

On the relationship between the short-end of the local and the implied volatility surfaces via Malliavin calculus

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Some questions on local and implied volatilities

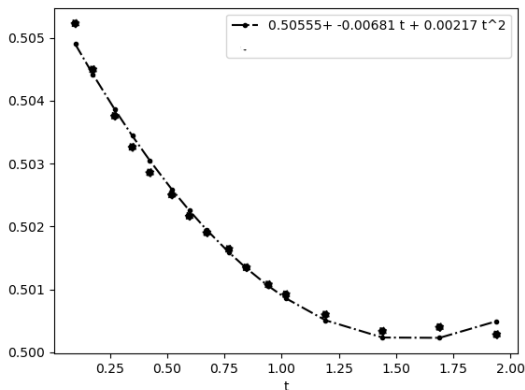


Figure – Empirical $\frac{\partial_k l_0(T, k^*)}{\partial_x \hat{\sigma}(T, X_T)}$ for market data of EUROSTOXX50E index.

- For which models is the one-half rule satisfied?
- What happens with rough volatilities? (see Bourgey, De Marco, Friz, and Pigato (2022))
- If rough volatilities do not satisfy the one-half rule, are these models non-adequate? How can we reconcile all this?
- What about the curvature?

Previous literature :

- Based on the expression of the implied volatility as an averaged local volatility : Derman, Kani and Zhou (1996), Gatheral (2006)
- From the direct comparison of local and implied volatility expansions : Lee (2001)
- Large deviations, Malliavin calculus, and rough paths, for a class of rough models : Bourgey, De Marco, Friz, and Pigato (2022)

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A primer on Malliavin calculus

One of the objectives of Malliavin calculus is to define, in an adequate sense, the concept of 'derivative' of a random variable with respect to the white noise.

Consider first a random variable of the form

$$W(h) = \int_0^T h(s) dW_s.$$

Intuitively, we should define a derivative operator D in such a way that

$$DW(h) = h.$$

Moreover, we would like this operator to satisfy the chain rule. That is, given a random variable A and a differentiable function f :

$$Df(A) = f'(A)DA.$$

All this observations lead us to define, for a random variable of the form $f(W(h))$,

$$D_f F = f'(W(h))h(t)$$

... and so on.

From the above definition, straightforward arguments allow us to see that the connection between Malliavin derivatives and Itô integrals and then we can see that, for an Itô process of the form

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t b_s dZ_s,$$

its Malliavin derivative is given by

$$D_u^Z X_t = \int_0^t D_u^Z a_s ds + b_u \mathbf{1}_{[0,t]}(u) + \int_0^t D_u^Z b_s dZ_s. \quad (1)$$

Malliavin derivatives are not just abstract random variables but they can be computed explicit and easily for most models in finance.

Malliavin derivatives can be computed for a wide class of random variables and processes. We do not need them to be semimartingales, nor Markovian... we can compute them for models based on the fractional Brownian motion.

Let us see some examples

- for the Black Scholes model, $S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)$,

$$D_u S_t = \sigma S_t \mathbf{1}_{[0,t]}(u).$$

- For the SABR volatility $\sigma_t = \alpha \exp\left(-\frac{\nu^2}{2}t + \nu W_t\right)$

$$D_u \sigma_t = \nu \sigma_t \mathbf{1}_{[0,t]}(u).$$

- For the Heston volatility (see Alòs and Ewald (2008))

$$dv_t = k(\theta - v_t)dt + \nu\sqrt{v_t}dW_t,$$

$$D_u v_t = \frac{\nu\sqrt{v_t}}{2} \mathbf{1}_{[0,t]}(u) \exp\left(\int_u^t \left(-\left(\frac{k\theta}{2} - \frac{\nu^2}{8}\right) \frac{1}{v_s} - \frac{k}{2}\right) ds\right),$$

- For volatility models of the type $\sigma_t = f(W_t^H)$, where W^H is a fBm $W^H = \int_0^t K_H(t,s)dW_s$, (with $K_H(t,s)$ 'similar' to $(t-s)^{H-\frac{1}{2}}$),

$$D_u \sigma_t = f'(W_t^H) K_H(t, u).$$

Notice that, for stochastic volatility models, $D_u \sigma_t$ tends to a finite limit as $u \rightarrow t$, while for rough volatilities $D_u \sigma_t \sim (t-u)^{H-\frac{1}{2}}$, for $H < \frac{1}{2}$.

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Malliavin calculus for local volatilities

Consider a local volatility model of the form

$$S_t = S_0 + \int_0^t \sigma(u, S_u) S_u dW_u, \quad (2)$$

where we consider some general hypotheses for σ and its derivatives.

According to (1), the Malliavin derivative of the random variable S_t is given by

$$D_r^W S_t = \sigma(r, S_r)S_r + \int_r^t a(u, S_u)D_r^W S_u dW_u,$$

where $r < t$ and $a(u, S_u) : \partial_x \sigma(u, S_u)S_u + \sigma(u, S_u)$. Burkholder's inequality and Gronwall's lemma allow us to see that, under (H1) and (H2), DS is in L^p for all $p > 1$.

Moreover

$$D_r^W S_t = \sigma(r, S_r)S_r \exp \left(-\frac{1}{2} \int_r^t a^2(u, S_u) du + \int_r^t a(u, S_u) dW_u \right) \quad (3)$$

The second Malliavin derivative is given by

$$\begin{aligned} D_{\theta}^W D_r^W S_t &= a(r, S_r) D_{\theta}^W S_r + \int_r^t \partial_x a(u, S_u) D_r^W S_u D_{\theta}^W S_u dW_u \\ &+ \int_r^t a(u, S_u) D_r^W D_{\theta}^W S_u dW_u \end{aligned} \quad (4)$$

for $\theta < r$,

and in a similar way we can compute the third derivative.

Notice that under some genral hypotheses, for $\theta, r \rightarrow t$,

$$D_r^W S_t \rightarrow \sigma(t, S_t) S_t, \quad (5)$$

and

$$D_\theta^W D_r^W S_t \rightarrow a(t, S_t) \sigma(t, S_t) S_t = \partial_x \sigma(t, S_t) S_t^2 + \sigma^2(t, S_t) S_t, \quad (6)$$

a.s.

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The skew : general results

Consider a risk-neutral probability model for asset prices of the form

$$dS_t = \sigma_t S_t (\rho dW_t + \sqrt{1 - \rho^2} dB_t), \quad (7)$$

where we assume the interest rate to be zero, and where σ is a stochastic process adapted to the filtration generated by W . Notice that (7) includes the cases of

- *local* volatilities,
- *classical stochastic volatility models* (where σ is assumed to be a diffusion), and
- *fractional volatilities* (where σ is driven by a fractional Brownian motion). In particular, it includes the case of *rough volatilities* ($H < \frac{1}{2}$)

Some notation

We denote by

$$C_{BS}(t, T, X_t, k, \sigma)$$

the Black-Scholes price at time t of a European call with time to maturity T , log-strike price $k := \ln K$, and by

$$I_t(T, k)$$

the corresponding Black-Scholes implied volatility.

Moreover, we denote as $k^* = \ln(S_t)$ the at-the-money strike

Theorem (Alòs, León and Vives, 1007)

Under some general hypotheses

$$\begin{aligned} & \lim_{T \rightarrow t} (T - t)^{\frac{1}{2} - H} \partial_k I_t(T, k^*) \\ &= \frac{\rho}{2\sigma_t^2} \lim_{T \rightarrow t} \frac{1}{(T - t)^{\frac{3}{2} + H}} \mathbb{E}_t \left(\int_t^T \left(\int_r^T D_r^W \sigma_u^2 du \right) dr \right). \quad (8) \end{aligned}$$

The above limit equality can be understood as

$$\partial_k I_t(T, k^*) \sim \frac{\rho}{2\sigma_t^2} \frac{1}{(T - t)^2} \mathbb{E}_t \left(\int_t^T \left(\int_r^T D_r^W \sigma_u^2 du \right) dr \right). \quad (9)$$

Notice that, for rough volatilities $D_r^W \sigma_u^2 \sim (u - r)^{H - \frac{1}{2}}$ with $H < \frac{1}{2}$ and then the RHS in (9) tends to infinite as $T \rightarrow t$.

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Regular local volatilities

We can apply the previous result to local volatilities that are bounded with bounded derivatives. In this case

$$D_r^W \sigma^2(u, S_u) = 2\sigma(u, S_u) \partial_x \sigma(u, S_u) D_r^W S_u$$

and, as $D_r^W S_u \rightarrow \sigma(0, S_0) S_0$ as $r, u \rightarrow 0$,

$$\begin{aligned} D_r^W \sigma^2(0, S_0) &\rightarrow 2\sigma^2(0, S_0) \partial_x \sigma(0, S_0) S_0 \\ &= 2\sigma^2(0, S_0) \partial_x \hat{\sigma}(0, X_0), \end{aligned} \tag{10}$$

where $\hat{\sigma}$ denotes the local volatility function in terms of the log-price X . Then we get (see Alòs and García-Lorite (2021))

$$\begin{aligned}
& \lim_{T \rightarrow 0} \partial_k l_0(T, k^*) \\
&= \frac{1}{2\sigma^2(0, S_0)} \lim_{T \rightarrow t} \frac{1}{T^2} \mathbb{E} \left(\int_0^T \left(\int_r^T D_r^W \sigma^2(u, S_u) du \right) dr \right) \\
&= \lim_{T \rightarrow t} \frac{1}{T^2} \partial_X \hat{\sigma}(0, X_0) \left(\int_0^T \left(\int_r^T dr \right) du \right) \\
&= \frac{1}{2} \partial_X \hat{\sigma}(0, X_0).
\end{aligned} \tag{11}$$

Remark

The above is consistent with the one-half rule, this heuristic relationship was introduced by Derman, Kani, and Zou (1996). This rule has been proved for stochastic volatility models via different techniques (see Derman, Kani, and Zou (1996), Gatheral (2006), Lee (2001), or Alòs and García-Lorite (2021)).

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The rough volatility case

Now, assume that (as in real market data) the local volatility has a singularity at $(0, S_0)$ (that is $\partial_x \hat{\sigma}(T, X_T) \sim T^{H-\frac{1}{2}}$), and then we refer to it as a 'rough local volatility'. Then

Theorem

Under some general hypotheses

$$\lim_{T \rightarrow 0} T^{\frac{1}{2}-H} \partial_k l_0(T, k^*) = \frac{1}{\frac{3}{2} + H} \lim_{T \rightarrow 0} T^{\frac{1}{2}-H} \partial_x \hat{\sigma}(T, X_T),$$

where $\hat{\sigma}$ denotes the local volatility function in terms of the log-price X .

Remark

The above result recovers the recent results by Bourgey, De Marco, Friz, and Pigato (2022).

Remark

This relationship between the local and implied volatility implies that, given a model for the volatility process σ , the local and the implied skew slopes follow the same power law.

Remark

The above result seems to be in contradiction with real market practice, where the one-half rule is observed. However, both facts are compatible. Even when a rough local volatility will satisfy the above $\frac{1}{H+\frac{3}{2}}$ rule, local volatilities computed in real market practice are never 'rough', but they are regularized near maturity, as we can see in the following example.

Example

In the plot the shortest market maturity is $T = 0.03$. Then, the computation of the local volatility from $T = 0$ up to $T = 0.03$ was computed via a regularization procedure, which leads in practice to a regular local volatility function, with a (even high) finite skew slope limit. Then, the one-half rule is satisfied, as predicted before.

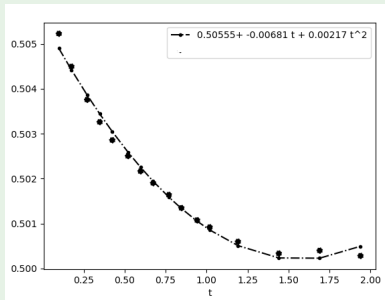


Figure – Empirical $\frac{\partial_k^2 I_0(T, k^*)}{\partial_x^2 \hat{\sigma}(T, X_T)}$ for market data of EUROSTOXX50E index, with date with date July 14 2023

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The curvature

Theorem (Alòs and León (2017))

Take $t \in [0, T]$. Then, under some general hypotheses

$$\begin{aligned} & \lim_{T \rightarrow t} (T - t)^{1-2H} \partial_{kk}^2 I_t(T, k^*) \\ &= \frac{1}{4\sigma_t^5} \lim_{T \rightarrow t} \frac{1}{(T - t)^{2+2H}} \mathbb{E} \left(\int_t^T \left(\mathbb{E}_r \int_r^T D_r^W \sigma_u^2 du \right)^2 dr \right) \\ & - \frac{3\rho^2}{2\sigma_t^5} \lim_{T \rightarrow t} \frac{1}{(T - t)^{3+2H}} \mathbb{E} \left(\int_t^T \left(\int_r^T D_r^W \sigma_u^2 du \right) dr \right)^2 \\ & + \frac{\rho^2}{\sigma_t^4} \lim_{T \rightarrow t} \frac{1}{(T - t)^{2+2H}} \mathbb{E} \left(\int_t^T \int_s^T D_s^W \left(\sigma_r \int_r^T D_r^W \sigma_u^2 du \right) dr ds \right). \end{aligned} \tag{12}$$

Theorem

Then we get

$$\begin{aligned} & \lim_{T \rightarrow 0} T^{1-2H} \partial_{kk}^2 I_0(T, k^*) \\ &= \frac{1}{\sigma(0, S_0)} \lim_{u \rightarrow 0} u^{1-2H} (\partial_x \hat{\sigma}(u, X_u))^2 \\ & \times \left[\frac{3}{(H + \frac{3}{2})(H + 1)} - \frac{6}{(H + \frac{3}{2})^2} + \frac{1}{2(H + 1)} \right] \\ &+ \frac{1}{2(1 + H)} \lim_{T \rightarrow 0} T^{1-2H} \partial_{xx}^2 \hat{\sigma}(T, X_T), \end{aligned} \tag{13}$$

Remark (Connexion with previous results)

Notice that, if $H = \frac{1}{2}$ (the stochastic volatility case), the above reduces to

$$\begin{aligned} \lim_{T \rightarrow 0} \partial_{kk}^2 I_0(T, k^*) &= -\frac{1}{6\sigma(0, S_0)} \lim_{u \rightarrow 0} (\partial_x \hat{\sigma}(u, X_T))^2 \\ &+ \frac{1}{3} \lim_{T \rightarrow 0} \partial_{xx}^2 \hat{\sigma}(T, X_T), \end{aligned} \quad (14)$$

according to Equation (8.4.3) in Alòs and García-Lorite (2021).

On the other hand, in the uncorrelated case $\rho = 0$ it reads

$$\lim_{T \rightarrow 0} \partial_{kk}^2 I_0(T, k^*) = \frac{1}{2(H+1)} \lim_{T \rightarrow 0} \partial_{xx}^2 \hat{\sigma}(T, X_T). \quad (15)$$

Remark (Cont.)

In particular, if $\rho = 0$ and $H = \frac{1}{2}$, we get

$$\lim_{T \rightarrow 0} \partial_{kk}^2 l_0(T, k^*) = \frac{1}{3} \lim_{T \rightarrow 0} \partial_{xx}^2 \hat{\sigma}(T, X_T), \quad (16)$$

according to the results in Hagan, Kumar, Lesniewski, and Woodward (2002).

Remark

As

$$\lim_{T \rightarrow t} T^{1-2H} (\partial_x \hat{\sigma}(T, X_T))^2 = 4 \lim_{T \rightarrow 0} T^{1-2H} (\partial_k l_0(T, k^*))^2$$

the result in the above theorem can be written as

$$\begin{aligned} & \frac{1}{2(1+H)} \lim_{T \rightarrow 0} T^{1-2H} \partial_{xx}^2 \hat{\sigma}(T, X_T) \\ &= \lim_{T \rightarrow 0} T^{1-2H} \partial_{kk}^2 l_0(T, k^*) \\ & - \frac{4}{\sigma(0, S_0)} \lim_{T \rightarrow 0} T^{1-2H} (\partial_k l_0(T, k^*))^2 \\ & \times \left[\frac{3}{(H + \frac{3}{2})(H + 1)} - \frac{6}{(H + \frac{3}{2})^2} + \frac{1}{2(H + 1)} \right]. \end{aligned} \quad (17)$$

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Numerical Results

Example (The skew in the Rough Bergomi model)

Consider the rough Bergomi model with the volatility process given by

$$\sigma_t = \sigma_0 \exp \left(\nu \sqrt{2H} W_t^H - \frac{1}{2} \nu^2 t^{2H} \right),$$

where $W_t^H = \int_0^t (t-s)^{H-\frac{1}{2}} dW_s$ (see Bayer, Friz, and Gatheral(2016)). The set of parameters used in the simulation is the following $S_0 = 100$, $\nu = 1.1$, $\sigma_0 = 0.3$, and $\rho = -0.6$. We estimate the corresponding at-the-money implied and local volatility skews of a European call option as a function of maturity.

Example (Cont.)

From our simulations, we compute the ratio of the implied volatility skew over the local volatility skew for $H = \frac{1}{2}$ and $H = 0.2$ and we show, that in the limit, $\frac{\partial_k l_0(T, k^*)}{\partial_x \hat{\sigma}(T, X_T)}$ goes to $\frac{1}{H + \frac{3}{2}} = 0.5$ and $\frac{1}{H + \frac{3}{2}} = 0.588$, respectively.

These coincide with the previous theoretical results. In the following figures we show the results that we have obtained from the Monte Carlo simulation for the ratio $\frac{\partial_k l_0(T, k^*)}{\partial_x \hat{\sigma}(T, X_T)}$ as a function of T ,

Example (Cont.)

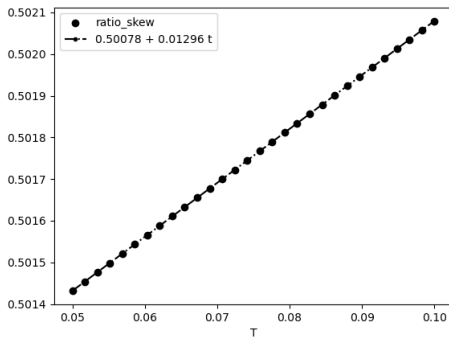


Figure – $H = 0.5$

Example (Cont.)

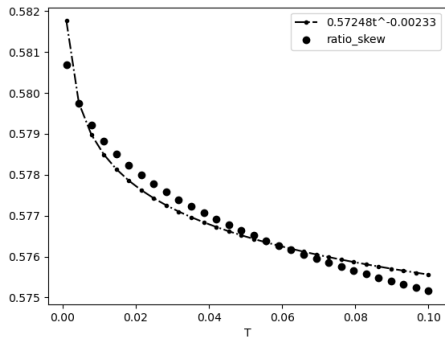


Figure – $H = 0.2$

Example (The curvature in the SABR model)

In the SABR model

$$\sigma_t = \alpha \exp \left(-\frac{\nu^2}{2} t + \nu W_t \right)$$

we have that

$$D_r \sigma_u^2 = 2\sigma_u^2 \rho \nu$$

which allows us to see that the limit of the IV skew slope is $\frac{\rho\nu}{2}$ and then the limit of the LV skew slope is $\rho\nu$. Then the difference of the curvatures in the short-end limit term is equal to $\frac{\rho^2\nu^2}{6\alpha}$.

Example (Cont)

In order to check it we run the Monte Carlo simulation with the following set of parameters for the SABR model

$\alpha = 0.3, \nu = 0.6, \rho = -0.6$ and $S_0 = 100$ (notice that then $\frac{\rho^2 \nu^2}{6\alpha} = \frac{0.6^4}{6 \times 0.3} = 0.072$). The results are presented in the following figure.

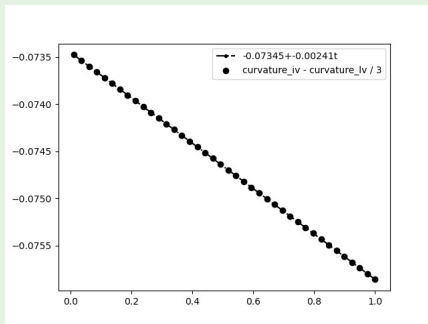


Figure – $\partial_{kk}^2 I_0(T, k^*) - \partial_{xx}^2 \hat{\sigma}(T, X_T)$ as a function of T

Example (Cont)

Recall that for $\rho = 0$

$$\lim_{T \rightarrow 0} \partial_{kk}^2 l_0(T, k^*) = \frac{1}{3} \partial_{xx}^2 \hat{\sigma}(T, X_T)$$

This property can be observed in the following figure.

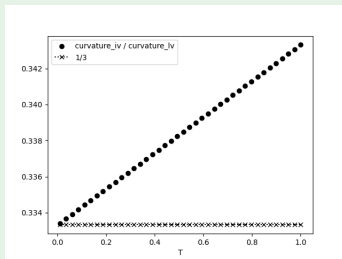


Figure – $\frac{\partial_{kk}^2 l_0(T, k^*)}{\partial_{xx}^2 \hat{\sigma}(T, X_T)}$ as a function of T .

Example (Curvatures for the rough Bergomi model)

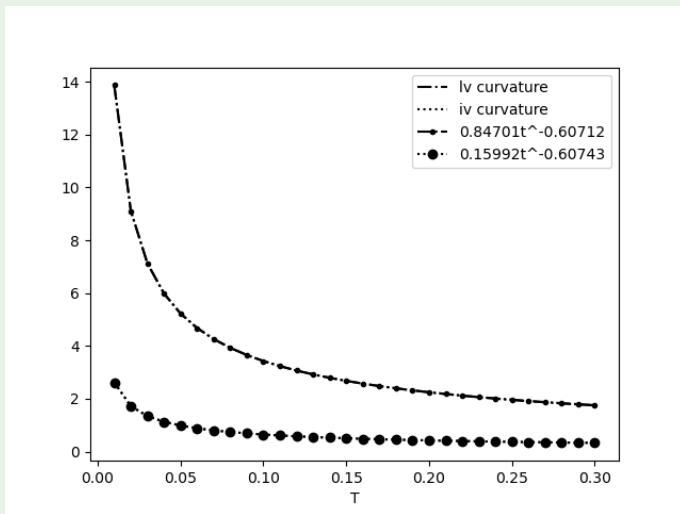


Figure – Short-end curvatures of the local volatility and the at-the-money implied volatility, $H=0.2$.