

Optimal control of SPDEs driven by the Brownian sheet

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Dedicated to the memory of Yuri Kondratiev

Abstract

In this paper we study a Pontryagin type stochastic maximum principle for the optimal control of a system, where the state dynamics satisfy a stochastic partial differential equation (SPDE) driven by a two-parameter (time-space) Brownian motion (also called Brownian sheet). We first discuss some properties of a Brownian sheet driven linear SPDE which models the growth of an ecosystem.

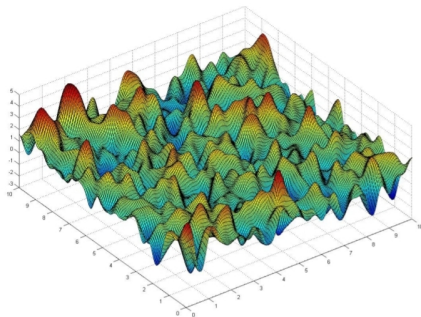
Further, applying time-space white noise calculus we derive sufficient conditions and necessary conditions of optimality of the control.

Finally, we illustrate our results by solving a linear quadratic control problem and an optimal harvesting problem in the plane. We also study possible applications to machine learning.

1.Introduction

The purpose of this paper is to study optimal control of systems driven by the Brownian sheet.

Throughout this work, we denote by $\{B(t, x) : t \geq 0, x \in \mathbb{R}\}$ a Brownian sheet and $(\Omega, \mathcal{F}, P)$ a complete probability space on which we define the (completed) σ -field $\mathcal{F}_{t,x}$ generated by $B(s, a), s \leq t, a \leq x$.



Wong & Zakai [WZ] generalised the notion of stochastic integrals with respect to 1-parameter Brownian motion to stochastic integrals driven by the two-parameter Brownian sheet.

Let us denote by $\mathbb{R}_+^2$ the positive quadrant of the plane and let $z \in \mathbb{R}_+^2$. We define a first type stochastic integral with respect to the two-parameter Brownian motion in Cairoli [C72] denoted by:

$$\int_{R_z} \phi(\zeta) B(d\zeta)$$

and a second type [WZ74] stochastic integral denoted by

$$\int_{R_z} \int_{R_z} \psi(\zeta, \zeta') B(d\zeta) B(d\zeta'),$$

where $R_z = [0, t] \times [0, x]$ if $z = (t, x)$.

In Wong & Zakai [WZ], an Itô formula for stochastic integrals in the plane is given.

3 motivating examples

Example

(An optimal harvesting problem)

A classical model for the growth of an ecosystem (e.g. a population or a forest) with value $Y(t)$ at time t in a random environment is the geometric Brownian motion, defined by the Itô stochastic differential equation (SDE)

$$dY(t) = \kappa Y(t)dt + \gamma Y(t)dB(t), \quad t \geq 0, \quad Y(0) > 0,$$

where κ and γ are given constants. Equivalently, in terms of white noise $\dot{B}$ and Wick product $\diamond$, the equation can be written

$$\frac{d}{dt}Y(t) = \kappa Y(t) + \gamma Y(t) \diamond \dot{B}(t), \quad Y(0) > 0,$$

where

$$\dot{B}(t) = \frac{d}{dt}B(t) \text{ is (time) white noise.}$$

A natural extension of this model to the case where the noise of the environment depends on both time t and position x , is the following SPDE in the value $Y(t, x)$ of the ecosystem at time t and position x :

$$(1.1) \quad \frac{\partial^2}{\partial t \partial x} Y(t, x) = \alpha_0(t, x) Y(t, x) + \beta_0(t, x) Y(t, x) \diamond \dot{B}(t, x), \quad Y(0, 0) > 0,$$

where $\alpha_0(t, x)$ and $\beta_0(t, x)$ are given bounded deterministic functions, and

$$(1.2) \quad \dot{B}(t, x) = \frac{\partial^2}{\partial t \partial x} B(t, x) \text{ is time-space white noise.}$$

Alternatively, if we use the following result:

$$(1.3) \quad \int_{R_z} \varphi(s, a) B(ds, da) = \int_{R_z} \varphi(s, a) \diamond \dot{B}(s, a) ds da, \quad \forall \varphi, t, x.$$

(see e.g. Holden et al [HØUZ]), we can rewrite the equation (1.1) as:

$$(1.4) \quad Y(t, x) = Y(0, 0) + \int_{R_z} \alpha_0(\zeta) Y(\zeta) d\zeta + \int_{R_z} \beta_0(\zeta) Y(\zeta) B(d\zeta).$$

Assume for simplicity that α_0 and β_0 are constants.

If we at (t, x) harvest from $Y(t, x)$ at the rate $u(t, x)$, the dynamics (1.1) becomes

$$\frac{\partial^2}{\partial t \partial x} Y_u(t, x) = \alpha_0 Y_u(t, x) - u(t, x) + \beta_0 Y_u(t, x) \diamond \dot{B}(t, x),$$

or, in integral form,

$$\begin{aligned} Y_u(t, x) = & Y(0, 0) + \int_0^t \int_0^x \{ \alpha_0 Y_u(s, a) - u(s, a) \} ds da \\ & + \int_0^t \int_0^x \beta_0 Y_u(s, a) B(ds, da). \end{aligned}$$

For given utility functions U_1, U_2 and given constants $T > 0, X > 0$ define the combined utility of the harvesting and the terminal population by

$$J(u) = E \left[\int_0^T \int_0^X U_1(u(s, a)) ds da + U_2(Y_u(T, X)) \right].$$

We want to find the harvesting strategy $u^*(s, x)$ which maximizes the utility of the harvest, i.e.

$$J(u^*) = \sup_{u \in \mathcal{A}} J(u).$$

Here the set of admissible controls is denoted by $\mathcal{A}$.

Example

(A linear-quadratic (LQ) problem)

Consider the following linear-quadratic (LQ) control problem for time-space random fields:

Suppose the state $Y(t, x)$ is given by

$$Y(t, x) = Y(0, 0) + \int_0^t \int_0^x u(s, a) ds da + \beta B(t, x), \quad t \geq 0, x \in \mathbb{R}.$$

We want to drive the state Y to 0 at time-space (T, X) with minimal use of energy. Hence we put

$$J(u) = -\frac{1}{2}E\left[\int_0^T \int_0^X u^2(s, a) ds da + \theta Y^2(T, X)\right],$$

where $\theta > 0$ is a given constant.

The problem is to find $u^* \in \mathcal{A}$ such that

$$(1.5) \quad J(u^*) = \sup_{u \in \mathcal{A}} J(u).$$

Example

(A machine learning problem)

Consider the following hyperbolic SPDE:

(1.6)

$$Y(t, x) = y - \int_0^t \int_0^x u(s, a) \nabla f(Y(s, a)) ds da + \sigma B(t, x), y \in \mathbb{R}^d, t, x \geq 0,$$

where B is a Brownian sheet in $\mathbb{R}^d$, $\sigma \in \mathbb{R}^{d \times d}$ and ∇f is the gradient of a function $f \in C^1(\mathbb{R}^d; \mathbb{R})$. Further, $u : \Omega \times [0, \infty)^2 \rightarrow [0, \infty)$ is a stochastic learning rate in time *and* space, which is assumed to be an adapted random field. By formally setting $u = \eta \delta_x$ in (1.6) for the Dirac delta function δ_x in a fixed point x and $\eta \geq 0$ we get (as a special case of (1.6) the SDE

$$(1.7) \quad dY_t = -\eta \nabla f(Y_t) dt + \sigma dB_t, Y_0 = x \in \mathbb{R}^d, t \geq 0.$$

The latter type of SDE (1.7) is used in machine learning in connection with the stochastic gradient descent method (SGD) to minimize or maximize the objective or loss function f . Since the dynamics (1.6) is more general than that of (1.7), one may replace for the sake of a deeper understanding of the classical SGD approach (at the possible expense of numerical tractability) the equation (1.7) by (1.6) and study an optimal control problem with respect to the (time-space) stochastic learning rate u .

In order to illustrate this application in a simplified framework, we approximate ∇f (for smooth f) by Taylor's expansion in the case of $d = 1$ and consider in this paper the controlled process

$$Y_u(t, x) = Y(0, 0) - \int_0^t \int_0^x u(s, a) Y_u(s, a) ds da \\ + \beta_0 B(t, x), Y(0, 0), \beta_0 \in \mathbb{R}, t, x \geq 0$$

with respect to (in this context) natural performance functional

$$J(u) = -E \left[\int_0^T \int_0^X u^2(s, a) ds da + \theta Y^2(T, X) \right]$$

for $\theta > 0$.

A general formulation

The examples mentioned above are special cases of the following general optimal stochastic control problem:

We study optimal control of solutions $Y(t, x)$, $t \geq 0$, $x \in \mathbb{R}$ of SPDEs of the form

(1.8)

$$Y_u(t, x) = Y(t_0, x_0) + \int_{R(t, x)} \alpha_u(Y_u(s, a)) ds da + \int_{R(t, x)} \beta_u(Y_u(s, a)) B(ds, da),$$

where

$$R(t, x) = R^{(t_0, x_0)}(t, x) = [t_0, t] \times [x_0, x], t \geq t_0, x \geq x_0,$$

and

$B(t, x)$ = Brownian sheet.

The differential form of (1.8) is

$$(1.9) \quad \frac{\partial^2}{\partial t \partial x} Y_u(t, x) = \alpha_u(Y_u(t, x)) + \beta_u(Y_u(t, x)) \diamond \dot{B}(t, x).$$

The identity of (1.8) and (1.9) comes from the fact that

$$(1.10) \quad \int_{R(t,x)} \varphi(s, a) B(ds, da) = \int_{R(t,x)} \varphi(s, a) \diamond \dot{B}(s, a) ds da, \quad \forall \varphi, t, x.$$

See e.g. Holden et al [HØUZ].

Remark

Hyperbolic SPDEs of the type (1.1) have been studied over the years by several authors. See e.g. Cairoli [C72] and Yeh [Y], who established strong existence and pathwise uniqueness of solutions Y of the equation

$$(1.11) \quad \begin{aligned} Y(t, x) = & y_0 + \int_{R(t, x)} b(s, a, Y(s, a)) ds da \\ & + \int_{R(t, x)} \sigma(s, a, Y(s, a)) B(ds, da), y_0 \in \mathbb{R}^d, \end{aligned}$$

when b and σ are Lipschitz continuous vector fields of linear growth.

Further, smoothness of solutions to (1.11) in the sense of Malliavin differentiability for sufficiently regular b and σ was analysed in Nualart & Sanz [NS]. See also Bogso et al [BDMPP], where the authors construct Malliavin differentiable unique solutions to (1.11), when the drift vector field b is merely bounded and measurable and σ is given by the unit matrix. As for other works in this direction (in the case of both weak and strong solutions), we also refer to Yeh [Y87] and [Y85].

In addition we refer to the book of Nualart [N].

Finally, we want to point out the interesting link between hyperbolic SPDEs and non-linear (random) wave equations, when $d = 1$ and $b, \sigma : \mathbb{R} \mapsto \mathbb{R}$:

By applying the orthogonal transformation $u = x + t, v = x - t$ to the SPDE (1.9) we see that the corresponding differential version of equation (1.11) can be transformed into the following non-linear stochastic wave equation:

$$\frac{\partial^2}{\partial t^2} Y(t, x) - \frac{\partial^2}{\partial x^2} Y(t, x) = \sigma(Y(t, x)) \frac{\partial^2}{\partial t \partial x} \tilde{B}(t, x) + b(Y(t, x)),$$

where $\tilde{B}$ is another Brownian sheet, obtained by applying the inverse orthogonal transformation to B . See e.g. Walsh [W] for further details.

In the sequel we assume that the performance functional $J_u(t_0, x_0)$ has the form

$$(1.12) \quad J_u(t_0, x_0) = E^{(t_0, x_0)} \left[\int_{t_0}^T \int_{x_0}^X f_u(Y_u(s, a)) ds da + g(Y_u(T, X)) \right],$$

and consider the following problem:

Problem

Let $\mathcal{A}$ be a given family of admissible controls. Find an optimal control $u^* \in \mathcal{A}$ and the value function Φ such that

$$(1.13) \quad \Phi(t_0, x_0) = J_{u^*}(t_0, x_0) = \sup_{u \in \mathcal{A}} J_u(t_0, x_0).$$

The control u^* is called an optimal control and the function Φ is referred to as the value function of this problem.

We handle those problems by using a maximum principle approach. Therefore we need to study the adjoint equation given by a BSPDE in the plane. The existence and the uniqueness of solutions to such a BSPDE were proven in Zaidi & Nualart [ZN] for a particular Lipschitz constant.

Here is an outline of the rest of the talk:

- ▶ In the next section we check properties of the solution of SPDE.
- ▶ In Section 3 we introduce some background about stochastic calculus of time-space white noise.
- ▶ In Section 4 we prove two types of maximum principles, a sufficient (verification theorem) and a necessary maximum principle.
- ▶ Finally, in Section 5 we apply the results of Section 4 to study the problems mentioned above.

A discussion of the solution of the SPDE

It is natural to ask what are the properties the solution $Y(t, x)$ of the equation

(2.1)

$$Y(t, x) = Y(0, 0) + \int_{R_z} \alpha_0(\zeta) Y(\zeta) d\zeta + \int_{R_z} \beta_0(\zeta) Y(\zeta) B(d\zeta).$$

For example, is $Y(t, x)$ always positive?

To study this question, let us suppose that the coefficients α_0, β_0 are bounded deterministic functions. Then it follows from (1.1) that

$$\widetilde{Y(t, x)}(z) = Y(0, 0) + \int_{R_{t, x}} \widetilde{K(s, a)}(z) \widetilde{Y(s, v)}(z) ds da,$$

where $\widetilde{Y(t, x)}(z)$ denotes the Hermite transform of $Y(t, x)$ for $z \in (\mathbb{C}^{\mathbb{N}})_c$ (set of all finite sequences in $\mathbb{C}^{\mathbb{N}}$) and where $R_{(t, x)} = [0, t] \times [0, x]$. Further,

$$K(s, a) = \alpha_0(s, a) + \beta_0(s, a) \dot{B}(s, a),$$

where

$$(2.2) \quad \dot{B}(s, a) = \frac{\partial^2}{\partial s \partial a} B(s, a) \text{ is white noise.}$$

See [HØUZ] for the properties of the Hermite transform.

Then, using Picard iteration, we find with $y_0 = Y(0, 0) > 0$, the semi-explicit solution

$$(2.3) \quad \widetilde{Y(t, x)}(z) = y_0 \sum_{n=0}^{\infty} \int_{R(t, x)} \int_{R(s_1, a_1)} \dots \int_{R(s_{n-1}, a_{n-1})} \prod_{j=1}^n \widetilde{K(s_j, a_j)}(z) ds_n da_n \dots ds_1 da_1,$$

for $z \in (\mathbb{C}^{\mathbb{N}})_c$, where $s_1 > \dots > s_n$, $a_1 > \dots > a_n$.

By Theorem 2.11.4 in Holden et al [HØUZ] $Y(t, x)$ is positive if and only if for all m the function

$$(2.4) \quad g(y) := \widetilde{Y(t, x)}(iy) e^{-\frac{1}{2}|y|^2}; \quad i = \sqrt{-1}, y = (y_1, y_2, \dots, y_m) \in \mathbb{R}^m$$

is positive definite.

After some computation we get the following conclusion:

Theorem

The solution $Y(t, x)$ of the equation (23) is not always positive

Remark

Using a different method, we mention that the latter result was obtained in Nualart [N87] in the special case of $\alpha_0 = 0$ and β_0 a constant. There Nualart proves that $Y(t, x) < 0$ uniformly in (t, x) on an open set, with positive probability.

Background

To study such optimal control problems, we will use a version of the Itô formula for such systems. First we introduce some notation from Wang & Zakai [WZ].

- ▶ We put $\zeta = (\zeta_1, \zeta_2) = (s, a) \in \mathbb{R} \times \mathbb{R}$ and $d\zeta = d\zeta_1 d\zeta_2 = ds da$,
- ▶ $B(t, x)$ is a Brownian sheet; $t \geq 0, x \in \mathbb{R}$,
- ▶ $z = (z_1, z_2) = (t, x), R_z = [0, z_1] \times [0, z_2]$,
- ▶ $\int_{R_z} \varphi(\zeta) B(d\zeta)$ denotes the Itô integral with respect to $B(\cdot)$ over R_z ,
- ▶ $\int_{R_z} \psi(\zeta) d\zeta$ is 2-dimensional Lebesgue integral of ψ ,
- ▶ if $a = (a_1, a_2), b = (b_1, b_2)$, then $a \vee b = (\max(a_1, b_1), \max(a_2, b_2))$.
- ▶ $I((a_1, a_2) \bar{\wedge} (b_1, b_2)) = 1$
if $a_1 \leq b_1$ & $a_2 \geq b_2$, 0 otherwise.

Theorem (Itô formula, Wang & Zakai [WZ])

Suppose

(3.1)

$$Y(z) = Y_0 + \int_{R_z} \alpha(\zeta) d\zeta + \int_{R_z} \beta(\zeta) B(d\zeta) + \iint_{R_z \times R_z} \psi(\zeta, \zeta') B(d\zeta) B(d\zeta').$$

Then, if $f : \mathbb{R} \rightarrow \mathbb{R}$ is smooth, we have

$$\begin{aligned}
f(Y(z)) &= f(Y_0) + \int_{R_z} f'(Y(\zeta))[\alpha(\zeta)d\zeta + \beta(\zeta)B(d\zeta)] \\
&+ \frac{1}{2} \int_{R_z} f''(Y(\zeta))\beta^2(\zeta)d\zeta \\
&+ \iint_{R_z \times R_z} \left\{ f''(Y(\zeta \vee \zeta'))u\tilde{u} + f'(Y(\zeta \vee \zeta'))\psi(\zeta, \zeta') \right\} B(d\zeta)B(d\zeta') \\
&+ \iint_{R_z \times R_z} \left\{ f''(Y(\zeta \vee \zeta')) \left(u\alpha(\zeta) + \psi(\zeta, \zeta')\tilde{u} \right) \right. \\
&+ \left. \frac{1}{2}f^{(3)}(Y(\zeta \vee \zeta'))u^2\tilde{u} \right\} d\zeta B(d\zeta') \\
&+ \iint_{R_z \times R_z} \left\{ f''(Y(\zeta \vee \zeta')) \left(\tilde{u}\alpha(\zeta') + \psi(\zeta, \zeta')u \right) \right. \\
&+ \left. \frac{1}{2}f^{(3)}(Y(\zeta \vee \zeta'))u^2\tilde{u} \right\} d\zeta B(d\zeta')
\end{aligned}$$

$$\begin{aligned}
& + \iint_{R_z \times R_z} I(\zeta \bar{\lambda} \zeta') \left\{ f''(Y(\zeta \vee \zeta')) \left(\alpha(\zeta') \alpha(\zeta) + \frac{1}{2} \psi^2(\zeta, \zeta') \right) \right. \\
& + f^{(3)}(Y(\zeta \vee \zeta')) u \tilde{u} \psi(\zeta, \zeta') + \frac{1}{2} f^{(3)}(Y(\zeta \vee \zeta')) [\alpha(\zeta') \tilde{u}^2 + \alpha(\zeta) u^2] \\
& \left. + \frac{1}{4} f^{(4)}(Y(\zeta \vee \zeta')) u^2 \tilde{u}^2 \right\} d\zeta d\zeta',
\end{aligned}$$

where

$$\begin{aligned}
u &= u(\zeta') = \beta(\zeta') + \int_{R_z} I(\zeta \bar{\lambda} \zeta') \psi(\zeta, \zeta') B(d\zeta), \\
\tilde{u} &= \tilde{u}(\zeta) = \beta(\zeta) + \int_{R_z} I(\zeta \bar{\lambda} \zeta') \psi(\zeta, \zeta') B(d\zeta').
\end{aligned}$$

It is proved in [WZ] that the double $B(d\zeta)B(d\zeta')$ -integrals, and the mixed $d\zeta B(d\zeta')$ and $B(d\zeta)d\zeta'$ -integrals are all weak martingales and hence have expectation 0. Therefore, by the Itô formula above we get the following:

Theorem
(Dynkin formula)

$$\begin{aligned} E[f(Y(z))] &= f(Y_0) + E\left[\int_{R_z} \left\{ \alpha(\zeta)f'(Y(\zeta)) + \frac{1}{2}\beta^2(\zeta)f''(Y(\zeta)) \right\} d\zeta \right. \\ &\quad + \iint_{R_z \times R_z} I(\zeta \bar{\wedge} \zeta') \left\{ f''(Y(\zeta \vee \zeta')) \left(\alpha(\zeta')\alpha(\zeta) + \frac{1}{2}\psi^2(\zeta, \zeta') \right) \right. \\ &\quad + f^{(3)}(Y(\zeta \vee \zeta')) u \tilde{u} \psi(\zeta, \zeta') \\ &\quad + \frac{1}{2}f^{(3)}(Y(\zeta \vee \zeta')) [\alpha(\zeta')\tilde{u}^2 + \alpha(\zeta)u^2] \\ &\quad \left. \left. + \frac{1}{4}f^{(4)}(Y(\zeta \vee \zeta')) u^2 \tilde{u}^2 \right\} d\zeta d\zeta' \right]. \end{aligned}$$

Lemma (Integration by parts)

Suppose that

$$Y_k(z) = Y_k(0) + \int_{R_z} \alpha_k(\zeta) d\zeta + \int_{R_z} \beta_k(\zeta) B(d\zeta) + \iint_{R_z \times R_z} \psi_k(\zeta, \zeta') B(d\zeta) B(d\zeta').$$

Then

$$\begin{aligned} E[Y_1(z)Y_2(z)] &= Y_1(0)Y_2(0) \\ &+ E\left[\int_{R_z} \left\{Y_1(\zeta)\alpha_2(\zeta) + Y_2(\zeta)\alpha_1(\zeta) + \beta_1(\zeta)\beta_2(\zeta)\right\} d\zeta\right. \\ &\left.+ \iint_{R_z \times R_z} I(\zeta\bar{\wedge}\zeta') \left\{\alpha_1(\zeta')\alpha_2(\zeta) + \alpha_1(\zeta)\alpha_2(\zeta') + \psi_1(\zeta, \zeta')\psi_2(\zeta, \zeta')\right\} d\zeta d\zeta'\right]. \end{aligned}$$

Proof. The proof follows from the Itô formula, Proposition 5.1 in Wang & Zakai [WZ]. $\square$

BSPDEs in the plane

Let us recall now the representation of square integrable martingales.

Theorem (Wong & Zakai [WZ])

If $M = \{M(z), \mathcal{F}_z, z \in \mathbb{R}_+^2\}$ is square integrable martingale, then for each $z \in \mathbb{R}_+^2$

(4.1)

$$M(z) = M(0) + \int_{R_z} \phi(\zeta) B(d\zeta) + \iint_{R_z \times R_z} \psi(\zeta, \zeta') B(d\zeta) B(d\zeta'),$$

where ϕ, ψ are adapted processes.

Let $Z = (T, X)$ and fix a rectangle $R_Z = [0, T] \times [0, X]$, and let ξ be an $\mathcal{F}_Z$ -measurable random variable and $h(\omega, \zeta, p, q)$ is a $\mathcal{P} \times \mathcal{B}_{\mathbb{R}} \times \mathcal{B}_{\mathbb{R}}$ -measurable function such that $\int_{R_Z} |h(\zeta, p(\zeta), q(\zeta))| d\zeta < \infty$. Then we want to find a triple of processes $(p, q, r) \in L^2_{a,1} \times L^2_{a,1} \times L^2_{a,2}$, which is a solution of the BSPDE in the plane

$$\begin{aligned}
 (4.2) \quad p(z) = & \xi - \int_{R_Z \setminus R_z} h(\zeta, p(\zeta), q(\zeta)) d\zeta - \int_{R_Z \setminus R_z} q(\zeta) B(d\zeta) \\
 & - \int_{R_Z \setminus R_z} \int_{R_Z \setminus R_z} r(\zeta, \zeta') B(d\zeta) B(d\zeta').
 \end{aligned}$$

Alternatively, let us introduce the notation

$$(4.3) \quad M_r(z) = \iint_{R_z \times R_z} r(\zeta, \zeta') B(d\zeta) B(d\zeta').$$

Then $M_r(z)$ is a martingale, and we can write the equation for (p, q, r) above in differential form as follows

$$(4.4) \quad \begin{aligned} p(dz) &= h(z, p(z), q(z))dz + q(z)B(dz) + M_r(dz), \quad z \leq Z, \\ p(Z) &= \xi. \end{aligned}$$

Assumptions We impose the following set of assumptions:

- (i) $\xi \in L^2(\Omega, \mathcal{F}_{z_0}, P)$,
- (ii) $h(\cdot, p, q) \in L^2(dP \times dz)$ and predictable for all $p, q \in \mathbb{R}$,
- (ii) $|h(\zeta, p, q) - h(\zeta', p', q')|^2 \leq K_1|p - p'|^2 + K_2|q - q'|^2$, for all $p, q, p', q' \in \mathbb{R}$ and $\zeta \in R_Z$.

Let f_0 be the Bessel function of order zero and $r_0 \approx 1.4458$ be the first nonnegative zero of J_0 :

$$r_0 = \inf \left\{ t > 0 : f_0(2\sqrt{t}) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!^2} t^j = 0 \right\}.$$

Theorem (Existence and Uniqueness Zaidi & Nualart [ZN])

Under the above assumptions (i)-(iii) and if the Lipschitz constant satisfies $K_1|z_0| < \sqrt{r_0}$ and $K_2|z_0| < \sqrt{r_0}$, there exists a unique solution of the BSPDE (4.2).

Maximum principles

We introduce the following notation:

Definition

For general functions $h, k : [0, T] \times [0, X] \mapsto \mathbb{R}$, we define

$$(5.1) \quad (h \star k)(t, x) = \int_0^x \int_t^T \{h(t, x)k(s, a) + h(s, a)k(t, x)\} ds da$$

Note that with this notation we have

$$(5.2) \quad \iint_{R_z \times R_z} I(\zeta \bar{\alpha} \zeta') \left\{ \alpha_1(\zeta') \alpha_2(\zeta) + \alpha_1(\zeta) \alpha_2(\zeta') \right\} d\zeta d\zeta' = \int_{R_z} (\alpha_1 \star \alpha_2)(\zeta) d\zeta,$$

and the integration by parts formula can be written

Lemma (Integration by parts (2))

Suppose that

$$Y_k(z) = Y_k(0) + \int_{R_z} \alpha_k(\zeta) d\zeta + \int_{R_z} \beta_k(\zeta) B(d\zeta), \quad k = 1, 2.$$

Then

$$\begin{aligned} E[Y_1(z)Y_2(z)] &= Y_1(0)Y_2(0) \\ (5.3) \quad &+ E\left[\int_{R_z} \left\{ Y_1(\zeta)\alpha_2(\zeta) + Y_2(\zeta)\alpha_1(\zeta) + \beta_1(\zeta)\beta_2(\zeta) + (\alpha_1 \star \alpha_2)(\zeta) \right\} d\zeta\right]. \end{aligned}$$

Given a subset U of $\mathbb{R}$ we denote by $\mathcal{U}$ the set of all $\mathcal{F}_{t,x}$ -adapted control processes $u = \{u(t, x), t < T, x < X\}$ valued in U . We then define the set of admissible control processes $\mathcal{A} \subset \mathcal{U}$ to be the collection of all $\mathcal{F}_{t,x}$ -adapted processes with values in U . Let f and g be given functions and consider the performance functional

$$J(u) = E \left[\int_{R_Z} f(\zeta, Y(\zeta), u(\zeta)) d\zeta + g(Y(Z)) \right],$$

where

$$R_Z = [0, T] \times [0, X],$$

with $Z = (T, X)$ for some given $T > 0, X > 0$, and the state Y of the system is described by the equation

(5.4)

$$Y(z) = Y(t, x) = Y(0) + \int_{R_z} \alpha(\zeta, Y(\zeta), u(\zeta)) d\zeta + \int_{R_z} \beta(\zeta, Y(\zeta), u(\zeta)) B(d\zeta),$$

where $R_z = [0, t] \times [0, x]$ when $z = (t, x)$, and u denotes a control process.

Problem

We want to find $\hat{u} \in \mathcal{A}$ such that

$$(5.5) \quad J(\hat{u}) = \sup_{u \in \mathcal{A}} J(u).$$

The maximum principle approach to this problem, adapted to the time-space situation, is to introduce the following associated Hamiltonian

$$H : R_Z \times \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

given by

$$(5.6) \quad H(z, y, u, p, q, L) = f(z, y, u) + p\alpha(z, y, u) + q\beta(z, y, u) + (L \star \alpha)(z),$$

where we recall that

$$(L \star \alpha)(z) = \int_0^x \int_t^T \left\{ L(z) \alpha(\zeta', y, u) + L(\zeta') \alpha(z, y, u) \right\} d\zeta'; \quad z = (t, x)$$

The *adjoint processes*

$(p, q, r, L) = (p(t, x), q(t, x), r(t, x, \cdot), L(t, x))$ are given by the following equations:

- ▶ $L : R_Z \mapsto \mathbb{R}$ is defined implicitly by the integral equation

(5.7)

$$L(z) = -\frac{\partial H}{\partial y}(z, Y(z), u(z), p(z), q(z), r(z, \cdot), L(z)); \quad z = (t, x)$$

- ▶ and (p, q, r) is the solution of the BSPDE

(5.8)

$$\begin{cases} p(dz) = -\frac{\partial H}{\partial y}(z, Y(z), p(z), q(z), r(z, \cdot), L(z))dz \\ \quad + q(z)B(dz) + M_r(dz), & 0 \leq (t, x) \leq (T, X), \\ p(Z) = \frac{\partial g}{\partial y}(Y(Z)), \end{cases}$$

or, in integrated form,

(5.9)

$$\begin{cases} p(z) &= p(0) - \int_{R_z} \frac{\partial H}{\partial y}(\zeta, Y(\zeta), p(\zeta), q(\zeta), r(\zeta, \cdot), L(\zeta)) d\zeta \\ &+ \int_{R_z} q(\zeta) B(d\zeta) + \iint_{R_z \times R_z} r(\zeta, \zeta') B(d\zeta) B(d\zeta'), \quad z \leq Z. \\ p(Z) &= \frac{\partial g}{\partial y}(Y(Z)) \end{cases}$$

There are two versions of the maximum principle for this problem, namely the so-called *sufficient maximum principle* and the *necessary maximum principle*. We present them both below.

The sufficient maximum principle

Theorem (Sufficient maximum principle)

Suppose $\hat{u} \in \mathcal{A}$ with corresponding solutions $\hat{Y}, (\hat{p}, \hat{q}, \hat{r}, \hat{L})$ of the equations above. Moreover, suppose that $y \mapsto g(y)$ is concave and $(y, u) \mapsto H(z, y, u, p, q, r, L)$ is concave for all z, p, q, r, L and that

$$(5.10) \quad \sup_{v \in A} H(z, \hat{Y}(\cdot), v, \hat{p}(z), \hat{q}(z), \hat{r}(z, \cdot), \hat{L}(z))$$

$$(5.11) \quad = H(z, \hat{Y}(\cdot), \hat{u}(z), \hat{p}(\cdot)(z), \hat{q}(z), \hat{r}(z, \cdot), \hat{L}(z)),$$

Then $\hat{u}$ is an optimal control for problem (5.5).

The necessary maximum principle

It is a drawback of the sufficient maximum principle that we have to assume that $y \mapsto g(y)$ and $(y, u) \mapsto H(z, y, u, p, q, L)$ are concave. The following result does not need concavity, but we have to add conditions of the set $\mathcal{A}$ of admissible controls instead, as follows:

(A1) $\mathcal{A}$ is a convex set

(A2) For all $z_0 = (t_0, x_0) < Z = (T, X)$ and all bounded $\mathcal{F}_{z_0}$ -measurable random variables θ_{z_0} , the control

$$u_{z_0}(\zeta) = \theta_{z_0} 1_{R_{z_0}}(\zeta)$$

is admissible, where

$$(5.12) \quad 1_{R_{z_0}}(\zeta) = \begin{cases} 1 & \text{if } \zeta \in R_{z_0} \\ 0 & \text{if } \zeta \notin R_{z_0} \end{cases}$$

is the indicator function of the rectangle $R_{z_0} = [t_0, T] \times [x_0, X]$.

Lemma

For all $u, v \in \mathcal{A}$ the derivative process

$$G(\zeta) := \lim_{\epsilon \rightarrow 0} \frac{Y^{u+\epsilon v}(\zeta) - Y^u(\zeta)}{\epsilon}$$

satisfies the equation

$$\begin{aligned} G(z) = & G(0) + \int_{R_{z_0}} \left\{ \frac{\partial \alpha}{\partial y}(\zeta) G(\zeta) + \frac{\partial \alpha}{\partial u}(\zeta) v(\zeta) \right\} d\zeta \\ & + \int_{R_{z_0}} \left\{ \frac{\partial \beta}{\partial y}(\zeta) G(\zeta) + \frac{\partial \beta}{\partial u}(\zeta) v(\zeta) \right\} B(d\zeta), \end{aligned}$$

where $\frac{\partial \alpha}{\partial y}(\zeta) = \frac{\partial \alpha}{\partial y}(\zeta, Y^u(\zeta), u(\zeta))$ etc.

Proof. This follows by the chain rule. □

Lemma

For all $u, v \in \mathcal{A}$, we have

$$\frac{d}{d\epsilon} J(u + \epsilon v)_{\epsilon=0} = E \left[\int_{R_Z} \frac{\partial H}{\partial u}(\zeta) v(\zeta) d\zeta \right].$$

Theorem (Necessary maximum principle)

Suppose $\hat{u} \in \mathcal{A}$ is optimal for Problem 2.5. Then

$$\frac{\partial H}{\partial u}(\zeta, \hat{Y}(\zeta), u, \hat{p}(\zeta), \hat{q}(\zeta), \hat{L}(\zeta))|_{u=\hat{u}(\zeta)} = 0 \text{ for a.a. } \zeta.$$

Applications

Return to the optimal harvesting problem in the plane

Suppose that the growth of a population at time t and position x with density $Y(t, x)$ satisfies

$$\begin{aligned} Y_u(t, x) = & Y(0, 0) + \int_0^t \int_0^x \{ \alpha_0 Y_u(s, a) - u(s, a) \} ds da \\ & + \int_0^t \int_0^x \beta_0 Y_u(s, a) B(ds, da), \end{aligned}$$

where α_0, β_0 are given constants and $Y(0, 0) > 0$.

For given constants $T > 0, X > 0$ such that $T > t, X > x$, define the combined utility of the harvesting and the terminal population by

$$J(u) = E \left[\int_0^T \int_0^X \ln(u^2(s, a)) ds da + \theta Y_u(T, X) \right],$$

where θ is a given bounded, $\mathcal{F}_Z$ -measurable random variable.

Problem

We want to find the harvesting strategy $u^*(s, x)$ which maximizes the utility of the harvest, i.e. we want to find $u^* \in \mathcal{A}$ such that

$$J(u^*) = \sup_{u \in \mathcal{A}} J(u).$$

The associated Hamiltonian for this case is

$$H(t, x, y, u, p, q, L) = \ln(u^2) + (\alpha_0 y - u)p + \beta_0 yq + (L \star (\alpha_0 y - u)).$$

The Hamiltonian has a maximum at u given by the equation

$$\frac{\partial H}{\partial u} = \frac{2}{u(z)} - p - (L \star 1)(z) = 0.$$

Hence we must have

$$u = \frac{2}{p + (L \star 1)},$$

where

$$(6.1) \quad (L \star 1)(z) = x(T - t)L(z) + \int_0^x \int_t^T L(\zeta') d\zeta'.$$

Since

$$(6.2) \quad \frac{\partial H}{\partial y} = \alpha_0 p + \beta_0 q + (L \star \alpha_0)$$

the corresponding adjoint equation is

$$(6.3) \quad \begin{cases} p(dz) &= -(\alpha_0 p + \beta_0 q + L \star \alpha_0)(z) dz \\ &+ q(z) B(dz) + M_r(dz), \quad 0 \leq (t, x) \leq (T, X), \\ p(Z) &= \frac{\partial g}{\partial y}(Y(Z)). \end{cases}$$

The adjoint variable L is defined by

$$\begin{aligned} L(z) &= -\frac{\partial H}{\partial y}(z) = -[\alpha_0 p(z) + \beta_0 q(z) + (L \star 1)(z)] \\ (6.4) \quad &= -[\alpha_0 p(z) + \beta_0 q(z) + x(T-t)L(z) + \int_0^x \int_t^T L(\zeta') d\zeta', \end{aligned}$$

which is an integral equation for $L(z)$.

We summarise this as follows:

Theorem

Let $z > 0$ and assume that $p(z) + (L \star 1)(z) \neq 0$. Then the optimal harvesting rate u^* is given by

$$u^*(t, x) = u^*(z) = \frac{2}{p(z) + (L \star 1)(z)},$$

with p and L given by (6.3) and (6.4), respectively.

Return to the linear-quadratic (LQ) problem in the plane

We now return to the linear-quadratic (LQ) control problem for time-space random fields discussed in the introduction:

Suppose the state $Y(t, x)$ is given by

(6.5)

$$Y(t, x) = Y(0, 0) + \int_0^t \int_0^x u(s, a) ds da + \beta B(t, x); \quad t \geq 0, x \in \mathbb{R}.$$

We want to drive the state $Y(t, x)$ to 0 at time T and point X with minimal use of energy. Hence we put

$$(6.6) \quad J(u) = -\frac{1}{2}E \left[\int_0^T \int_0^X u^2(s, a) ds da + \theta Y^2(T, X) \right],$$

where $\theta > 0$ is a given constant.

Problem

We want to find $u^* \in \mathcal{A}$ such that

$$(6.7) \quad J(u^*) = \sup_{u \in \mathcal{A}} J(u).$$

In this case we have $\nabla_y H = 0$, so we can simplify the Hamiltonian to

$$(6.8) \quad H_0(t, x, y, u, p, q) = -\frac{1}{2}u^2 + up + \beta q$$

The maximum of $u \mapsto H_0(u)$ is obtained when $\frac{\partial H_0}{\partial u} = -u + p = 0$, i.e. when

$$(6.9) \quad u = p.$$

The adjoint equation is

$$(6.10) \quad \begin{aligned} p(dz) &= q(z)B(dz); \quad z < Z = (T, X) \\ p(T, X) &= \theta Y(T, X) \end{aligned}$$

Let us try to put

$$(6.11) \quad p(t, x) = \lambda(t, x) Y(t, x)$$

for some deterministic function λ . Then by the Itô formula

$$(6.12) \quad \begin{aligned} \lambda(t, x) Y(t, x) &= \lambda(0, 0) Y(0, 0) \\ &+ \int_0^t \int_0^x \left\{ Y(s, a) \frac{\partial^2 \lambda}{\partial s \partial x}(s, a) + \lambda(s, a) u(s, a) \right\} ds da \\ &+ \text{terms containing } B(ds, da). \end{aligned}$$

Using the concept of quadratic variation of 2-parameter martingales (see e.g. Imkeller [12]), one finds that the decomposition of a (continuous) 2-parameter "semimartingale", which is given by a sum of a 2-parameter process of bounded variation and a 2-parameter martingale, is unique. So, comparing the latter equation with the adjoint equation, we see that we must have

$$(6.13) \quad Y(t, x) \frac{\partial^2 \lambda}{\partial t \partial x}(t, x) + \lambda(t, x) u(t, x) = 0 \text{ for all } t, x.$$

Combining this with (6.11) we get

$$(6.14) \quad Y(t, x) \left[\frac{\partial^2 \lambda}{\partial t \partial x}(t, x) + \lambda^2(t, x) \right] = 0,$$

with terminal condition

$$(6.15) \quad \lambda(T, X) = \theta.$$

In addition we get from (6.11) the other boundary condition

$$(6.16) \quad \lambda(0,0) = \frac{E[\theta Y(T,X)]}{Y(0,0)}.$$

With this choice of u, p, λ we see that all the conditions of the sufficient maximum principle are satisfied, and we have proved the following:

Theorem

The optimal control $\hat{u}$ for the LQ problem (22) is given in feedback form by

$$(6.17) \quad \hat{u}(t,x) = \lambda(t,x)Y(t,x); \quad t \leq T, x \leq X,$$

where $\lambda(t,x)$ solves the time-space Riccati equation

$$(6.18) \quad \begin{cases} \frac{\partial^2 \lambda}{\partial t \partial x}(t,x) + \lambda^2(t,x) = 0; & 0 \leq t \leq T, 0 \leq x \leq X, \\ \lambda(T,X) = \theta, \\ \lambda(0,0) = \frac{E[\theta Y(T,X)]}{Y(0,0)}. \end{cases}$$

Remark

Let φ_1 be a solution to the Riccati equation

$$\dot{\varphi}_1(t) = (\varphi_1(t))^2, \varphi_1(0) = 1, 0 \leq t \leq T$$

and φ_2 be a solution to

$$\dot{\varphi}_2(x) = -(\varphi_2(x))^2, \varphi_2(0) = \theta, 0 \leq x \leq X.$$

Define $\lambda(t, x) = \alpha_1(t)\alpha_2(x)$, where $\alpha_1(t) := \varphi_1(T - t)$ and $\alpha_2(x) := \varphi_2(X - x)$. Then

$$\frac{\partial^2 \lambda(t, x)}{\partial t \partial x} = \dot{\varphi}_1(T - t) \dot{\varphi}_2(X - x) = (\alpha_1(t))^2 (-(\alpha_2(t))^2) = -(\lambda(t, x))^2$$

with $\lambda(T, X) = \varphi_1(0)\varphi_2(0) = \theta$.

By solving the Riccati equations, we find that λ given by

$$\lambda(t, x) = \frac{1}{(1 - T + t)(\theta^{-1} + X - x)}$$

is an explicit solution to the above hyperbolic PDE with boundary condition $\lambda(T, X) = \theta$ for $0 < T < 1$.

Let us now have a look at the other condition

$$\lambda(0,0) = \frac{\theta E[Y(T,X)]}{Y(0,0)}$$

We observe that

$$E[Y(t,x)] = Y(0,0) + \int_0^t \int_0^x \lambda(s,a) E[Y(s,a)] ds da.$$

So, if we use Picard iteration combined with the fact that λ can be written as a product of a function in t and another function in x , we see that the solution of the latter equation has the representation

$$E[Y(t,x)] = Y(0,0) F\left(\int_0^t \int_0^x \lambda(s,a) ds da\right),$$

where the function $F : \mathbb{R} \mapsto \mathbb{R}$ is defined by

$$F(y) = \sum_{n \geq 0} \frac{y^n}{(n!)^2}.$$

On the other hand,

$$\begin{aligned}\int_0^T \int_0^X \lambda(s, a) ds da &= \int_0^T \int_0^X \frac{1}{(1 - T + s)(\theta^{-1} + X - a)} ds da \\ &= -\log(1 - T) \log(1 + X\theta).\end{aligned}$$

So the condition

$$\lambda(0, 0) = \frac{\theta E[Y(T, X)]}{Y(0, 0)}$$

is equivalent to

$$\frac{1}{(1 - T)(\theta^{-1} + X)} = \theta F(-\log(1 - T) \log(1 + X\theta))$$

or

$$(6.19) \quad 1 = (1 - T)(1 + X\theta)F(-\log(1 - T) \log(1 + X\theta)).$$

For given $T < 1$ and $\theta > 0$ the expression on the right hand side of the latter equation converges to $(1 - T)$ for $X \rightarrow 0$. For $X \rightarrow \infty$, this expression converges to ∞ . Because of continuity, we then see that there exists a $X = X(T, \theta) > 0$ such that the equation (6.19) is satisfied. Using such a time horizon X , gives the other boundary condition.

Example related to machine learning

In machine learning the (continuous-time) stochastic gradient descent method (see e.g. [MS] and the references therein) is used to minimize an objective function $f : \mathbb{R}^d \rightarrow \mathbb{R}$. Compared to the classical gradient descent method without noise, this approach is especially computationally efficient, when the dimension d in practical optimization problems is high. If the objective function is sufficiently smooth the critical points of f with respect to local or global minima may be found by means of solutions to SDEs of the type

$$(6.20) \quad dY_t = -\eta \nabla f(Y_t) dt + \beta_0 dB_t, Y_0 = x \in \mathbb{R}^d, t \geq 0,$$

where $\eta \geq 0$ is the learning rate (or step size), $\beta_0 \in \mathbb{R}^{d \times d}$, $B_t, t \geq 0$ a Brownian motion and where ∇ denotes the gradient of a function. In general, the selection of an "optimal" learning rate η , which determines the optimal step size towards a minimum in the sense of speed, is in general difficult. If η is chosen too small, the solution may converge too slowly to a critical point. On the other hand, a too large η could result in overshoot or divergence. In order to gain a deeper understanding of the latter problem, one may consider instead of the SDE (6.20) a more general framework (at the possible expense of computational cost) in connection with the following type of hyperbolic SPDE:

(6.21)

$$Y(t, x) = y - \int_0^t \int_0^x u(s, a) \nabla f(Y(s, a)) ds da + \beta_0 B(t, x), y \in \mathbb{R}^d, t, x \geq 0,$$

where $u : \Omega \times [0, \infty)^2 \rightarrow [0, \infty)$ is a stochastic learning rate in time and space given by an adapted random field and where B is a Brownian sheet in $\mathbb{R}^d$. Formally, by choosing in (6.21) $u = \eta \delta_x$ for the Dirac delta function δ_x in a fixed point x and $\eta \geq 0$ we obtain an SDE of the type (6.20). So the random field dynamics (6.21) provides a more general framework than that in the one-parameter case (6.20) for finding the critical points of f .

On the other hand, we may view the integral term

$$\int_0^x u(s, a) \nabla f(Y(s, a)) da$$

in (6.21) for a fixed x and a certain class of stochastic 2-parameter learning rate processes as an (weighted) average of $\nabla f(Y(s, a))$, $0 \leq a \leq x$ in (6.20). Here $Y(s, a)$, $0 \leq a \leq x$ can be interpreted as a group of mountain hikers in the optimisation landscape who communicate with each other with respect to (average) gradient information in order to find the descent to the valley (i.e. minimum).

The latter, combined with the "exploration ability" of the Brownian sheet with respect to the spatial parameter direction in the optimisation landscape, suggests a solution that converges to rather flat minima, while escaping from sharp minima. The convergence to flat minima, however, is in many applications a favorable feature from a machine learning point of view (see [HS]). In order to construct optimal stochastic 2-parameter learning rate processes one may e.g study stochastic control problems based on the stochastic maximum principle for SPDEs driven by a Brownian sheet with respect to certain performance functionals as e.g.

$$(6.22) \quad J(u) = -E \left[\int_0^T \int_0^X u^2(s, a) ds da + f(Y(T, X)) \right],$$

where one minimises the expected value of $f(Y(T, X))$, while the "energy invested" in u is kept minimal.

Using the first order Taylor expansion, we can also approximate ∇f in (6.21) by an affine function g given by $g(x) = a + Ax$ for $a \in \mathbb{R}^d$, $A \in \mathbb{R}^{d \times d}$ and obtain a more simplified framework for our stochastic control problem with respect to u . In this setting, let us now consider the case $d = 1$ and the following controlled process:

$$Y_u(t, x) = Y(0, 0) - \int_0^t \int_0^x u(s, a) Y_u(s, a) ds da + \int_0^t \int_0^x \beta_0 B(ds, da).$$

Problem

We want to maximise the performance functional

$$J(u) = -E \left[\int_0^T \int_0^X u^2(s, a) ds da + \theta Y^2(T, X) \right].$$

In this case the associated Hamiltonian is

$$H(t, x, y, u, p, q, L) = -u^2 - yup + \beta_0 q - (L \star (uy)),$$

which gives

$$(6.23) \quad \frac{\partial H}{\partial y} = -up - (L \star u)$$

and hence the adjoint variable L is given by

$$(6.24) \quad L = -\frac{\partial H}{\partial y} = up + (L \star u)$$

which is an integral equation for $L(z)$.

The adjoint BSDE is

$$p(z) = p(0) + \int_{R_z} (u(\zeta)p(\zeta) + (L \star u))(\zeta) d\zeta$$

(6.25)

$$- \int_{R_z} q(\zeta) B(d\zeta) - \iint_{R_z \times R_z} r(\zeta, \zeta') B(d\zeta) B(d\zeta'), \quad z \leq (T, X)$$

(6.26)

$$p(T, X) = -2\theta Y(T, X).$$

In order to maximize H with respect to u , we first compute

$$\frac{\partial H}{\partial u} = -2u - yp - (L \star y)$$

which gives the optimal control

$$u = u^* = \frac{1}{2}(yp + (L \star y))$$

We have proved:

Theorem

The optimal control $u(z) = u^*(x)$ for Problem 25 is given as the 4th component of the solution $(Y(z), L(z), p(z), u(z))$ of the coupled system of equations, consisting of

(6.27)

$$u(z) = u^*(z) = \frac{1}{2}[Y(z)p(z) + (L \star Y)(z)],$$

(6.28)

$$Y(t, x) = Y(0, 0) - \int_0^t \int_0^x u(s, a) Y_u(s, a) ds da + \int_0^t \int_0^x \beta_0 B(ds, da),$$

(6.29)

$$L(z) = u(z)p(z) + (L \star u)(z)$$

and

(6.30)

$$\begin{cases} p(z) &= p(0) + \int_{R_z} (u(\zeta)p(\zeta) + (L \star u))(\zeta) d\zeta \\ &\quad - \int_{R_z} q(\zeta) B(d\zeta) - \iint_{R_z \times R_z} r(\zeta, \zeta') B(d\zeta) B(d\zeta'), \quad z \leq (T, X) \\ p(T, X) &= -2\theta Y(T, X). \end{cases}$$

Remark

In the more general case, when $\nabla f(x) = Ax$ for $A \in \mathbb{R}^{d \times d}$, one can show a similar result about the optimal control u^* with respect to the controlled process (6.21) and performance functional (6.22)

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