How many were killed in Guatemala, 1978–1996?

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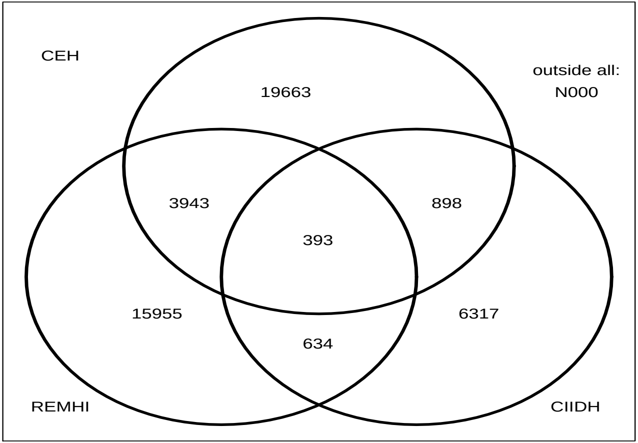
NordStat, Göteborg, June 2023

Counting the not counted



Lists REMHI, CEH, CIIDH:
$n_{1,1,1} = 393$, $n_{1,1,0} = 3943$, $n_{1,0,0} = 15955$, $n_{1,0,1} = 634$,
$n_{0,1,1} = 898$, $n_{0,1,0} = 19663$, $n_{0,0,1} = 6317$. How big is N000?

Today (with more details elsewhere)

Clearly we need to assume something, to construct estimates of $N_{0,0,0}$ and the total

$$\begin{aligned} N &= N_{1,1,1} + N_{1,1,0} + N_{1,0,1} + N_{0,1,1} + N_{1,0,0} + N_{0,1,0} + N_{0,0,1} + XXX \\ &= N_{\mathrm{counted}} + N_{0,0,0}. \end{aligned}$$

Easiest start assumption is list independence, which means $\mathrm{Pr}(\mathrm{counted\ in\ } 1, 2, 3) = pqr$ etc. Then $2^3 - 1 = 7$ probabilities are modelled via 3 parameters.

I will develop a log-likelihood profile method, with $\ell_{\mathrm{prof}}(N)$, giving $\widehat{N}$ and also a full confidence curve $\mathrm{cc}(N)$ – and apply this for Guatemala lists.

The methodology works also for other submodels (and for more than three lists).

I'm building a FIC for N000, a Focused Information Criterion that sorts through candidate models and finds the best.

Two lists: N = N11 + N10 + N01 + how many more?

Multinomial setup, with $(N_{0,0}, N_{0,1}, N_{1,0}, N_{1,1})$ having sum N, and probabilities

$$p_{i,j} = \Pr(X = i, Y = j) \quad \text{for } i, j = 0, 1,$$

1-0 for counted and not-counted. Under list independence:

$$p_{0,0} = (1-p)(1-q), \quad p_{0,1} = (1-p)q, \quad p_{1,0} = p(1-q), \quad p_{1,1} = pq.$$

Two quantities aiming for the same pq:

$$\frac{N_{1,1}}{N} \quad \text{and} \quad \frac{N_{1,0} + N_{1,1}}{N} \frac{N_{0,1} + N_{1,1}}{N}.$$

Equating these gives the Petersen estimator (counting fish in Limfjorden, 1896):

$$N^* = \frac{(N_{1,0} + N_{1,1})(N_{0,1} + N_{1,1})}{N_{1,1}} = \frac{N_{1,\cdot} N_{\cdot,1}}{N_{1,1}}.$$

Behaviour of N^*

May work with the four multinomial ratios $\widehat{p}_{i,j} = N_{i,j}/N$:

$$N^{1/2}(\widehat{p}_{i,j} - p_{i,j}) \to_d A_{i,j},$$

a mean-zero four-normal with a clear covariance matrix.
Delta method yields:

$$\frac{N^* - N}{\sqrt{N}} = N^{1/2}(\frac{N^*}{N} - 1) \to_d U = \frac{A_{1,0} + A_{1,1}}{p} + \frac{A_{0,1} + A_{1,1}}{q} - \frac{A_{1,1}}{pq}.$$

We learn

$$N^*/\sqrt{N} - \sqrt{N} \approx_d \mathrm{N}(0, \tau^2), \quad \tau^2 = \frac{(1-p)(1-q)}{pq}.$$

Can construct confidence intervals etc. using this.

Note that p, q small implies high uncertainty (& vice versa).

Via log-likelihood profiling

It's fruitful to work with log-likelihood and profiling: results will be
(a) it gives $\widehat{N}$ almost equivalent to Petersen estimator N^*;
(b) there is a useful χ^2_1 recipe;
(c) matters generalise to $k \geq 3$ lists (where $\nexists$ Petersen).

With $N_{0,1}, N_{1,0}, N_{1,1}$ and hence $S = N_{0,1} + N_{1,0} + N_{1,1}$ observed, but $N = S + N_{0,0}$ unknown:

$$L(N,p,q) = \frac{N!}{(N-S)!\, N_{1,0}!\, N_{0,1}!\, N_{1,1}!}\{(1-p)(1-q)\}^{N-S} \\ \{(1-p)q\}^{N_{0,1}}\{p(1-q)\}^{N_{1,0}}(pq)^{N_{1,1}}.$$

Taking log, and maximising over p, q:

$$\ell_{\rm prof}(N) = \log(N!) - \log((N-S)!) + NH(\widehat{p}_N) + NH(\widehat{q}_n),$$

in terms of $\widehat{p}_N = N_{1,\cdot}/N$ and $\widehat{q}_N = N_{\cdot,1}/N$, and

$$H(r) = r\log r + (1-r)\log(1-r).$$

A chi-squared theorem for two independent lists

Some analysis, involving approximations, limiting normality, information, etc., and the quantity

$$J = \frac{1 - p_{0,0}}{p_{0,0}} - \frac{p}{1-p} - \frac{q}{1-q} = \frac{pq}{(1-p)(1-q)},$$

leads to

$$D(N_0) = 2\{\ell_{\mathrm{prof,max}} - \ell_{\mathrm{prof}}(N_0)\} \to_d U^2/J \sim \chi^2_1$$

at the true (but still unknown) N_0.

Confidence interval: $\{N_0 \colon D(N_0) \le 1.96^2\}$, etc.

Full confidence curve:

$$\mathrm{cc}(N_0) = \Gamma_1(D(N_0)),$$

with $\Gamma_1(\cdot)$ the χ^2_1 c.d.f.

Srebrenica 1995

From Brunborg, Lyngstad, Urdal (2003): ICRC and PHR lists:
$N_{1,1} = 5712$, $N_{1,0} = 1586$, $N_{0,1} = 192$.
Estimate 7543; interval $[7528, 7560]$; $\widehat{p} = 0.967$, $\widehat{q} = 0.783$.

Three lists

First: Assuming list independence:

$$p_{i,j,k} = p_{i,\cdot,\cdot}\, p_{\cdot,j,\cdot}\, p_{\cdot,\cdot,k} \quad \text{for } i,j,k = 0,1.$$

No clear generalisation of the Petersen estimator.
But log-likelihood profiling works well:

$$\ell_{\rm prof}(N) = \log(N!) - \log((N-S)!) + N\{H(\widehat{p}_N) + H(\widehat{q}_n) + H(\widehat{r}_N)\},$$

with the same $H(x) = x\log x + (1-x)\log(1-x)$ and

$$\widehat{p}_N = N_{1,\cdot,\cdot}/N, \quad \widehat{q}_N = N_{\cdot,1,\cdot}/N, \quad \widehat{r}_N = N_{\cdot,\cdot,1}/N.$$

Also, a crucial quantity

$$J = \frac{1-p_{0,0,0}}{p_{0,0,0}} - \frac{p}{1-p} - \frac{q}{1-q} - \frac{r}{1-r}$$

is at work. Theorem:

$$D(N_0) = 2\{\ell_{\rm prof,max} - \ell_{\rm prof}(N_0)\} \to_d U^2/J \sim \chi^2_1$$

at the true (but still unknown) N_0.

Fun to do: simulate, estimate, learn

Your fish population: $\{1, \ldots, N\}$.

Go fishing, with mark-release, probabilities p_1, p_2, p_3. This gives subsets A_1, A_2, A_3. Then do all of the above, with quite simple `R` tools

```
setdiff
intersect
union
length
```

One learns about the importance of p_1, p_2, p_3, the value of fishing even more (!), the somewhat skewed distributions of $\widehat{N}$, etc.

May also put priors into the game.

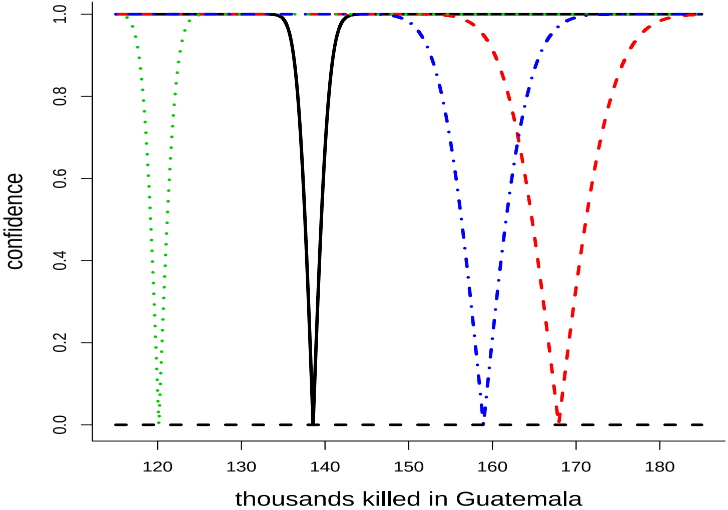
Guatemala

From Lum, Price, Banks (2013): Lists REMHI, CEH, CIIDH:

$n_{1,1,1} = 393$, $n_{1,1,0} = 3943$, $n_{1,0,0} = 15955$, $n_{1,0,1} = 634$, $n_{0,1,1} = 898$, $n_{0,1,0} = 19663$, $n_{0,0,1} = 6317$.

Using list independence (first): total estimate 138,576; 95 percent interval 135,794 to 141,453; low detection rates $(\widehat{p}, \widehat{q}, \widehat{r}) = (0.151, 0.179, 0.069)$.

Can do two lists at a time and the three lists jointly (looking for biases?).



With list independence assumption: Three two-sources curves, three-sources $\mathrm{cc}(N)$ in the middle.

With dependence among the lists

The log-likelihood profile machinery still works, for any $p_{i,j,k}(\theta)$; need $\dim(\theta) \le 6$. A class of four-parameter models:

$$
\begin{aligned}
p_{0,0,0} &= (1-p)(1-q)(1-r)/s \\
p_{0,0,1} &= (1-p)(1-q)r\,\gamma/s \\
p_{0,1,0} &= (1-p)q(1-r)/s \\
p_{0,1,1} &= (1-p)qr/s \\
p_{1,0,0} &= p(1-q)(1-r)/s \\
p_{1,0,1} &= p(1-q)/s \\
p_{1,1,0} &= pq(1-r)/s \\
p_{1,1,1} &= pqr/s
\end{aligned}
$$

where the γ is a parameter associated with cell 001, modifying independence in that direction; s is the factor to give sum 1.

This is the best of 8 similar choices. Then a clear leap in log-likelihood, and much better Pearson statistic

$$K = \sum_{i,j,k} (N_{i,j,k} - \widehat{N}\widehat{p}_{i,j,k})^2 / (\widehat{N}\widehat{p}_{i,j,k}).$$

A five-parameter model

Starting with independence equations, then modifying, in two directions:

$$
\begin{aligned}
p_{0,0,0} &= (1-p)(1-q)(1-r)/s \\
p_{0,0,1} &= (1-p)(1-q)r\,\gamma_1/s \\
p_{0,1,0} &= (1-p)q(1-r)/s \\
p_{0,1,1} &= (1-p)qr/s \\
p_{1,0,0} &= p(1-q)(1-r)/s \\
p_{1,0,1} &= p(1-q)r/s \\
p_{1,1,0} &= pq(1-r)/s \\
p_{1,1,1} &= pqr\,\gamma_2/s
\end{aligned}
$$

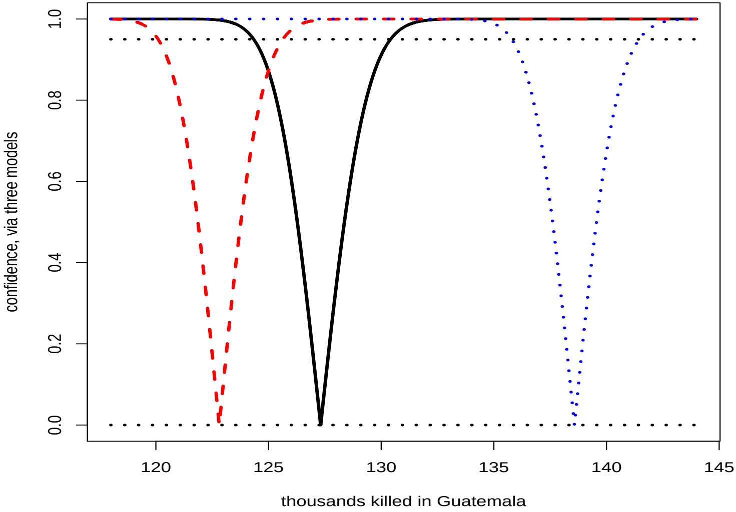
with s scale to get sum $p_{0,0,0} + \cdots + p_{1,1,1} = 1$.

The best cell for modification 1, with γ_1, is 001; and the best cell for modification 2, with γ_2, is 111.

So with modification parameters γ_1 placed at cell 001 and γ_2 placed at cell 111, I have a quite good model, with data fitting the model well (when it comes to the seven observed cells in the Venn diagram; can never check the 000 box).

The modifications amount to upward pushes at these two cells, with $\widehat{\gamma}_1 = 1.85$ and $\widehat{\gamma}_2 = 2.32$.

	obs3	obs5	expect3	expect5	pearson3	pearson5
n000	90772	79522	90772.223	79521.883	-0.001	0.000
n100	15955	15955	16144.571	15905.047	-1.492	0.396
n010	19663	19663	19880.329	19713.270	-1.541	-0.358
n001	6317	6317	5740.255	6317.001	7.612	0.000
n110	3943	3943	3535.877	3942.820	6.847	0.003
n101	634	634	1020.951	684.094	-12.110	-1.915
n011	898	898	1257.193	847.890	-10.130	1.721
n111	393	393	223.602	392.995	11.328	0.000



3-para: 138,576, with 135,794 to 141,453 (width 5,659)
4-para: 122,812, with 120,100 to 125,634 (width 5,534)
5-para: 127,314, with 124,341 to 130,415 (width 6,074)
Ball (1999): 132,174 (with a standard error of 6,568 ?).

Things To Do: bigger models, more sources

Looking for biases.

Inventing and using other models for the

$$p_{i,j,k}(\theta) = \Pr(X = i, Y = j, Z = k) \quad \text{for } i, j, k = 0, 1.$$

As long as $2^3 - 1 = 7$ probabilities in terms of θ of dimension 6 or lower, we're in business and can do log-likelihood profiling etc. Can search systematically (or 'logically') through

$$p_{i,j,k}(\theta) = p^{\rm ind}_{i,j,k} \exp(d_1 e_{i,j,k} + d_2 f_{i,j,k})/\mathrm{sum}.$$

Insights $\Longrightarrow$ covariates, or priors; will be helpful.

Yes, we can attack situations with $k \geq 4$ lists, but then need more care, for both modelling; principles giving shorter lists of candidate models; and clever algorithms for identifying and travelling through the most important ones.

Bayesian versions.

Comparing models: Building a FIC for N

Consider a candidate model, $p_{i,j,k}(\theta)$ for $2^3 = 8$ probabilities. With $S = \sum_{\text{not }(0,0,0)} N_{i,j,k}$ the sum of 7 observed counts,

$$(N_{1,0,0}, \ldots, N_{1,1,1}) \,|\, S \sim \mathrm{multin}(S, q_{1,0,0}(\theta), \ldots, q_{1,1,1}(\theta)),$$

with

$$q_{i,j,k}(\theta) = p_{i,j,k}(\theta)/\{1 - p_{0,0,0}(\theta)\}$$

for the 7 cells. I estimate θ from this, then giving

$$\widehat{N} = \frac{S}{1 - p_{0,0,0}(\widehat{\theta})}.$$

How to sift through and rank different candidate models?

For each candidate model, need to assess, approximate, estimate

$$\mathrm{E}_{\text{true model}}(\widehat{N}/N_{\text{true}} - 1)^2.$$

This is a tall order, needing a list of clarifying lemmas and proofs.

Two of the required lemmas: Assume there is an underlying true multinomial $p_{i,j,k}$ for the 8 probabilities. First,

$$\frac{\widehat{N}}{N_{\rm true}} = \frac{S/N_{\rm true}}{1 - p_{0,0,0}(\widehat{\theta})} \rightarrow_{\rm pr} \gamma = \frac{1 - p_{0,0,0}}{1 - p_{0,0,0}(\theta_0)},$$

involving the least false θ_0 for the parametric model. Second,

$$N_{\rm true}^{1/2}(\widehat{N}/N_{\rm true} - \gamma) \rightarrow_d {\rm N}(0, \tau^2),$$

with a long and semi-complicated formula for τ^2, involving both the true model and the candidate model.

From these:

$${\rm E}_{\rm true}\,(\widehat{N}/N_{\rm true} - 1) \doteq \gamma - 1, \quad {\rm Var}_{\rm true}\,(\widehat{N}/N_{\rm true} - 1) \doteq \tau^2/N_{\rm true},$$

so need to estimate

$${\rm mse} = {\rm E}_{\rm true}\,(\widehat{N}/N_{\rm true} - 1)^2 \doteq (\gamma - 1)^2 + \tau^2/N_{\rm true}.$$

This is somewhat complicated but doable: ${\rm fic} = \widehat{\rm mse}$.

Carrying out these things, and implementing FIC, requires selecting a wide model for the 8 probabilities $p_{i,j,k}$, deemed plausible, under which biases and variances can be quantified and estimated.

For Guatemala: I've carried model fitting and N estimation for $1 + 8 + 28 = 37$ candidate models, using a plausible 5-parametric model as the wide model.

The winning model ... is this five-parameter model:

$$
\begin{aligned}
p_{0,0,0} &= (1-p)(1-q)(1-r)/s \\
p_{0,0,1} &= (1-p)(1-q)r\,\gamma_1/s \\
p_{0,1,0} &= (1-p)q(1-r)/s \\
p_{0,1,1} &= (1-p)qr/s \\
p_{1,0,0} &= p(1-q)(1-r)/s \\
p_{1,0,1} &= p(1-q)r/s \\
p_{1,1,0} &= pq(1-r)/s \\
p_{1,1,1} &= pqr\,\gamma_2/s
\end{aligned}
$$

with s scale to get sum $p_{0,0,0} + \cdots + p_{1,1,1} = 1$.

$\widehat{N} =$127,314, with 124,341 to 130,415 (width 6,074).

Concluding remarks

- Constructing the FIC for N is a more ... delicate operation than for most other contexts and setups, as a plausible wide model needs to be put up.
- There are applications of 'how many deads' with e.g. 20 lists, i.e. 2^{20} probabilities to be modelled (see reports from Patrick Ball).
- Many other application domains: from multinomial $(N_0, N_1, \ldots, N_k)$, we only observe $N_1, \ldots, N_k$, and need to estimate N_0 – counting the not counted.
- Bayesian versions may be developed (both for estimation and for model selection), e.g. with a start prior for $p_{0,0,0}$ or $N_{0,0,0}$.
- There is a need for extension to setups with covariates.

(Some) references

P Ball. Lots of papers and reports.

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