

Multi-fractional Stochastic Dominance

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Classical Stochastic Dominance

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Definition (First Degree SD): $F \preceq_{FSD} G$ if

$$F(x) \geq G(x), \quad \text{or equivalently} \quad \bar{F}(x) \leq \bar{G}(x), \quad \forall x \in \mathbb{R},$$

where $\bar{F} := 1 - F$ denotes the so-called *survival function* of F .

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Definition (Second Degree SD): $F \preceq_{SSD} G$ if

$$\int_{-\infty}^t (F(x) - G(x)) dx \geq 0, \quad \forall t \in \mathbb{R}, \text{ i.e.}$$
$$\int_{-\infty}^t (F(x) - G(x))_- dx \leq \int_{-\infty}^t (F(x) - G(x))_+ dx \quad \forall t \in \mathbb{R}$$

Mann and Whitney (1947) and Lehmann (1955) in statistics; Karlin (1960) in operation research; Von-Neumann and Morgenstern (1947), Friedman and Savage (1948) and Markowitz (1952) in economics ...

Theorem (Expected utility counterpart):



$$F \preceq_{FSD} G \quad \text{iff} \quad \mathbb{E}_F[u] \leq \mathbb{E}_G[u]$$

for all utility functions $u \in \mathcal{U}_1 := \{u : \mathbb{R} \rightarrow \mathbb{R} : \text{increasing}\}$
(decision makers who prefer more to less or insatiable).

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Let $N \geq 1$. The **N th integer-degree** stochastic dominance is defined (Ekern 1980):

$$F \preceq_{N\text{-th degree SD}} G \quad \text{iff} \quad \mathbb{E}_F[u] \leq \mathbb{E}_G[u]$$

for all $u \in \mathcal{U}_N := \{u : (-1)^{n+1}u^{(n)}(x) \geq 0, \forall x, \text{ and } n = 1, \dots, N\}$.

Definition (Fractional SD): For a given $\gamma \in [0, 1]$, $F \preceq_{(1+\gamma)\text{-SD}} G$ if

$$\int_{-\infty}^t (F(x) - G(x))_- dx \leq \gamma \int_{-\infty}^t (F(x) - G(x))_+ dx \quad \forall t \in \mathbb{R},$$

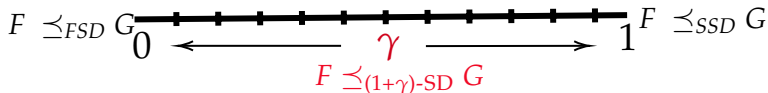
[Müller et al., (2017)]

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- Interpolation between first and the second SD:

$$F \preceq_{FSD} G \xrightarrow{\gamma} F \preceq_{SSD} G$$

- Fishburn (1976, 1980): $\alpha \in [0, 1]$, $u_{\alpha,t}(x) = -(t - x)_{+}^{\alpha}$,

$$F \preceq_{FSD} G \quad \text{iff} \quad \mathbb{E}[u_{\alpha,t}(X)] \leq \mathbb{E}[u_{\alpha,t}(Y)], \forall t.$$

Multifractional Brownian motion

Definition (fBm): A Gaussian process $B^H = (B^H(t))$ is called a fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1]$, if it has mean zero and covariance function

$$\mathbb{E} \left[B^H(t) B^H(s) \right] = \frac{\sigma^2}{2} \left\{ |t|^{2H} + |s|^{2H} - |t - s|^{2H} \right\}, \quad s, t \in \mathbb{R}, \sigma^2 > 0.$$

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Definition (Multifractional Brownian motion): to replace Hurst parameter H with a suitable (Hölder continuous) function

$$H : \mathbb{R} \rightarrow (0, 1).$$

Peltier and Lévy-Véhel (1995); Benassiet al. (1998), Cohen (1999).

Multi-fractional Stochastic Dominance (MFSD)

Definition (Multi-fractional SD): Let $\gamma : \mathbb{R} \rightarrow [0, 1]$ be an **arbitrary non-decreasing** function, $F \preceq_{(1+\gamma)\text{-SD}}^{\text{MF}} G$ if

$$\int_{-\infty}^t (F(x) - G(x))_- dx \leq \gamma(t) \int_{-\infty}^t (F(x) - G(x))_+ dx, \quad \forall t \in \mathbb{R}. \quad (1)$$

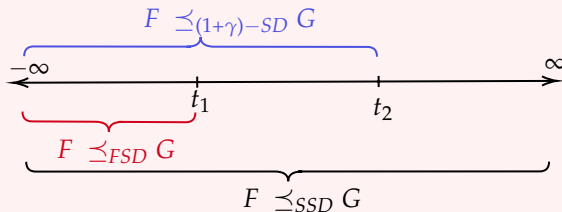
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Example (Local interpolation): Assume that $F \preceq_{(1+\gamma)\text{-SD}}^{\text{MF}} G$ with

$$\gamma(t) = \begin{cases} 0 & \text{if } t < t_1 \\ \gamma & \text{if } t_1 \leq t \leq t_2 \\ 1 & \text{if } t > t_2. \end{cases}$$



Universe of MFSD

Theorem (Universal inclusion): For every $\gamma \in [0, 1]$ and, non-decreasing function $\gamma : \mathbb{R} \rightarrow [0, 1]$, let introduce

$$\mathcal{E}_F(\gamma) := \left\{ (F, G) \in \mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R}) : F \preceq_{(1+\gamma)\text{-SD}} G \right\}$$
$$\mathcal{E}_{MF}(\gamma) := \left\{ (F, G) \in \mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R}) : F \preceq_{(1+\gamma)\text{-SD}}^{MF} G \right\}.$$

Then,

$$\bigcup_{\gamma \in [0, 1)} \mathcal{E}_F(\gamma) \subsetneq \bigcup_{\substack{\gamma: \mathbb{R} \rightarrow [0, 1] \\ \text{non-decreasing} \\ \gamma \neq 1}} \mathcal{E}_{MF}(\gamma) = \mathcal{E}_{SSD}. \quad (2)$$

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Hence, one advantage is that distributions with **equal means** can be distinguishable in the sense of MFSD whilst **impossible via any** fractional SD (apart from SSD).

Utility Class

Theorem (Utility Counterpart Fractional Degree SD): For a fixed $\gamma \in [0, 1]$, $F \preceq_{(1+\gamma)\text{-SD}} G$ if and only if

$$\mathbb{E}_F[u] \leq \mathbb{E}_G[u], \quad \forall u \in \mathcal{U}_\gamma^*. \quad (3)$$

where

$$\mathcal{U}_\gamma^* := \left\{ u : 0 \leq \gamma \frac{u(x_4) - u(x_3)}{x_4 - x_3} \leq \frac{u(x_2) - u(x_1)}{x_2 - x_1}, \forall x_1 < x_2 \leq x_3 < x_4 \right\}.$$

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Proposition: For any $\gamma \in (0, 1]$,

$$\mathcal{U}_\gamma^* = \{u : \text{both-sided diff.}, 0 \leq \gamma u'_\pm(y) \leq u'_\pm(x), \forall x \leq y\}.$$

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The most natural set to be considered as a candidate for the MFSD utility class is

$$\{u : \text{both-sided diff.}, 0 \leq \gamma(y) u'_\pm(y) \leq u'_\pm(x), \forall x \leq y\}.$$

However, this is not the case ...

We define $\mathcal{U}_\gamma^{\text{MF}}$ in two steps as follows:

i) For $t \in \mathbb{R}$, define

$$\mathcal{U}_{\gamma(t)}^t := \left\{ u : \mathbb{R} \rightarrow \mathbb{R} : u(x) = u(s), \forall x \geq t \text{ and,} \right. \\ \left. \forall x_1 < x_2 \leq x_3 < x_4, 0 \leq \gamma(t) \frac{u(x_4) - u(x_3)}{x_4 - x_3} \leq \frac{u(x_2) - u(x_1)}{x_2 - x_1} \right\} \supsetneq \mathcal{U}_{\gamma(t)}^*.$$

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We define $\mathcal{U}_Y^{\text{MF}}$ in two steps as follows:

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ii) Let $\mathcal{U}_Y^{\text{Union}} := \bigcup_{t \in [-\infty, \infty]} \mathcal{U}_{Y(t)}^t$. Denote

$$\mathcal{U}_Y^{\text{MF}} := \underbrace{\text{span}_{\mathbb{R}_+} \left\{ u : u \in \mathcal{U}_Y^{\text{Union}} \right\}}_{\text{Convex cone}} = \left\{ u = \sum_{\text{finite}} \lambda_k u_k, \lambda_k \geq 0, u_k \in \mathcal{U}_Y^{\text{Union}} \right\}.$$

Theorem (Utility Counterpart Multi-fractional Degree SD): Let $\gamma : \mathbb{R} \rightarrow [0, 1]$ be a non-decreasing function. Then,

$$F \preceq_{(1+\gamma)\text{-SD}}^{\text{MF}} G \quad \text{iff} \quad \mathbb{E}_F[u] \leq \mathbb{E}_G[u], \quad \forall u \in \mathcal{U}_\gamma^{\text{MF}}. \quad (4)$$

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Proposition:

$$\mathcal{U}_\gamma^{\text{MF}} \subsetneq \{u : \text{both-sided diff.}, 0 \leq \gamma(y) u'_\pm(y) \leq u'_\pm(x), \forall x \leq y\}.$$

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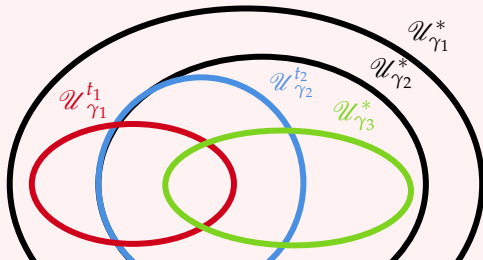
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Example (Step function):

$$\mathcal{U}_\gamma^{\text{Union}} = \mathcal{U}_{\gamma_3}^* \cup \mathcal{U}_{\gamma_2}^{t_2} \cup \mathcal{U}_{\gamma_1}^{t_1}.$$

$$\gamma(t) = \begin{cases} \gamma_1 & \text{if } t < t_1 \\ \gamma_2 & \text{if } t_1 \leq t < t_2 \\ \gamma_3 & \text{if } t \geq t_2. \end{cases}$$



Decision Theoretic Aspects

Definition (Chateauneuf et al. (2005), *Econom. Theory*): The index of greediness (or nonconcavity) of a utility function as

$$\begin{aligned} G(u) &:= \sup_{x_1 < x_2 \leq x_3 < x_4} \left(\frac{u(x_4) - u(x_3)}{x_4 - x_3} / \frac{u(x_2) - u(x_1)}{x_2 - x_1} \right) \\ &= \sup_{y < z} \frac{u'(z)}{u'(y)} \quad (\text{whenever } u \text{ is differentiable}) \end{aligned}$$

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- ▶ Always $G(u) \geq 1$ and $G(u) = 1$ iff u is concave.
- ▶ The index $G(u)$ **globally** determines the maximum magnitude of the increase in the slope of the utility function u over its **entire** domain.
- ▶ $G(u) \leq \frac{1}{\gamma}$ for all $u \in \mathcal{U}_\gamma^*$ in the fractional utility class.

The Local Greediness

We work on a larger utility class:

$$\mathcal{U}_Y^{\text{cv}} := \{u : \text{one-sided diff. \& cont.}, 0 \leq \gamma(y) u'_\pm(y) \leq u'_\pm(x), \forall x \leq y\} \supsetneq \mathcal{U}_Y^{\text{MF}}.$$

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Proposition (Local Greediness): Let $u \in \mathcal{U}_\gamma^{\text{cv}}$. For every $x \in \mathbb{R}$,

$$\begin{aligned} G_{\text{loc}}(u; x) &:= \sup_{x < x_1 < x_2 \leq x_3 < x_4} \left(\frac{u(x_4) - u(x_3)}{x_4 - x_3} / \frac{u(x_2) - u(x_1)}{x_2 - x_1} \right) \\ &= \sup_{x < y < z} \frac{u'_\pm(z)}{u'_\pm(y)} \leq \frac{1}{\gamma(x)}. \end{aligned}$$

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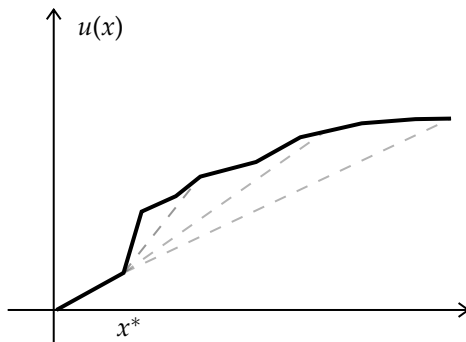
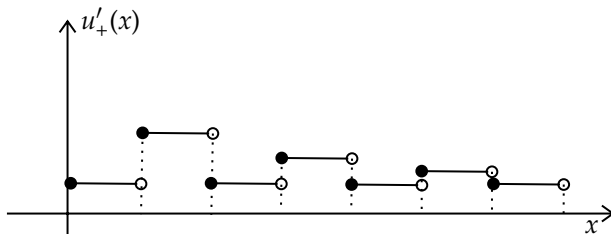
Proposition (ARA coefficient): Suppose that $u \in \mathcal{U}_\gamma^{\text{cv}}$ and twice differentiable then

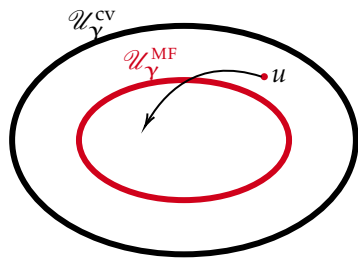
$$\int_x^y r_u(z) dz \geq \ln(\gamma(y)), \quad \forall x \leq y$$

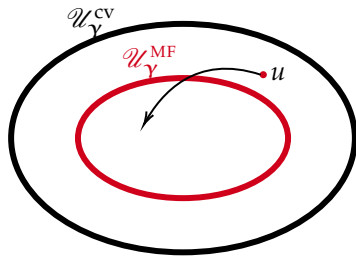
where the absolute risk aversion coefficient

$$r_u(x) := -\frac{u''(x)}{u'(x)}, \quad x \in \mathbb{R}.$$

The multi-fractional utility space $\mathcal{U}_Y^{\text{MF}}$, in general, contains utilities having non-concave segments.



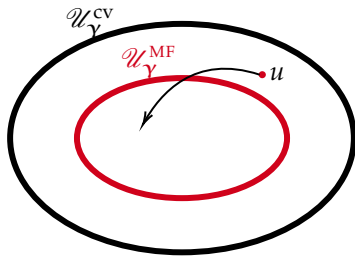




Theorem: Let $u \in \mathcal{U}_\gamma^{\text{cv}}$ and $\exists x_0, x_1 \in \mathbb{R}$ such that $0 < \gamma(x_0) \neq \gamma(x_1) < 1$.
Let

$$G_{\text{loc}}(u; x_k) = \frac{1}{\gamma(x_k)}, \quad k = 0, 1. \quad (5)$$

Then, $u \notin \mathcal{U}_\gamma^{\text{MF}}$.



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Theorem: Let there exists open $U \subseteq \mathbb{R}$ where γ is strictly increasing on U .
Suppose that $u \in \mathcal{U}_\gamma^{\text{cv}}$ and $\exists x_0 \in U$ such that

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Then, $u \notin \mathcal{U}_\gamma^{\text{MF}}$.

Definition (Greediness metric): Let $u, v \in \mathcal{U}_\gamma^{\text{cv}}$. The greediness distance d_G is defined

$$d_G(u, v) := \sup_{x \in \mathbb{R}} \left| G_{\text{loc}}(u; x) - G_{\text{loc}}(v; x) \right|.$$

Observe that $(\mathcal{U}_\gamma^{\text{cv}}, d_G)$ is a pseudo-metric space.

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Observe that $(\mathcal{U}_\gamma^{\text{cv}}, d_G)$ is a pseudo-metric space.

A few open problems:

- ▶ Does the metric d_G turn $\mathcal{U}_\gamma^{\text{cv}}$ into a Banach space?
- ▶ What are (necessary) and sufficient conditions on function γ so that

$$\overline{\mathcal{U}_\gamma^{\text{MF}}}^{\|\cdot\|_{d_G}} = \mathcal{U}_\gamma^{\text{cv}}?$$

- ▶ Are there other analytical tools available to characterize multi-fractional utility functions?
- ▶ Does multi-fractional SD admit a **smooth** generator (like fractional SD)?

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Happy anniversary Yuliia! Thank you so much for over five decades of original contributions to the world of mathematics, along with your invaluable dedication to educating and inspiring future generations!!