NLH-Memories & Grid-free Spatial Modelling

by

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NTNU, Trondheim, Norway

Oslo 04.12.2023

Why am I invited ?

NLH employment :

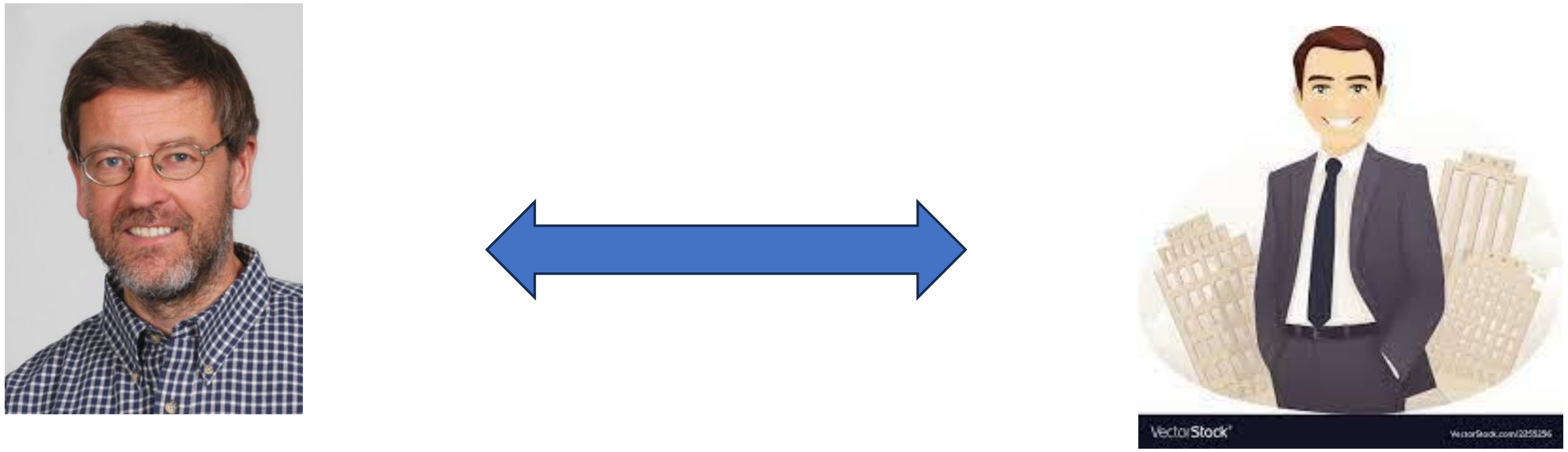
Uni Tromsø	Academia	'Verna arbeidsplass'	Support personell
Norwegian Computing Center	Applied research	Research Consulting	Bosses
Uni Oslo	Academia	'Verna arbeidsplass'	Support personell

My experiences as NLH-Boss !

Timeliste:


Timeliste støtteperson m.m.

PROJECT WORK



PROJECT WORK

PROJECT WORK

BRAVISSIMO !!!

BRAVO !!!

PROJECT WORK

BRAVISSIMO !!!

BRAVO !!!

HM ???

NLH is a Champion in a Noble Art !!!

NLH is a Champion in a Noble Art !!!

The Noble Art of making Simple Things – very, very Complicated !!!

Quizzz !!

Which Problem is NLH solving here ???

Consequently

$$\max_{\mu=\mu_k} L(z,x) = (2\pi e)^{-(N+1)d/2} |\widetilde{\Sigma}_{(k)}|^{-(N+1)/2},$$

where $\widetilde{\Sigma}_{(k)}$ is the adjusted maximum likelihood estimator of Σ obtained by including x as an observation from class k, i.e.

$$\widetilde{\Sigma}_{(k)} = \frac{1}{N+1}(A_1 + \cdots + \widetilde{A}_k + \cdots + A_K),$$

where

$$\widetilde{A}_k = \sum_{j=1}^{n_k} \left(x_j^{(k)} - \widetilde{\mu}_k\right)\left(x_j^{(k)} - \widetilde{\mu}_k\right)' + (x - \widetilde{\mu}_k)(x - \widetilde{\mu}_k)',$$

$$\widetilde{\mu}_k = \frac{1}{n_k+1}\left(\sum_{j=1}^{n_k} x_j^{(k)} + x\right) = \widehat{\mu}_k + \frac{1}{n_k+1}(x - \mu_k).$$

Some manipulations give

$$\widetilde{A}_k = A_k + \frac{n_k}{n_k+1}(x - \widehat{\mu}_k)(x - \widehat{\mu}_k)', \qquad (6.22)$$

implying

$$\widetilde{\Sigma}_{(k)} = \frac{N}{N+1}\left[\widehat{\Sigma} + \frac{1}{N}\frac{n_k}{n_k+1}(x - \widehat{\mu}_k)(x - \widehat{\mu}_k)'\right].$$

By the identity

$$\begin{aligned} |B + cyy'| &= |B(I + B^{-1}cyy')| \\ &= |B|(1 + cy'B^{-1}y), \end{aligned} \qquad (6.23)$$

which can be seen to follow from (8.4.11) in Box and Tiao (1973),

$$\max_{\mu=\mu_k} L(z,x) = (2\pi e)^{-(N+1)d/2}\left(\frac{N+1}{N}\right)^{(N+1)d/2} |\widehat{\Sigma}|^{-(N+1)/2} \left\{1 + \frac{1}{N}\frac{n_k}{n_k+1}(x - \widehat{\mu}_k)'\widehat{\Sigma}^{-1}(x - \widehat{\mu}_k)\right\}^{-(N+1)/2},$$

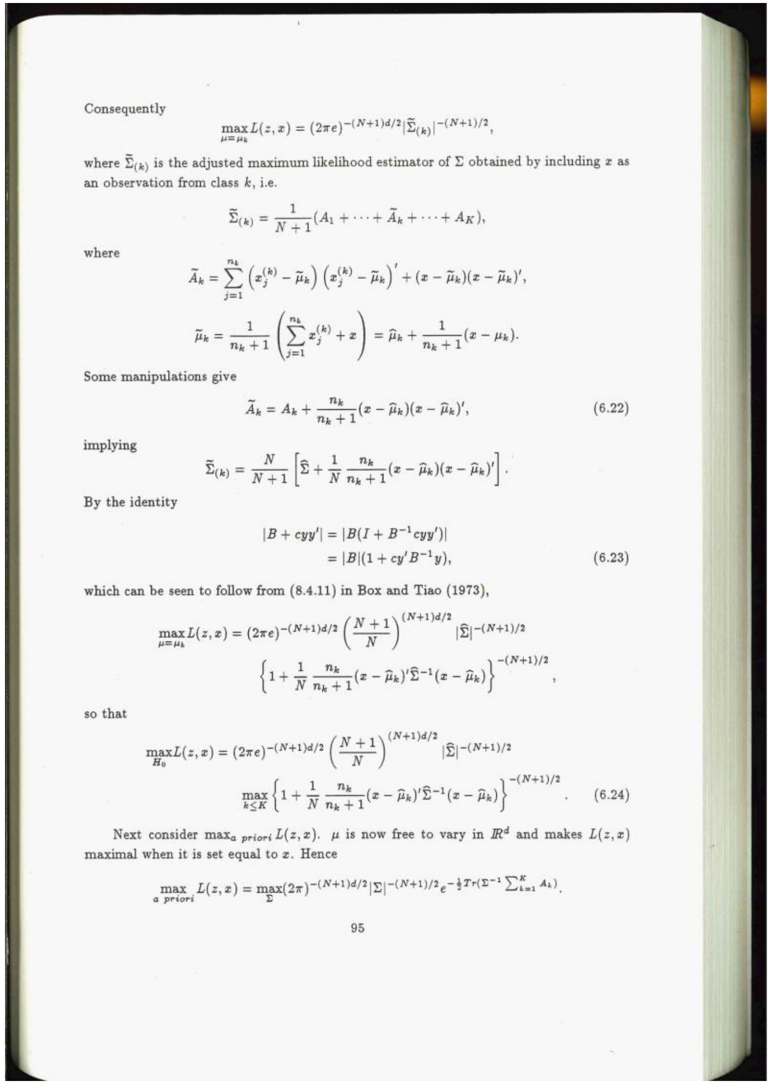
so that

$$\max_{H_0} L(z,x) = (2\pi e)^{-(N+1)d/2}\left(\frac{N+1}{N}\right)^{(N+1)d/2} |\widehat{\Sigma}|^{-(N+1)/2} \max_{k \le K}\left\{1 + \frac{1}{N}\frac{n_k}{n_k+1}(x - \widehat{\mu}_k)'\widehat{\Sigma}^{-1}(x - \widehat{\mu}_k)\right\}^{-(N+1)/2}. \qquad (6.24)$$

Next consider $\max_{a\ priori} L(z,x)$. μ is now free to vary in $\mathbb{R}^d$ and makes $L(z,x)$ maximal when it is set equal to x. Hence

$$\max_{a\ priori} L(z,x) = \max_{\Sigma} (2\pi)^{-(N+1)d/2} |\Sigma|^{-(N+1)/2} e^{-\frac{1}{2}Tr(\Sigma^{-1}\sum_{k=1}^{K} A_k)}.$$

Which Problem is NLH solving here ???



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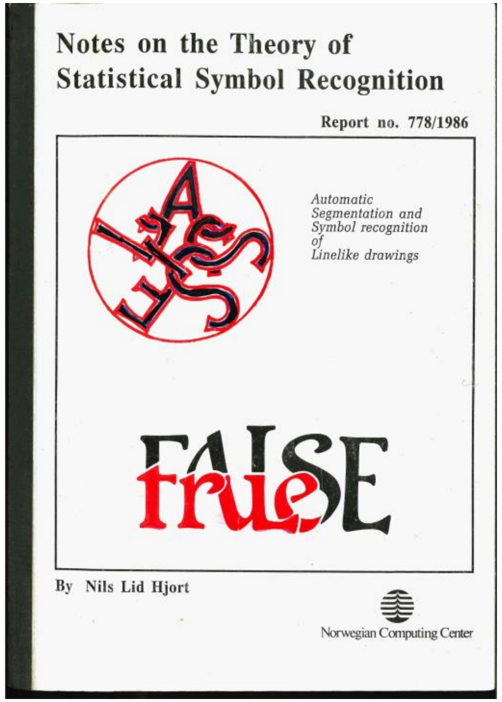
95



Which Problem is NLH solving here ???

To obtain explicit formulae we will consider the situation where the pixel crosses are chosen so far apart that their spectral vectors can be considered uncorrelated. The central pixels of the crosses can for example be chosen as every 4th pixel in the training set. As estimator for μ_k we will use the mean of the vectors from only the crosses belonging to class k:

$$\bar{X}^{(k)} = \frac{1}{5m_k} \sum_{i=1}^{m_k} (X_i^{(k)} + X_{iN}^{(k)} + X_{iE}^{(k)} + X_{iS}^{(k)} + X_{iW}^{(k)}).$$

The corresponding estimators $A^{(k)}(\bar{X}^{(k)})$, $B^{(k)}(\bar{X}^{(k)})$, $C^{(k)}(\bar{X}^{(k)})$ will just be denoted $A^{(k)}$, $B^{(k)}$ and $C^{(k)}$.

Let us consider $EA^{(k)}$. Introducing

$$U_i = X_i^{(k)} - \mu_k, \quad U_{iN} = X_{iN}^{(k)} - \mu_k, \text{ etc.},$$

we find for the first term

$$E(X_{iN}^{(k)} - \bar{X}^{(k)})(X_i^{(k)} - \bar{X}^{(k)})' = E\, U_{iN} U_i' - \frac{1}{5m_k} \sum_{j=1}^{m_k} E(U_j + U_{jN} + U_{jE} + U_{jS} + U_{jW})U_i' +$$

$$- \frac{1}{5m_k} \sum_{j=1}^{m_k} E\, U_{iN}(U_j + U_{jN} + U_{jE} + U_{jS} + U_{jW})'$$

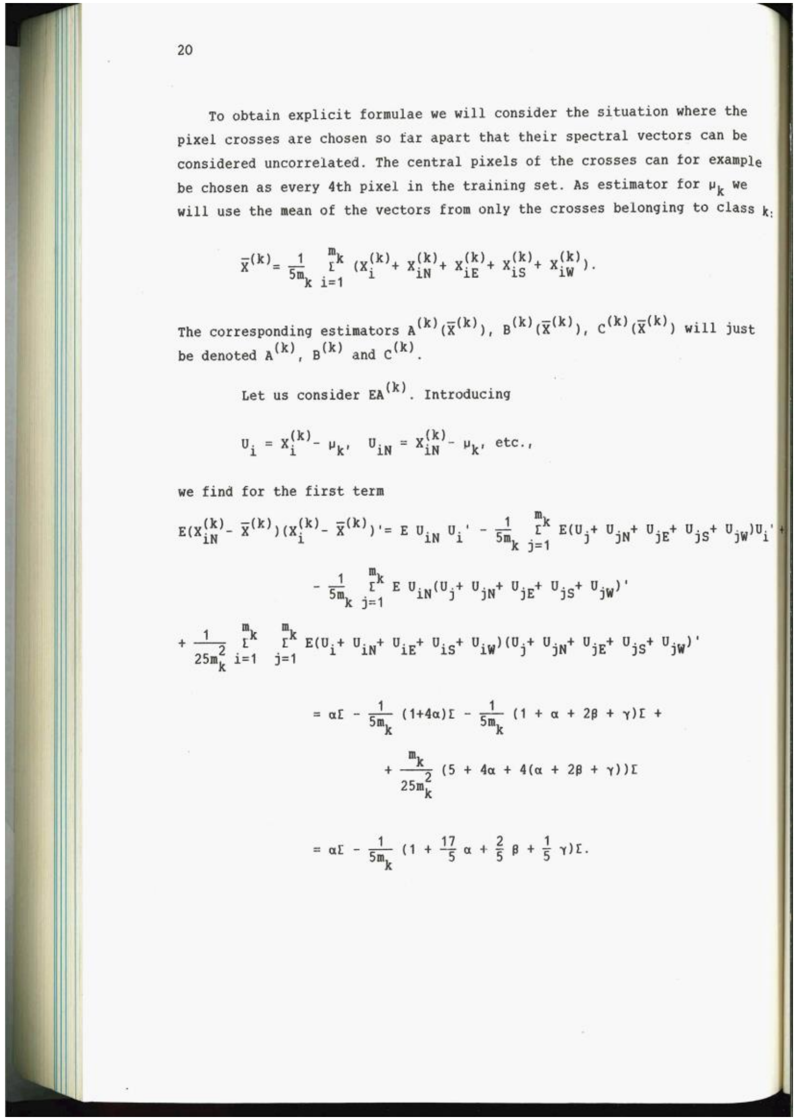
$$+ \frac{1}{25m_k^2} \sum_{i=1}^{m_k} \sum_{j=1}^{m_k} E(U_i + U_{iN} + U_{iE} + U_{iS} + U_{iW})(U_j + U_{jN} + U_{jE} + U_{jS} + U_{jW})'$$

$$= \alpha\Sigma - \frac{1}{5m_k}(1+4\alpha)\Sigma - \frac{1}{5m_k}(1 + \alpha + 2\beta + \gamma)\Sigma +$$

$$+ \frac{m_k}{25m_k^2}(5 + 4\alpha + 4(\alpha + 2\beta + \gamma))\Sigma$$

$$= \alpha\Sigma - \frac{1}{5m_k}(1 + \frac{17}{5}\alpha + \frac{2}{5}\beta + \frac{1}{5}\gamma)\Sigma.$$

Which Problem is NLH solving here ???



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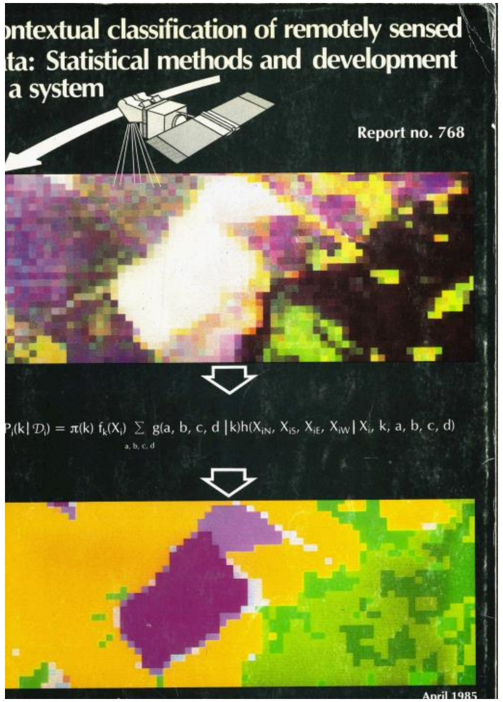

Which Problem is NLH solving here ???

is an unbiased estimator of B. Using the fact that $W_{1,r}^{(k)} = \widehat{\Delta}_{k,r}$, we see that the first term can be written

$$\frac{1}{K}\sum_{k=1}^{K}\frac{1}{R}\sum_{r=1}^{R} W_{1,r}^{(k)} = \frac{1}{K}\sum_{k=1}^{K}\widehat{\Delta}_k = \frac{1}{K}\widehat{\theta}_K,$$

where $\widehat{\theta}_K$ is the SBIL based estimator of the accumulated thickness of the whole section. Hence

$$\widehat{B} = \frac{1}{K}(\widehat{\theta}_K - \widetilde{\theta}_K) - \frac{1}{2}(\widehat{A} + \widehat{C}). \tag{7.16}$$

In order to find the variance of $\widehat{B}$ we first express its four components via the noise terms $\varepsilon_{i,r}^{(k)}$. Using (7.11) and (7.12) we find

$$\widehat{A} = A + \frac{1}{K}\sum_{k=1}^{K}[-\bar{\varepsilon}_1^{(k+1)} + \bar{\varepsilon}_2^{(k+1)}],$$

$$\widehat{C} = C + \frac{1}{K}\sum_{k=1}^{K}[-\bar{\varepsilon}_3^{(k)} + \bar{\varepsilon}_4^{(k)}],$$

$$\frac{1}{K}\widehat{\theta}_K = \frac{1}{K}\theta_K + \tfrac{1}{2}A + B + \tfrac{1}{2}C - \frac{1}{K}\sum_{k=1}^{K}[-\bar{\varepsilon}_1^{(k+1)} - \bar{\varepsilon}_2^{(k+1)} + \bar{\varepsilon}_3^{(k)} + \bar{\varepsilon}_4^{(k)}],$$

$$\begin{aligned}\frac{1}{K}\widetilde{\theta}_K = \frac{1}{K}\theta_K &+ \frac{1}{2K}[3\overline{\psi}^{(1)} - \overline{\psi}^{(2)} - \overline{\psi}^{(K)} - \overline{\psi}^{(K+1)}] \\ &+ \frac{1}{2K}[\bar{\varepsilon}_3^{(1)} + \bar{\varepsilon}_4^{(1)}] + \frac{1}{4K}[\bar{\varepsilon}_1^{(2)} + \bar{\varepsilon}_2^{(2)} - \bar{\varepsilon}_3^{(2)} - \bar{\varepsilon}_4^{(2)}] \\ &- \frac{1}{4K}[\bar{\varepsilon}_1^{(K+1)} + \bar{\varepsilon}_1^{(K+1)} + \bar{\varepsilon}_3^{(K+1)} + \bar{\varepsilon}_4^{(K+1)}].\end{aligned}$$

Here the various $\overline{\psi}^{(k)}$'s are independent, Gaussian variables with mean value zero and variance λ^2/R, and are also independent of the $\bar{\varepsilon}_i^{(k)}$'s. Futhermore, $\operatorname{Var}\bar{\varepsilon}_i^{(k)} = \sigma_k^2/R$.

Being linear functions of independent Gaussian varables, the four components of $\widehat{B}$ have a multinormal distribution, and a standard calculation shows that the covariance matrix is

$$\begin{pmatrix} \operatorname{Var}\widehat{A} & 0 & 0 & 0 \\ 0 & \operatorname{Var}\widehat{C} & 0 & 0 \\ 0 & 0 & \operatorname{Var}\frac{\widehat{\theta}_K}{K} & \operatorname{cov}(\frac{\widehat{\theta}_K}{K}, \frac{\widetilde{\theta}_K}{K}) \\ 0 & 0 & \operatorname{cov}(\frac{\widehat{\theta}_K}{K}, \frac{\widetilde{\theta}_K}{K}) & \operatorname{Var}\frac{\widetilde{\theta}_K}{K} \end{pmatrix}.$$

$\operatorname{Var}\widehat{A}$ and $\operatorname{Var}\widehat{C}$ are given by (7.14), while

$$\operatorname{Var}\frac{\widehat{\theta}_K}{K} = \frac{1}{RK^2}\sum_{k=1}^{K}\tfrac{1}{2}(\sigma_k^2 + \sigma_{k+1}^2) = \tfrac{1}{4}(\operatorname{Var}\widehat{A} + \operatorname{Var}\widehat{C}),$$

$$\operatorname{Var}\frac{\widetilde{\theta}_K}{K} = \frac{1}{4RK^2}[2\sigma_1^2 + \sigma_2^2 + \sigma_{K+1}^2] + \frac{\lambda^2}{RK^2}h(\rho),$$

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$$\begin{pmatrix} \mathrm{Var}\,\widehat{A} & 0 & 0 & 0 \\ 0 & \mathrm{Var}\,\widehat{C} & 0 & 0 \\ 0 & 0 & \mathrm{Var}\,\frac{\widehat{\theta}_K}{K} & \mathrm{cov}\,(\frac{\widehat{\theta}_K}{K}, \frac{\widetilde{\theta}_K}{K}) \\ 0 & 0 & \mathrm{cov}\,(\frac{\widehat{\theta}_K}{K}, \frac{\widetilde{\theta}_K}{K}) & \mathrm{Var}\,\frac{\widetilde{\theta}_K}{K} \end{pmatrix}.$$

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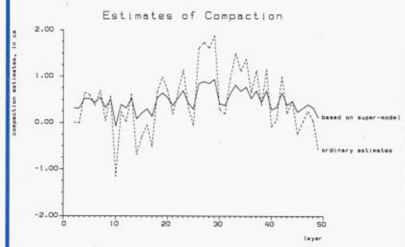
Estimating Compaction Using the Formation Subsidence Monitoring Tool:

Development of New Models and Methods

Norsk Regnesentral

RAPPORT



Norwegian Computing Center

SAND/08/88

by

Nils Lid Hjort
Erik Mohn
Geir Storvik

Oslo
March 1988

NLH was a BIG success at NR because:

- He is a very likeable person – friendly to us all
- He is interested in everything – absolutely everything
- His reports worked – and better than everything else.

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Topics in Spatial Statistics*

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University of Oslo

HENNING OMRE

Norwegian Institute of Technology

ABSTRACT. An overview is given over a fair range of topics within spatial and spatial–temporal statistics. The theory presented is motivated by and illustrated with actual applications to real world problems. We describe and discuss models for three basic types of spatial processes: continuous random surfaces, mosaic phenomena, and events-against-background processes. Various combinations of these sometimes occur naturally in applications, like Gaussian noise on top of a Markov random field in image restoration problems. Some of these combinations are also discussed. The applications we discuss are drawn from the areas of medical image analysis, pollution monitoring, characterisation of oil reservoirs, estimation of fish and whale stock, forestry surveillance via satellite, statistical meteorology, and symbol recognition.

Key words: Bayesian methods, covariance function, event processes, hidden Markov fields, image restoration, Kriging, marked point processes, Markov random fields, parameter estimation, pseudo-likelihood, quasi-likelihood, semi-Markov random fields, spatial classification, spatial sampling strategy, stochastic simulation

Grid-free Spatial Modelling

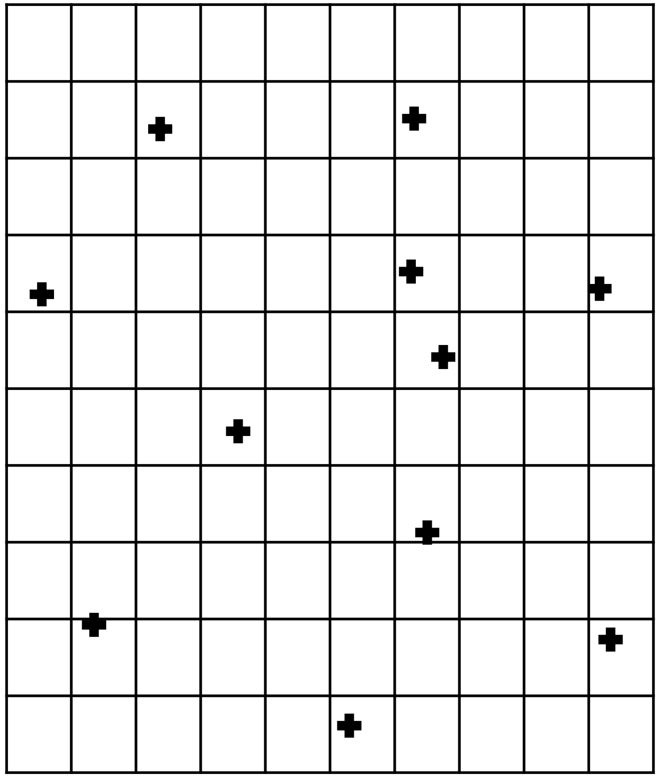
by

Henning Omre
Mina Spremic

Department of Mathematical Sciences,
NTNU, Trondheim, Norway

A Mickey Mouse Study !!!

2D area: [1, 100]^2



Observations: m = 10

Grid: n = 100^2 = 10^4

Stationary Gaussian RF model: $\mu_r \quad \sigma_r^2 \quad \rho_r(x-x')$

Grid-representations: $r:\{r(x); x \in L \in D \subset \mathbb{R}^3\}$

Predictors:

Kriging predictor:

$$\hat{r} = \Gamma_{rd}\Sigma_d^{-1}d = w^x d = \sum_{i=1}^{m} w_i^x d_i$$ **- GRID**

Gaussian Markov predictor:

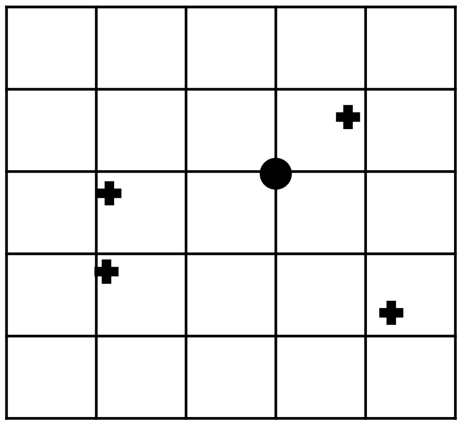
$$\hat{r} = \Psi_r^{-1}H_o^T\Psi_d d = w^x d = \sum_{i=1}^{m} w_i^x d_i$$ **- GRID**

Basis-function predictor:

$$\hat{r} = \Sigma_a F_d^T \Sigma_d^{-1} d = \sum_{i=1}^{n} \mu_{a_i|d}\, f_{x_i}(x)$$ **- GRID**

Journel and Huijbregts (1978); Rue and Held (2005); Cressie and Johannesson (2008)

The Kernel Predictor



Stationary Gaussian RF model: $\{r(x); x \in D\}$

$$\mu_r = 0.0 \quad \sigma_r^2 = 1.0 \quad \rho_r(x - x')$$

Predictor:

$$\hat{r}_0 = \sigma_{0d}^T \Sigma_d^{-1} d$$

Kriging predictor:

$$\hat{r}_0 = w^{xT} d = \sum_{i=1}^{m} w_i^x d_i \qquad w^x = \sigma_{0d}^T \Sigma_d^{-1}$$

Kernel predictor:

$$\hat{r}_0 = \sigma_{0d}^T w^d = \sum_{i=1}^{m} w_i^d \rho_r(x_0 - x_i) \qquad w^d = \Sigma_d^{-1} d$$

Grid-repr

infill **asymp**

Functional repr

Parameters:

$$\sigma_{0d} \quad \Sigma_d$$

$$\left\{ \hat{r}(x) = \sum_{i=1}^{m} w_i^d \rho_r(x - x_i); x \in D \right\}$$

Prediction variance: Ok !!

Dual Kriging predictor ! Matheron (1971)

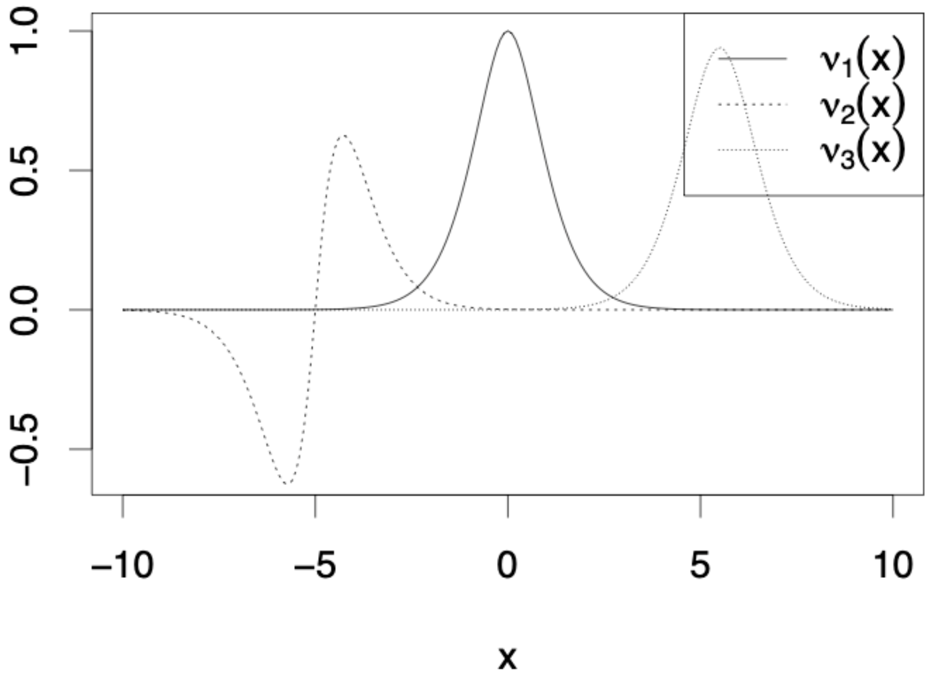
Example: $\{r(x); x \in [-10,10] \subset \mathbb{R}\}$

Normalized Gaussian RF model:

$$\rho_r(\tau) = [1 + 5^{1/2}\tau + 5/3\tau^2]\exp\{-5^{1/2}\tau\}; \tau = |x' - x''|/\tau_M.$$

Kernel spatial predictor:

$$\left\{\hat{r}(x) = \sum_{i=1}^{3} w_i^d v_i(x) \ ; x \in [-10,10] \subset \mathbb{R}\right\}$$



Observations:

$$d_1 = r(0)$$

$$d_2 = d/dx\ r(x)|_{x=-5}$$

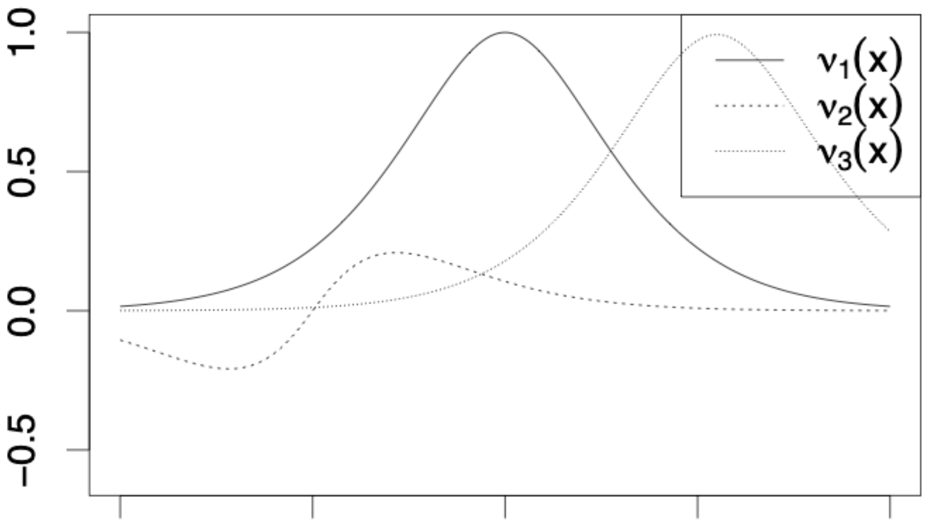
$$d_3 = \int_5^6 r(u)du.$$

Kernel functions:

$$\nu_1(x) = \rho_r(|x - 0|/\tau_M)$$

$$\nu_2(x) = d/du\ \rho_r(|x - u|/\tau_M)|_{u=-5}$$

$$\nu_3(x) = \int_5^6 \rho_r(|x - u|/\tau_M)du$$



Example: $\left\{ r(x); x \in [-10, 10] \subset \mathbb{R} \right\}$

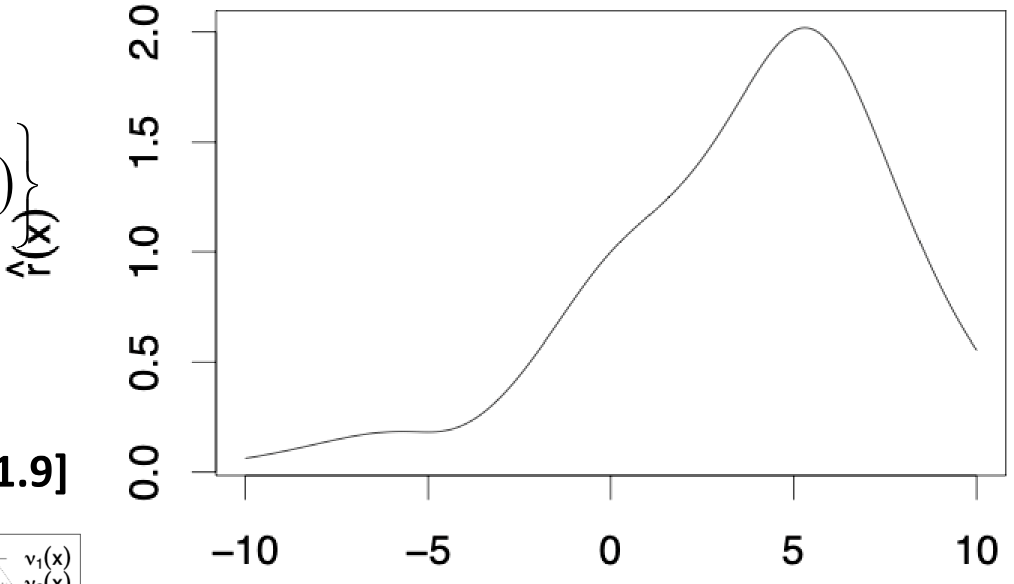
0 1 2

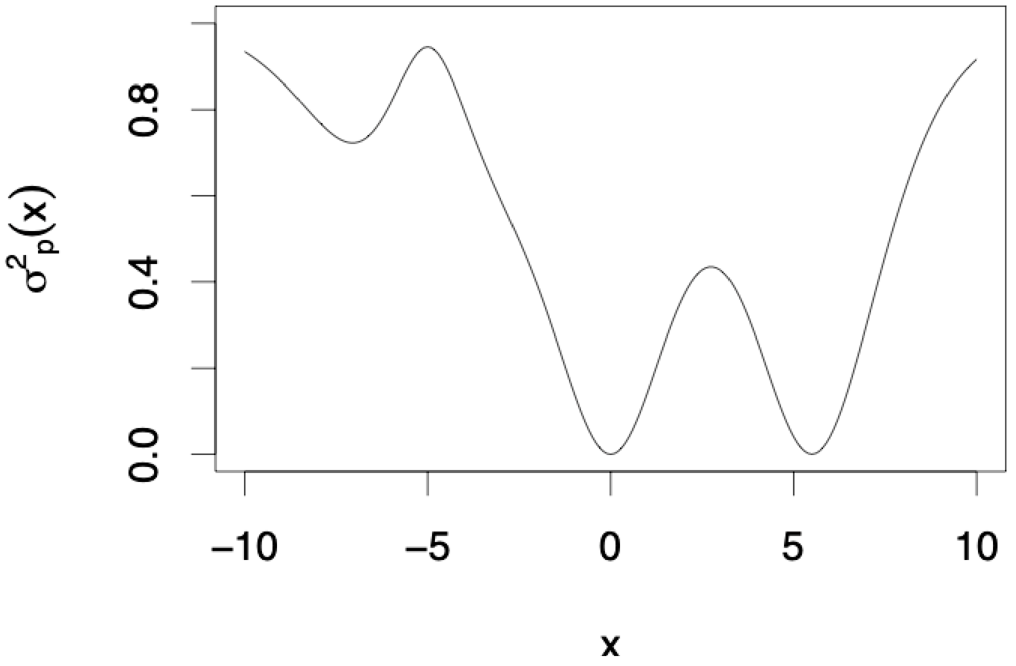
Kernel predictor:

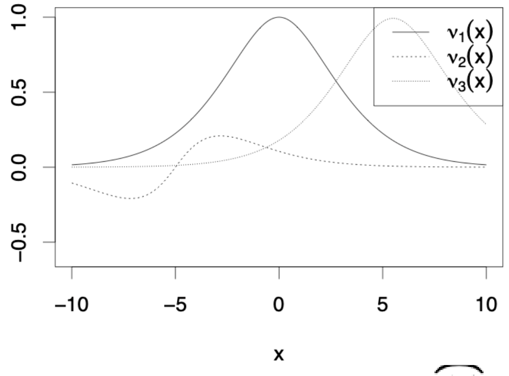
$$\left\{ \hat{r}(x) = \sum_{i=1}^{3} w_i^d \nu_i(x) \right\}$$

$$w^d = \Sigma_d^{-1} d$$

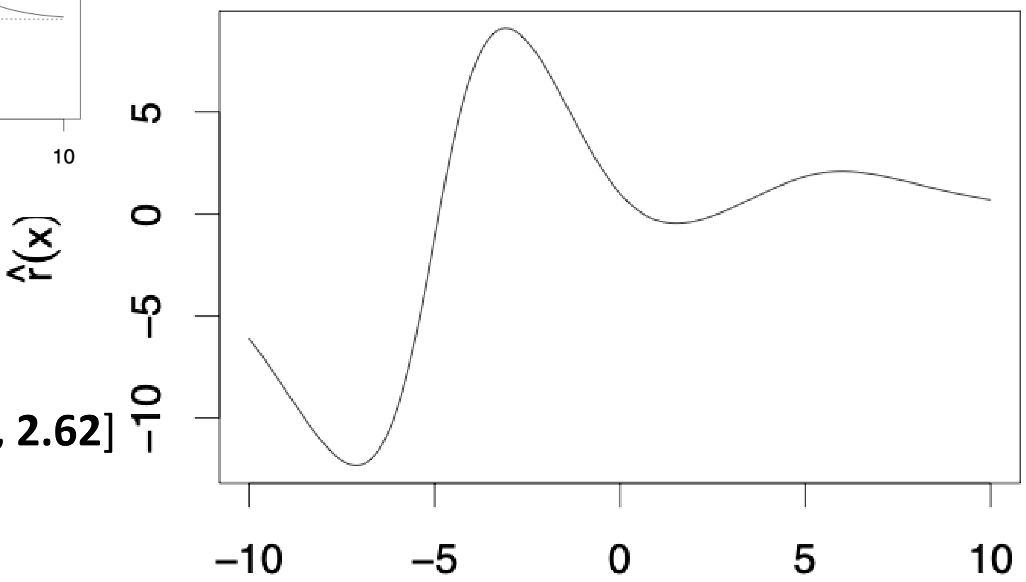
[0.71, -0.47, 1.9]

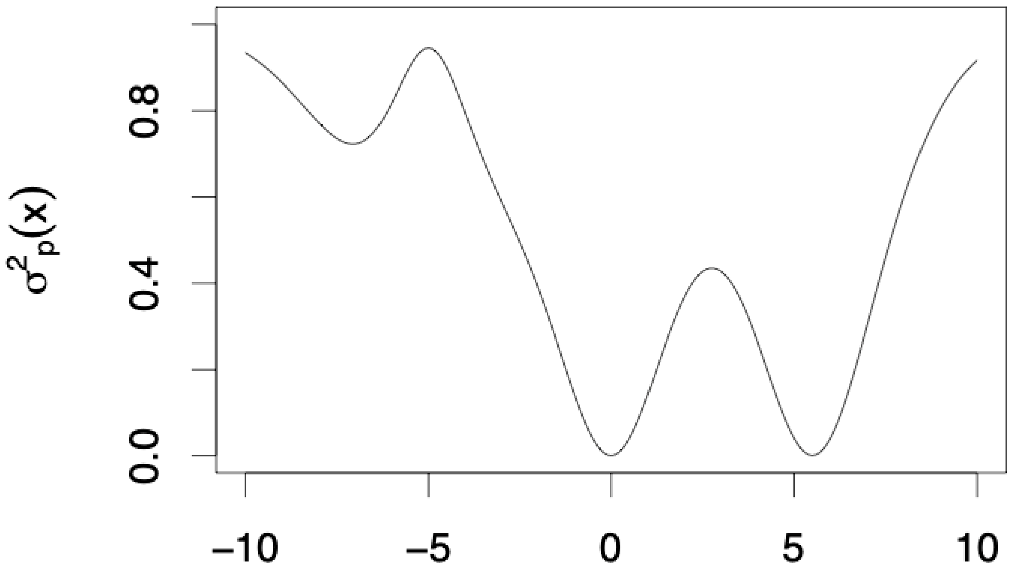






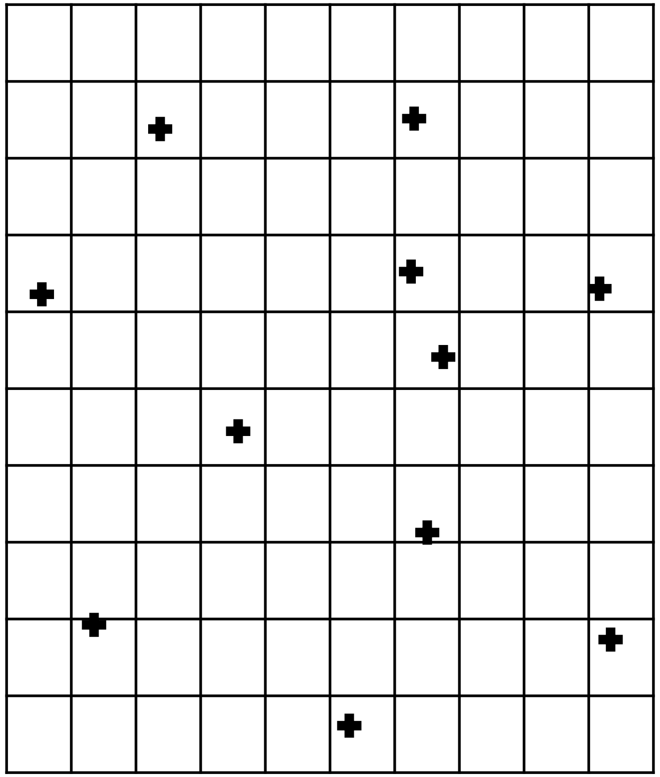
10 1 2





[-5.48, 57.02, 2.62]

3D volume: [1, 1000]^3



Observations: m = 1000

Grid: n = 1000^3 = 10^9

Rel. eff. = 10^6 – and func repr

Grid-representations: $r:\{r(x); x \in L \in D \subset \mathbb{R}^3\}$

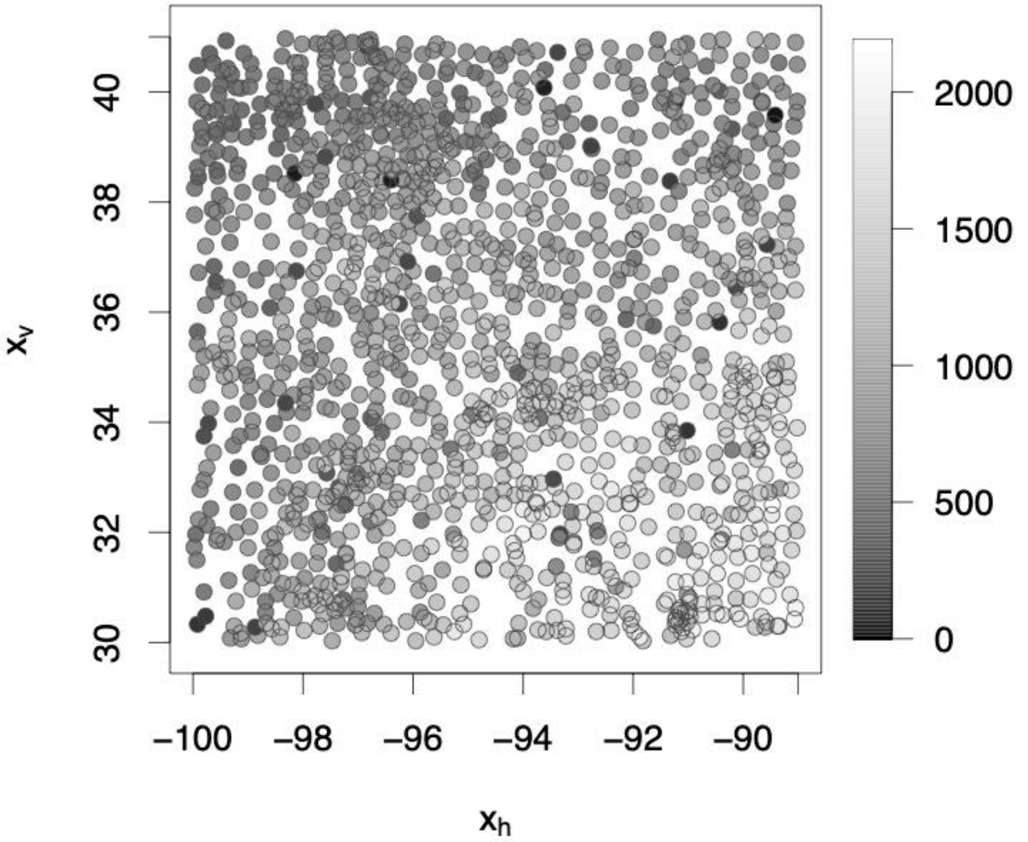
Predictor	Comp.eff.	2D-Comp Sparse	3D-Comp Sparse
Kriging predictor: $\hat{r} = \Gamma_{rd}\Sigma_d^{-1}d$	m^3	$m^{3/2}$	m^2
Gaussian Markov predictor: $\hat{r} = \Psi_r^{-1}H_o^T\Psi_d d$	n^3	$n^{3/2}$	n^2
Basis-function predictor: $\hat{r} = \Sigma_a F_d^T \Sigma_d^{-1} d$	m^3	$m^{3/2}$	m^2
Kernel predictor: $\hat{r} = \Gamma_{rd}\Sigma_d^{-1}d$	m^3	$m^{3/2}$	m^2
Loc-Kriging predictor:	n	n	n
Loc-Kernel predictor:	m	m	m

Example: $\left\{ r(x); x \in D \subset \mathbb{R}^2 \right\}$

Stationary Gaussian RF model – finite range:

$$\mu_r \quad \sigma_r^2 \quad \rho_r(x - x')$$

Observations: m = 1330



Localized kernel predictor:

$$\left\{ \hat{r}^*(x) = \mu_r^* + \sum_{i=1}^{1330} w_i^{d*} \rho_r(x - x_i); x \in D \right\}$$

$$w^{d*} = \left[\Sigma_d^\rho \right]^{-1*} d$$

Algorithm 1 (Observation inter-correlation matrix approximation)

Initiate,

Localization range: $\Delta \in \mathbb{R}_+$

Support matrix: $\mathbf{\Psi} = 0\mathbf{I}_m$ - *dim* $(m \times m)$

For $\mathbf{x}_i^d; i = 1, \ldots, m$

Define neighborhood set: $M_{\mathbf{x}_i^d}^\Delta = \{\mathbf{y} | \mathbf{y} \in M, |\mathbf{x}_i^d - \mathbf{y}| < \Delta\}$ - *dim* $m_{\mathbf{x}_i^d}^\Delta$

Construct matrix: $\mathbf{\Sigma}_{d_{\mathbf{x}_i^d}^\Delta}^\rho =$ *Sub-matrix* $\{\mathbf{\Sigma}_d^\rho; M_{\mathbf{x}_i^d}^\Delta\}$ - *dim* $(m_{\mathbf{x}_i^d}^\Delta \times m_{\mathbf{x}_i^d}^\Delta)$

Compute: $[\mathbf{\Sigma}_{d_{\mathbf{x}_i^d}^\Delta}^\rho]^{-1}$ - *dim* $(m_{\mathbf{x}_i^d}^\Delta \times m_{\mathbf{x}_i^d}^\Delta)$

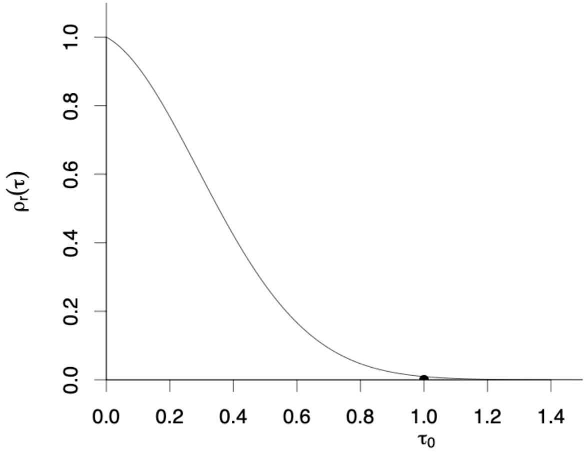
Copy: $[\mathbf{\Psi}]_{ij} = [[\mathbf{\Sigma}_{d_{\mathbf{x}_i^d}^\Delta}^\rho]^{-1}]_{\mathbf{x}_i^d \mathbf{y}}$ - *for corresponding* j *and* $\mathbf{y} \in M_{\mathbf{x}_i^d}^\Delta$ *entries*

End For

Define: $[\mathbf{\Sigma}_d^\rho]^{-1*} = 1/2 \times [\mathbf{\Psi} + \mathbf{\Psi}^T]$

$\longrightarrow \left[\Sigma_d^\rho \right]^{-1*}$

Correlation function:



Gneiting (2002); Omre and Spremic (2023)

Example:

Localized Kernel predictor:

$$\left\{ \hat{r}^*(x) = \mu_r^* + \sum_{i=1}^{1330} w_i^{d*} \rho_r(x - x_i); x \in D \right\}$$

$$w^{d*} = \left[\Sigma_d^{\rho} \right]^{-1*}$$

Display grid: [1101, 1101]

n = 1 212 201

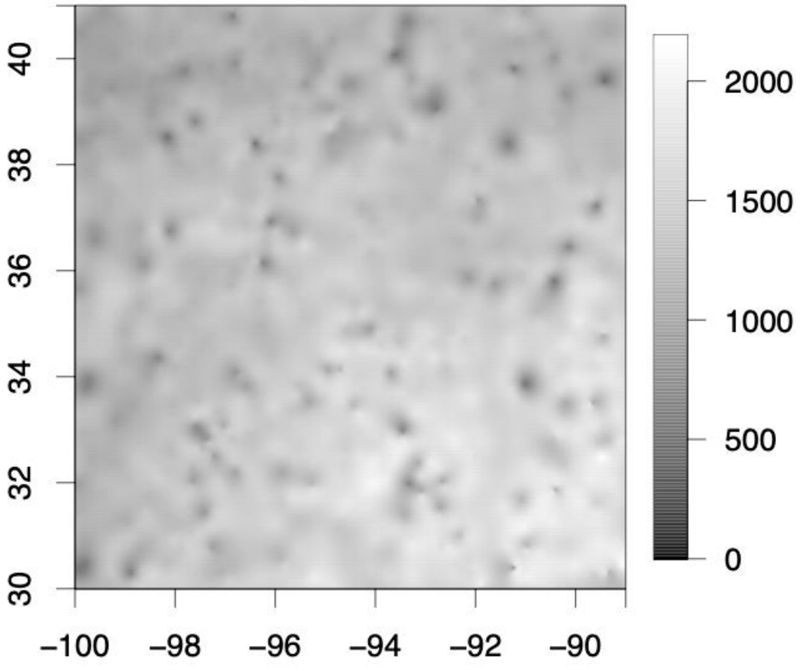
Rel. eff. = 911

Loc Kernel pred: 1 min

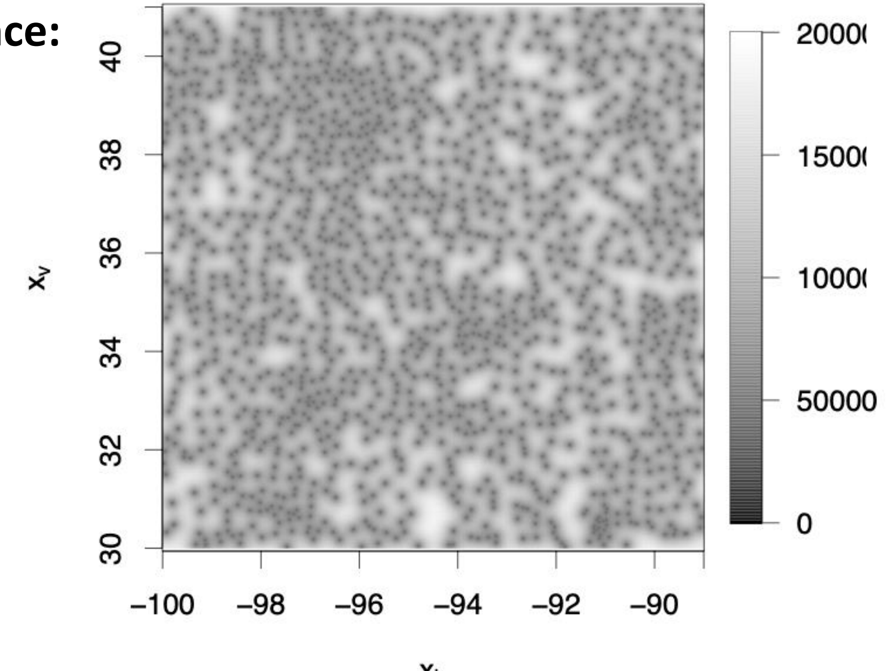
Loc Kriging pred: 15 hours

NOTE: Better in 3D and [3+1]D !!!

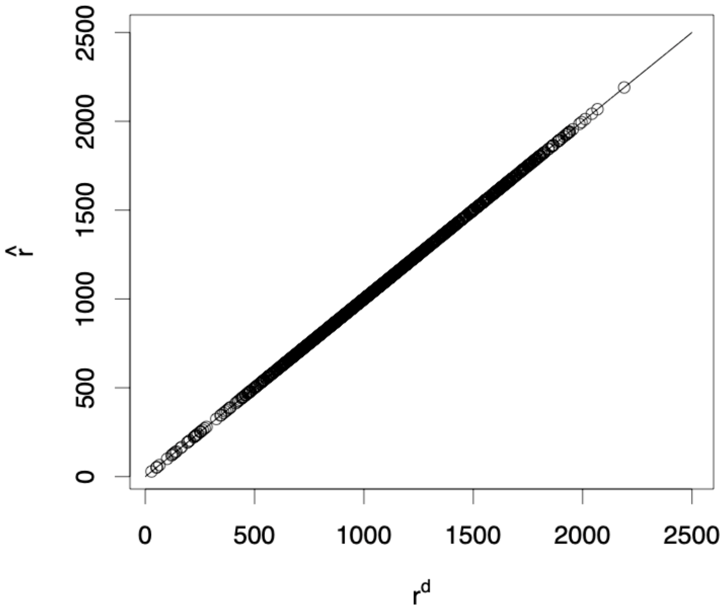
Prediction:



Variance:



Cross-plot in obs loc:



FINAL QUESTION :

Why have I not seen this before ????

Omre and Spremic (2023); arXiv