n\xi$, $\nu_n = \sqrt{n}\nu$ and that $0 < \beta \leq \beta_n \rightarrow \infty$ as $n \rightarrow \infty$. Then, for each $t \in [0, T]$,

$$\frac{1}{n} \Lambda_{n,t} \xrightarrow{p} \xi t, \quad \text{and} \quad \frac{1}{n} [\sigma^2, \lambda_n]_t \xrightarrow{p} \rho \gamma \nu \xi^{1/2} \int_0^t \sigma_s ds, \quad (\text{B.1})$$

as $n \rightarrow \infty$. We now prove (B.1): The expectation of the intensity is $E \lambda_{n,t} = \xi_n$, and

$$\begin{aligned} E \left| \frac{1}{n} (\lambda_{t,n} - \xi_n) \right|^2 &= \frac{1}{n^2} E \left| \nu_n \int_0^t \lambda_{n,s}^{1/2} e^{-\beta_n(t-s)} dB_s \right|^2 = \frac{\nu_n^2}{n^2} E \int_0^t \lambda_{n,s} e^{-2\beta_n(t-s)} ds \\ &= \frac{\nu_n^2}{n^2} \int_0^t \xi_n e^{-2\beta_n(t-s)} ds = \frac{\nu_n^2}{n^2} \frac{\xi_n}{2\beta_n} (1 - e^{-2\beta_n t}). \end{aligned} \quad (\text{B.2})$$

Note that

$$\begin{aligned} E |\lambda_{t,n}/n|^2 &= E |\lambda_{t,n}/n - \xi + \xi|^2 = E |\lambda_{t,n}/n - \xi|^2 + 2 E |\lambda_{t,n}/n - \xi| \xi + \xi^2 \\ &\leq E |\lambda_{t,n}/n - \xi|^2 + 2 (E |\lambda_{t,n}/n - \xi|^2)^{1/2} \xi + \xi^2, \end{aligned}$$

and from (B.2), $E |\lambda_{t,n}/n - \xi|^2 = \nu^2 (2\beta_n)^{-1} (1 - e^{-2\beta_n t})$, from which it follows that for each t , $\sup_n E |\lambda_{t,n}/n|^2 < \infty$. By Chebyshev's inequality (see eq. (25.13) in Billingsley (1995, p. 338)) we get that $\lim_{C \rightarrow \infty} \sup_n E |\lambda_{t,n}/n| I_{\{|\lambda_{t,n}/n| > C\}} = 0$, that is, for each t the sequence of random variables $\{\lambda_{t,n}/n\}_{n \geq 1}$ are uniformly integrable. Moreover, from the above we see that $E |\lambda_{t,n}/n| \leq (\nu^2/\beta + \xi^2)^{1/2}$ for all t and n , and the right hand side is trivially integrable on $[0, T]$. Hence,

the sequence of stochastic processes $\{\lambda_{n,s}/n\}_{n \geq 1}$ satisfies the conditions of Andersen et al. (1993, Proposition II.5.2, p. 85), and the first part of (B.1) follows. For the second part we have that

$$\begin{aligned} \mathbb{E} |\sigma_t \lambda_{n,t}^{1/2} / \sqrt{n}|^2 &= \mathbb{E} \sigma_t^2 \lambda_{n,t} / n = \mathbb{E} |(\sigma_t^2 - \alpha + \alpha)| |(\lambda_{n,t}/n - \xi + \xi)| \\ &= \mathbb{E} |(\sigma_t^2 - \alpha)(\lambda_{n,t}/n - \xi)| + \mathbb{E} |(\sigma_t^2 - \alpha)| \xi + \mathbb{E} |(\lambda_{n,t}/n - \xi)| \alpha + \alpha \xi, \end{aligned}$$

which by three applications of Hölder's inequality and the Itô isometry is seen to be bounded by a constant, hence $\sup_n \mathbb{E} |\sigma_t \lambda_{n,t}^{1/2} / \sqrt{n}|^2 < \infty$, and uniform integrability of the random variables $\sigma_t \lambda_{n,t}^{1/2} / \sqrt{n}$ follows by the same argument as above. Since $\mathbb{E} |\sigma_s \lambda_{n,s}^{1/2} / \sqrt{n}| \leq (\mathbb{E} |\sigma_s \lambda_{n,s}^{1/2} / \sqrt{n}|^2)^{1/2}$ for all s and n , and a constant is integrable on $[0, T]$, so the second part of (B.1) follows by the same argument as above.

C Notes on Theorem 1

The proof follows with trivial adjustments from Mykland and Zhang (2017a, Theorem 3, p. 208). Note that the convergence rates change due to our Theorem 2. Recall the setup in (5), that is

$$\hat{\Theta}_{(s,t]} - \Theta_{(s,t]} = M_{n,t}^\theta - M_{n,s}^\theta + e_{n,t}^\theta - e_{n,s}^\theta, \text{ and } \hat{\Lambda}_{(s,t]} - \Lambda_{(s,t]} = M_{n,t}^\lambda - M_{n,s}^\lambda + e_{n,t}^\lambda - e_{n,s}^\lambda.$$

Mykland and Zhang (2017a, Theorem 3, p. 208) and the convergence rates from Theorem 2 give

$$\begin{aligned} \text{QV}_{B,k}(\hat{\Theta}, \hat{\Lambda}) &= \overline{\text{QV}}_{B,k}(\hat{\Theta}, \hat{\Lambda}) + R_{n,k}(\Theta, \Lambda) \\ &\quad + O_p((k_n \Delta_n + n^{-\alpha}) R_{n,k}(\Lambda)^{1/2}) + O_p((k_n \Delta_n + n^{-\beta}) R_{n,k}(\Theta)^{1/2}), \end{aligned} \quad (\text{C.3})$$

where $R_{n,k}(\Theta) = R_{n,k}(\Theta, \Theta)$ and

$$R_{n,k}(\Theta, \Lambda) = \frac{1}{k} \sum_{i=k}^{B-k} (e_{n,t_{i+k}}^\theta - e_{n,t_i}^\theta - (e_{n,t_i}^\theta - e_{n,t_{i-k}}^\theta))(e_{n,t_{i+k}}^\lambda - e_{n,t_i}^\lambda - (e_{n,t_i}^\lambda - e_{n,t_{i-k}}^\lambda)),$$

while $\overline{\text{QV}}_{B,k}(\hat{\Theta}, \hat{\Lambda})$ is given by

$$\begin{aligned} \overline{\text{QV}}_{B,k}(\hat{\Theta}, \hat{\Lambda}) &= 2[M_n^\theta, M_n^\lambda]_{\tau-} + \frac{2}{3}(k\Delta t)^2 \left(1 - \frac{1}{k_n^2}\right) [\theta, \lambda]_{\tau-} + O_p(n^{-(\alpha+\beta)}(k_n \Delta_n)^{1/2}) \\ &\quad + \Delta_n^2 \int_0^{\tau-} \left\{ \left(\frac{t^*(s) - s}{\Delta_n} \right)^2 + \left(\frac{s - t_*(s)}{\Delta_n} \right)^2 \right\} d[\theta, \lambda]_s + O_p((k\Delta t)^{5/2}) \\ &\quad + \Delta_n \int_0^{\tau-} \left(1 - 2 \frac{s - t_*(s)}{\Delta_n} \right) d[\theta, M_n^\lambda]_s + O_p(n^{-\beta}(k_n \Delta_n)^{3/2}) \\ &\quad + \Delta_n \int_0^{\tau-} \left(1 - 2 \frac{s - t_*(s)}{\Delta_n} \right) d[\lambda, M_n^\theta]_s + O_p(n^{-\alpha}(k_n \Delta_n)^{3/2}), \end{aligned}$$

and $t_*(s) = \max\{t_i : t_i < s\}$ and $t^*(s) = \min\{t_i : t_i \geq s\}$.

We now consider two different sets of restrictions on the edge effect. All other cases can be deduced from (C.3). For all t on a given grid,

$$\begin{aligned} \text{Case (1): } e_t^\theta &= o_p((k_n \Delta_n)^{1/2} n^{-\alpha}), \quad \text{and} \quad e_t^\lambda = o_p((k_n \Delta_n)^{1/2} n^{-\beta}); \\ \text{Case (2): } e_t^\theta &= o_p((k_n \Delta_n)^{3/4} n^{-\alpha}), \quad \text{and} \quad e_t^\lambda = o_p((k_n \Delta_n)^{3/4} n^{-\beta}). \end{aligned}$$

Under Case (1) we have that (C.3) is

$$\text{QV}_{B,k}(\hat{\Theta}, \hat{\Lambda}) = 2[M_n^\theta, M_n^\lambda]_{\tau-} + \frac{2}{3}(k\Delta_n)^2[\theta, \lambda]_{\tau-} + o_p((k_n \Delta_n)^2) + o_p(n^{-(\alpha+\beta)}).$$

While under Case (2) we find that (C.3) is

$$\text{QV}_{B,k}(\hat{\Theta}, \hat{\Lambda}) = 2[M_n^\theta, M_n^\lambda]_{\tau-} + \frac{2}{3}(k\Delta_n)^2[\theta, \lambda]_{\tau-} + O_p((k_n \Delta_n)^{5/2}) + O_p((k_n \Delta_n)^{1/2} n^{-(\alpha+\beta)}).$$

It thus appears that the more stringent conditions on the edge effects in Case (2) are needed for the convergence rates of Theorem 2 to ‘enter’ Theorem 1. Do note, however, that this may be an artefact of the Cauchy–Schwarz inequality used in deriving (C.3).

D Proof of Theorem 2

Define

$$t_{*,l} = t_{*,l}(s) = \max\{t_i : t_{i+k} \leq s : i \equiv l[2k]\}.$$

It is enough to show the result when the sequences $\alpha^{(n)}$ and $\beta^{(n)}$ are local square integrable martingales. Let

$$\begin{aligned} Z_{n,l}(s) &= \sum_{t_{i+k} \leq s : i \equiv l[2k]} (\alpha_{t_{i+k}}^{(l,n)} - \alpha_{t_{i-k}}^{(l,n)})(\beta_{t_{i+k}}^{(l,n)} - \beta_{t_{i-k}}^{(l,n)}) \\ &\quad + (\alpha_s^{(l,n)} - \alpha_{t_{*,l}}^{(l,n)})(\beta_s^{(l,n)} - \beta_{t_{*,l}}^{(l,n)}) - [\alpha^{(l,n)}, \beta^{(l,n)}]_s, \end{aligned}$$

and set $Z_n(s) = k^{-1} \sum_{l=1}^{2k} Z_{n,l}(s)$. Let the stopping time τ and the nonrandom constant c be such that, for $t \leq \tau$, $d\langle \alpha^{(n)}, \alpha^{(n)} \rangle_t / dt \leq c$ and $d\langle \beta^{(n)}, \beta^{(n)} \rangle_t \leq c$. In particular, $|d\langle \alpha^{(n)} \beta^{(n)} \rangle_t / dt| \leq c$, by the Kunita–Watanabe inequality (see e.g., Protter (2004, Theorem II.25, p. 69)). By Itô’s lemma

we have that

$$\begin{aligned}
\langle Z_{n,l_1}, Z_{n,l_2} \rangle_\tau &= \int_0^\tau (\alpha_s^{(l_1,n)} - \alpha_{t^*,l_1}^{(l_1,n)})(\alpha_s^{(l_2,n)} - \alpha_{t^*,l_2}^{(l_2,n)}) d\langle \beta^{(l_1,n)}, \beta^{(l_2,n)} \rangle_s \\
&\quad + \int_0^\tau (\beta_s^{(l_1,n)} - \beta_{t^*,l_1}^{(l_1,n)})(\beta_s^{(l_2,n)} - \beta_{t^*,l_2}^{(l_2,n)}) d\langle \alpha^{(l_1,n)}, \alpha^{(l_2,n)} \rangle_s \\
&\quad + \int_0^\tau (\alpha_s^{(l_1,n)} - \alpha_{t^*,l_1}^{(l_1,n)})(\beta_s^{(l_2,n)} - \beta_{t^*,l_2}^{(l_2,n)}) d\langle \beta^{(l_1,n)}, \alpha^{(l_2,n)} \rangle_s \\
&\quad + \int_0^\tau (\beta_s^{(l_1,n)} - \beta_{t^*,l_1}^{(l_1,n)})(\alpha_s^{(l_2,n)} - \alpha_{t^*,l_2}^{(l_2,n)}) d\langle \alpha^{(l_1,n)}, \beta^{(l_2,n)} \rangle_s.
\end{aligned}$$

From which

$$\begin{aligned}
\mathbb{E} \langle Z_n, Z_n \rangle_\tau &= \frac{1}{4k^2} \sum_{l_1=1}^{2k} \sum_{l_2=1}^{2k} \mathbb{E} \langle Z_{n,l_1}, Z_{n,l_2} \rangle_\tau \\
&= \frac{1}{4k^2} \sum_{l_1=1}^{2k} \sum_{l_2=1}^{2k} \mathbb{E} \left\{ \int_0^\tau (\alpha_s^{(l_1,n)} - \alpha_{t^*,l_1}^{(l_1,n)})(\alpha_s^{(l_2,n)} - \alpha_{t^*,l_1}^{(l_2,n)}) d\langle \beta^{(l_1,n)}, \beta^{(l_2,n)} \rangle_s \right. \\
&\quad + \int_0^\tau (\beta_s^{(l_1,n)} - \beta_{t^*,l_1}^{(l_1,n)})(\beta_s^{(l_2,n)} - \beta_{t^*,l_1}^{(l_2,n)}) d\langle \alpha^{(l_1,n)}, \alpha^{(l_2,n)} \rangle_s \\
&\quad + \int_0^\tau (\alpha_s^{(l_1,n)} - \alpha_{t^*,l_1}^{(l_1,n)})(\beta_s^{(l_2,n)} - \beta_{t^*,l_1}^{(l_2,n)}) d\langle \beta^{(l_1,n)}, \alpha^{(l_2,n)} \rangle_s \\
&\quad \left. + \int_0^\tau (\beta_s^{(l_1,n)} - \beta_{t^*,l_1}^{(l_1,n)})(\alpha_s^{(l_2,n)} - \alpha_{t^*,l_1}^{(l_2,n)}) d\langle \alpha^{(l_1,n)}, \beta^{(l_2,n)} \rangle_s \right\} \\
&= \frac{1}{4k^2} \sum_{l_1=1}^{2k} \sum_{l_2=1}^{2k} \mathbb{E} \left\{ \int_0^\tau (\alpha_s^{(l_1,n)} - \alpha_{t^*,l_1}^{(l_1,n)}) g_s^{(l_1,n)} (\alpha_s^{(l_2,n)} - \alpha_{t^*,l_1}^{(l_2,n)}) g_s^{(l_2,n)} d\langle \beta^{(n)}, \beta^{(n)} \rangle_s \right. \\
&\quad + \int_0^\tau (\beta_s^{(l_1,n)} - \beta_{t^*,l_1}^{(l_1,n)}) f_s^{(l_1,n)} (\beta_s^{(l_2,n)} - \beta_{t^*,l_1}^{(l_2,n)}) f_s^{(l_2,n)} d\langle \alpha^{(n)}, \alpha^{(n)} \rangle_s \\
&\quad + \int_0^\tau (\alpha_s^{(l_1,n)} - \alpha_{t^*,l_1}^{(l_1,n)}) g_s^{(l_1,n)} (\beta_s^{(l_2,n)} - \beta_{t^*,l_1}^{(l_2,n)}) f_s^{(l_2,n)} d\langle \beta^{(n)}, \alpha^{(n)} \rangle_s \\
&\quad \left. + \int_0^\tau (\beta_s^{(l_1,n)} - \beta_{t^*,l_1}^{(l_1,n)}) f_s^{(l_1,n)} (\alpha_s^{(l_2,n)} - \alpha_{t^*,l_1}^{(l_2,n)}) g_s^{(l_2,n)} d\langle \alpha^{(n)}, \beta^{(n)} \rangle_s \right\}.
\end{aligned}$$

Changing the order of summation and integration we have that (being a bit careful with the stopping time τ),

$$\begin{aligned}
& \sum_{l_1=1}^{2k} \sum_{l_2=1}^{2k} \mathbb{E} \int_0^\tau (\alpha_s^{(l_1,n)} - \alpha_{t_*,l_1}^{(l_1,n)}) g_s^{(l_1,n)} (\alpha_s^{(l_2,n)} - \alpha_{t_*,l_1}^{(l_2,n)}) g_s^{(l_2,n)} d\langle \beta^{(n)}, \beta^{(n)} \rangle_s \\
&= \int_0^T \mathbb{E} \sum_{l_1=1}^{2k} \sum_{l_2=1}^{2k} (\alpha_{s \wedge \tau}^{(l_1,n)} - \alpha_{t_*,l_1 \wedge \tau}^{(l_1,n)}) g_s^{(l_1,n)} (\alpha_{s \wedge \tau}^{(l_2,n)} - \alpha_{t_*,l_1 \wedge \tau}^{(l_2,n)}) g_s^{(l_2,n)} d\langle \beta^{(n)}, \beta^{(n)} \rangle_s \\
&= \int_0^T \mathbb{E} \left[\sum_{l=1}^{2k} (\alpha_{s \wedge \tau}^{(l,n)} - \alpha_{t_*,l \wedge \tau}^{(l,n)}) g_s^{(l,n)} \right]^2 d\langle \beta^{(n)}, \beta^{(n)} \rangle_s \leq c \int_0^T \mathbb{E} \left(\sum_{l=1}^{2k} (\alpha_{s \wedge \tau}^{(l,n)} - \alpha_{t_*,l \wedge \tau}^{(l,n)}) g_s^{(l,n)} \right)^2 ds,
\end{aligned}$$

and similarly for the second term, where we use that $d\langle \beta^{(n)}, \beta^{(n)} \rangle_s/ds \leq c$ and $d\langle \alpha^{(n)}, \alpha^{(n)} \rangle_s/ds \leq c$ when $s \leq \tau$. For the third and fourth term we use that $d\langle \beta^{(n)}, \alpha^{(n)} \rangle_s/ds \leq c$ for $s \leq \tau$ and Hölder's inequality,

$$\begin{aligned}
& \sum_{l_1=1}^{2k} \sum_{l_2=1}^{2k} \mathbb{E} \int_0^\tau (\alpha_s^{(l_1,n)} - \alpha_{t_*,l_1}^{(l_1,n)}) g_s^{(l_1,n)} (\beta_s^{(l_2,n)} - \beta_{t_*,l_1}^{(l_2,n)}) f_s^{(l_2,n)} d\langle \beta^{(n)}, \alpha^{(n)} \rangle_s \\
&= \int_0^T \mathbb{E} \sum_{l_1=1}^{2k} \sum_{l_2=1}^{2k} (\alpha_{s \wedge \tau}^{(l_1,n)} - \alpha_{t_*,l_1 \wedge \tau}^{(l_1,n)}) g_s^{(l_1,n)} (\beta_{s \wedge \tau}^{(l_2,n)} - \beta_{t_*,l_1 \wedge \tau}^{(l_2,n)}) f_s^{(l_2,n)} d\langle \beta^{(n)}, \alpha^{(n)} \rangle_s \\
&= \int_0^T \mathbb{E} \left(\sum_{l_1=1}^{2k} (\alpha_{s \wedge \tau}^{(l_1,n)} - \alpha_{t_*,l_1 \wedge \tau}^{(l_1,n)}) g_s^{(l_1,n)} \right) \left(\sum_{l_2=1}^{2k} (\beta_{s \wedge \tau}^{(l_2,n)} - \beta_{t_*,l_1 \wedge \tau}^{(l_2,n)}) f_s^{(l_2,n)} \right) d\langle \beta^{(n)}, \alpha^{(n)} \rangle_s \\
&\leq c \int_0^T \mathbb{E} \left(\sum_{l_1=1}^{2k} (\alpha_{s \wedge \tau}^{(l_1,n)} - \alpha_{t_*,l_1 \wedge \tau}^{(l_1,n)}) g_s^{(l_1,n)} \right) \left(\sum_{l_2=1}^{2k} (\beta_{s \wedge \tau}^{(l_2,n)} - \beta_{t_*,l_1 \wedge \tau}^{(l_2,n)}) f_s^{(l_2,n)} \right) ds \\
&\leq c \int_0^T \left\{ \mathbb{E} \left(\sum_{l_1=1}^{2k} (\alpha_{s \wedge \tau}^{(l_1,n)} - \alpha_{t_*,l_1 \wedge \tau}^{(l_1,n)}) g_s^{(l_1,n)} \right)^2 \mathbb{E} \left(\sum_{l_2=1}^{2k} (\beta_{s \wedge \tau}^{(l_2,n)} - \beta_{t_*,l_1 \wedge \tau}^{(l_2,n)}) f_s^{(l_2,n)} \right)^2 \right\}^{1/2} ds,
\end{aligned}$$

and similarly for the fourth term. Now, all the action takes place in expressions of the form

$$\begin{aligned}
& \mathbb{E} \left(\sum_{l=1}^{2k} (\alpha_{s \wedge \tau}^{(l,n)} - \alpha_{t_*,l \wedge \tau}^{(l,n)}) g_s^{(l,n)} \right)^2 = \mathbb{E} \sum_{l=1}^{2k} (\alpha_{s \wedge \tau}^{(l,n)} - \alpha_{t_*,l \wedge \tau}^{(l,n)}) g_s^{(l,n)} \sum_{l=1}^{2k} (\alpha_{s \wedge \tau}^{(l,n)} - \alpha_{t_*,l \wedge \tau}^{(l,n)}) g_s^{(l,n)} \\
&= \sum_{l_1=1}^{2k} \sum_{l_2=1}^{2k} \mathbb{E} (\alpha_{s \wedge \tau}^{(l_1,n)} - \alpha_{t_*,l_1 \wedge \tau}^{(l_1,n)}) g_s^{(l_1,n)} (\alpha_{s \wedge \tau}^{(l_2,n)} - \alpha_{t_*,l_2 \wedge \tau}^{(l_2,n)}) g_s^{(l_2,n)} \\
&= \sum_{l_1=1}^{2k} \sum_{l_2=1}^{2k} \mathbb{E} (\alpha_{s \wedge \tau}^{(l_1,n)} - \alpha_{t_*,l_1 \wedge \tau}^{(l_1,n)}) (\alpha_{s \wedge \tau}^{(l_2,n)} - \alpha_{t_*,l_2 \wedge \tau}^{(l_2,n)}) g_s^{(l_1,n)} g_s^{(l_2,n)}.
\end{aligned} \tag{D.4}$$

Since $|f_s^{(l,n)}| \leq 1$ and $|g_s^{(l,n)}| \leq 1$, we have that

$$\begin{aligned}
(D.4) &= \sum_{l_1=1}^{2k} \sum_{l_2=1}^{2k} \mathbb{E} \{ (\langle \alpha^{(l_1,n)}, \alpha^{(l_2,n)} \rangle_{s \wedge \tau} - \langle \alpha^{(l_1,n)}, \alpha^{(l_2,n)} \rangle_{(t_{*,l_1} \vee t_{*,l_2}) \wedge \tau}) g_s^{(l_1,n)} g_s^{(l_2,n)} \} \\
&= \sum_{l_1=1}^{2k} \sum_{l_2=1}^{2k} \mathbb{E} \left\{ \int_{(t_{*,l_1} \vee t_{*,l_2}) \wedge \tau}^{s \wedge \tau} f_u^{(l_1,n)} f_u^{(l_2,n)} d\langle \alpha^{(n)}, \alpha^{(n)} \rangle_u g_u^{(l_1,n)} g_u^{(l_2,n)} \right\} \\
&\leq c \sum_{l_1=1}^{2k} \sum_{l_2=1}^{2k} (s - (t_{*,l_1} \vee t_{*,l_2})).
\end{aligned} \tag{D.5}$$

Define

$$t_{**,l}(s) = t_{**,l} = \max\{t_{i-k} : t_{i+k} \leq s, i \equiv l[2k]\}.$$

Clearly,

$$t_{**,l_1}(s) \leq t_{*,l_1}(s) \vee t_{*,l_2}(s), \quad \text{for } l_1, l_2 = 1, \dots, 2k. \tag{D.6}$$

For $l = 1, \dots, 2k$, define

$$h_s^{(l,n)} = \sum_{i \equiv l[2k]} (s - t_{i-k}) I\{t_{i-k} \leq s < t_{i+k}\},$$

and notice that

$$(s - t_{**,l}) = (s - t_{**,l}(s)) = h_s^{(l,n)}.$$

Substituting the bound in (D.5) and the three similar ones into $\mathbb{E} \langle Z_n, Z_n \rangle_\tau$ and use the inequality in (D.6), then

$$\begin{aligned}
\mathbb{E} \langle Z_n, Z_n \rangle_\tau &\leq \frac{c^2}{k^2} \sum_{l_1=1}^{2k} \sum_{l_2=1}^{2k} \int_0^T (s - (t_{*,l_1} \vee t_{*,l_2})) ds \\
&= \frac{c^2}{k^2} \sum_{l_1=1}^{2k} \left\{ \int_0^T (s - (t_{*,l_1} \vee t_{*,1})) ds + \dots + \int_0^T (s - (t_{*,l_1} \vee t_{*,2k})) ds \right\} \\
&\leq \frac{c^2}{k^2} \sum_{l_1=1}^{2k} \left\{ \int_0^T (s - t_{**,l_1}) ds + \dots + \int_0^T (s - t_{**,l_1}) ds \right\} = \frac{2c^2}{k} \sum_{l=1}^{2k} \int_0^T (s - t_{**,l}) ds \\
&= \frac{2c^2}{k} \sum_{l=1}^{2k} \int_0^T h_s^{(l,n)} ds = \frac{2c^2}{k} \sum_{l=1}^{2k} \sum_{i \equiv l[2k]} \int_{t_{i-k}}^{t_{i+k}} (s - t_{i-k}) ds \\
&= \frac{2c^2}{k} \sum_{i=k}^{B-k} \int_{t_{i-k}}^{t_{i+k}} (s - t_{i-k}) ds = \frac{4c^2}{k} \sum_{i=k}^{B-k} (k\Delta_n)^2 = 4c^2 T k \Delta_n.
\end{aligned}$$

Let $\tau_n = \inf\{t \in [0, T] : \langle \alpha^{(n)}, \alpha^{(n)} \rangle_t > ct \text{ or } \langle \beta^{(n)}, \beta^{(n)} \rangle_t > ct\}$. By Condition 5 $\tau_n \rightarrow T$ as $n \rightarrow \infty$. Let $\varepsilon > 0$ and choose $c > 0$ sufficiently large, so that $P(\tau_n \neq T) \leq \varepsilon/2$. Then

$$\begin{aligned} P(\langle Z_n, Z_n \rangle_T / (4c^2 T k \Delta_n) > M) &\leq P(\langle Z_n, Z_n \rangle_{\tau_n} / (4c^2 T k \Delta_n) > M) + P(\tau_n \neq T) \\ &\leq M^{-1} \mathbb{E}[\langle Z_n, Z_n \rangle_{\tau_n} / (4c^2 T k \Delta_n)] + P(\tau_n \neq T) \\ &= M^{-1} + \varepsilon/2 \leq \varepsilon, \end{aligned}$$

provided $M \geq 2/\varepsilon$. This shows that $\langle Z_n, Z_n \rangle_T / (4c^2 T k \Delta_n)$ is tight, so $\langle Z_n, Z_n \rangle_T = O_p(4c^2 T k \Delta_n) = O_p(k_n \Delta_n)$. By Lenglart's inequality (Andersen et al. 1993, p. 86), for any $\delta > 0$ and $M > 0$

$$P(\sup_{0 \leq t \leq T} |Z_n(t)| > \delta) \leq \frac{M}{\delta^2} + P(\langle Z_n, Z_n \rangle_T > M).$$

With the same $\delta = M$ and the same M as above, $P(\sup_{0 \leq t \leq T} |Z_n(t)| > \delta) \leq (3/2)\varepsilon$, from which we conclude that

$$\sup_{0 \leq t \leq T} |Z_n(t)| = O_p((k \Delta_n)^{1/2}).$$