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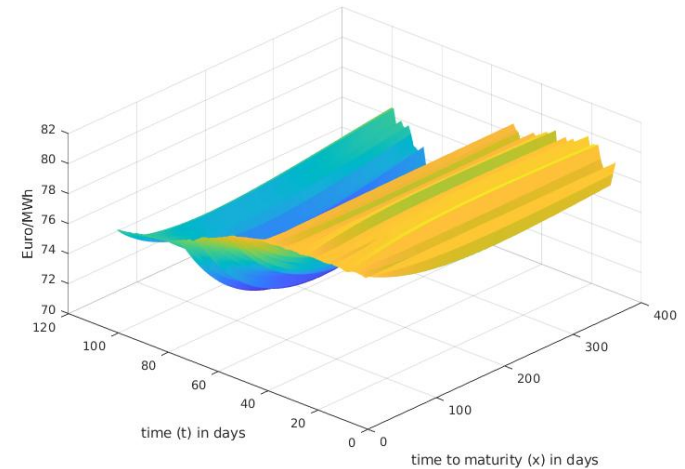
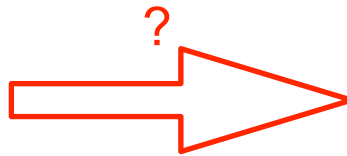
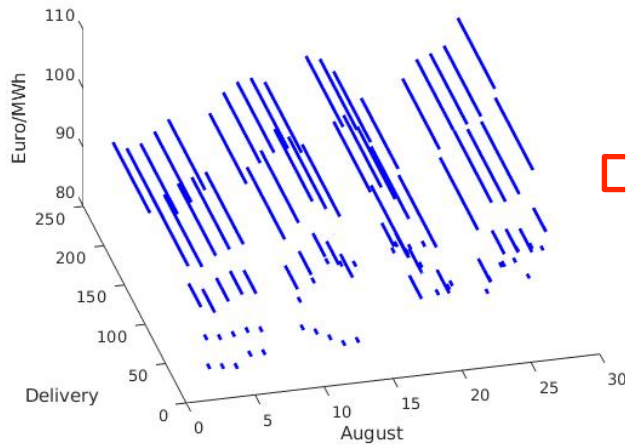
Recent advances on forward curve modeling and applications

Fred Espen Benth

Joint work with **Nils Detering** (Santa Barbara), **Luca Galimberti** (Trondheim), **Paul Kruhner** (Wien), **Carlo Sgarra** (Bari)



What is the problem and aim?



- Stochastic model for the term-structure evolution

$$(t, T_1, T_2) \mapsto \widehat{F}(t, T_1, T_2), \quad t \leq T_1 < T_2$$

$$(t, T) \mapsto F(t, T), \quad t \leq T$$

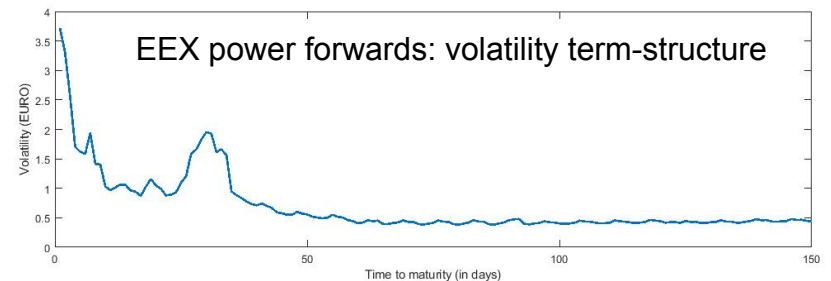
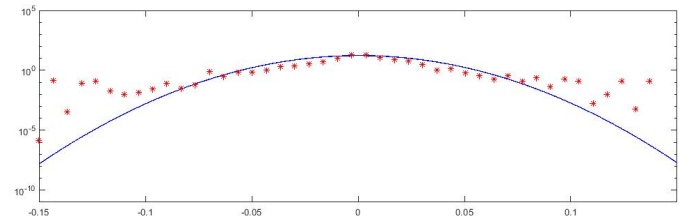
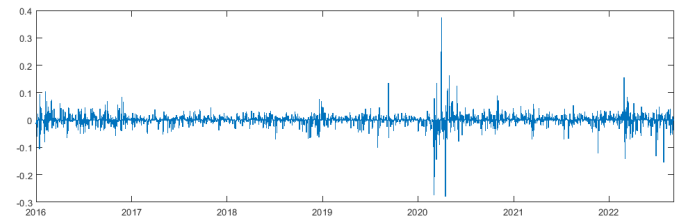
Challenges

- Many factors needed
 - PCA analysis reveals: 5-10 factors in power, 4 factors in metals
- Stochastic volatility
 - Volatility clustering
- Non-Gaussian “returns”
 - Or noise residuals
- Term-structure of volatility
 - Samuelson effect

Koekebakker and Ollmar (2005)

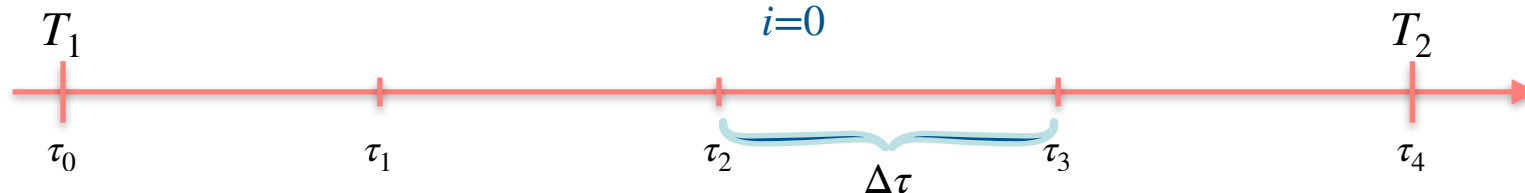
Feron and Gruet (2020) Geman (2005)

Brent Crude Oil front month: daily log returns



1. Fixed vs. periodic delivery: $F(t, T)$ vs. $\widehat{F}(t, T_1, T_2)$

- Power price denominated in EUR/MWh.
 - A short position pays $(T_2 - T_1) \widehat{F}(t, T_1, T_2)$
- No-arbitrage relationship

$$(T_2 - T_1) \widehat{F}(t, T_1, T_2) = \sum_{i=0}^{n-1} (\tau_{i+1} - \tau_i) \widehat{F}(t, \tau_i, \tau_{i+1})$$


- Taking the limit $\Delta\tau \rightarrow 0$

$$\widehat{F}(t, T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \widehat{F}(t, \tau, \tau) d\tau := \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} F(t, \tau) d\tau$$

1. Fixed vs. periodic

- We only need stochastic models for fixed delivery forwards!

$$(t, T) \mapsto F(t, T), \quad t \leq T$$

1. Fixed vs. periodic....

- Musiela parametrisation: $x = T - t$
 - *Time TO delivery*, instead of *time OF delivery*

$$f(t, x) := F(t, t + x), \quad t \geq 0, x \geq 0$$

- Periodic delivery forwards in Musiela parametrisation:

$$\widehat{F}(t, t + x, t + x + d) =: \hat{f}_d(t, x) = \int_0^\infty w_d(x, y) f(t, y) dy$$

$$x = T_1 - t \quad d = T_2 - T_1$$

2. Classical models

- Factor spot models (Lucia & Schwartz (2002))

$$S(T) = X_1(T) + X_2(T) \quad dX_i(t) = -\alpha_i X_i(t)dt + \sigma_i dB_i(t)$$

$$\text{Corr}(dB_1, dB_2) = \rho dt$$

- Forward dynamics

$$F(t, T) = \mathbb{E}[S(T) \mid \mathcal{F}_t]$$

$$dF(t, T) = \sigma_1 e^{-\alpha_1(T-t)} dB_1(t) + \sigma_2 e^{-\alpha_2(T-t)} dB_2(t)$$

- Musiela dynamics

$$df(t, x) = \partial_x f(t, x)dt + \underbrace{\begin{bmatrix} \sigma_1 e^{-\alpha_1 x} & \sigma_2 e^{-\alpha_2 x} \end{bmatrix}}_{\text{Samuelson effect}} \begin{bmatrix} \rho & \sqrt{1 - \rho^2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix}$$

Correlation matrix Indep. BM's

2. Classical models

- 5-factor dynamics

$$df(t, x) = \partial_x f(t, x) dt + \left[\sigma_1 e^{-\alpha_1 x} \quad \dots \quad \sigma_5 e^{-\alpha_5 x} \right] C \begin{matrix} \text{Independent BM's} \\ \left[\begin{array}{c} dW_1(t) \\ \vdots \\ dW_5(t) \end{array} \right] \end{matrix}$$

- 5×5 correlation matrix C :
 - $4+3+2+1=10$ unknown cross-correlations, with restrictions
- **High number** of unknowns to estimate
 - Correlations and parameters of Samuelson effect
- **Note:** 5 contracts spans out the whole forward market
 - Any maturity can be “hedged” using 5 contracts with given maturities

2. Classical models

- Extend to infinite factor models?
- Infinite number of parameters?

3. Infinite dimensional models

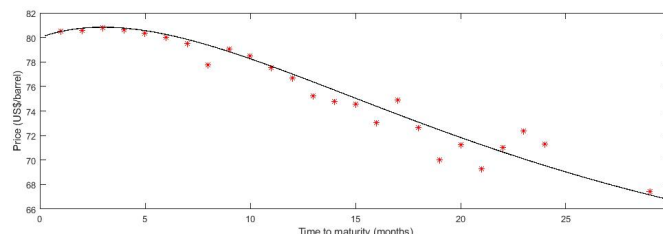
- HJM-model on Hilbert space

Samuelson effect



$$df(t) = \partial_x f(t)dt + \mathcal{M}dW(t)$$

- Hilbert space H of functions on positive real line



$$f(t) \in H, x \mapsto f(t, x), x \geq 0$$

- $W(t)$ is Wiener process
 - Covariance operator

$$\langle Qf, g \rangle = \mathbb{E}[\langle W(1), g \rangle \langle W(1), h \rangle]$$

3. Infinite dimensional models: covariance

- **Infinite** number of parameters in Q ?

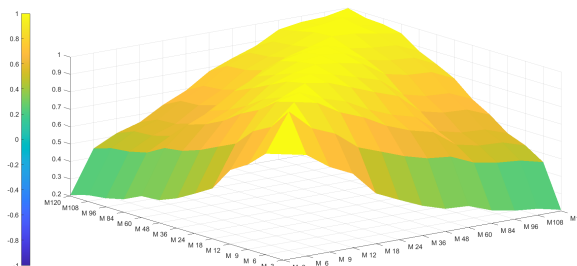
$$Qf(x) \sim \int_0^\infty q(x, y)f(y)dy$$

$$q(x, y) = \mathbb{E}[W(1, x)W(1, y)]$$

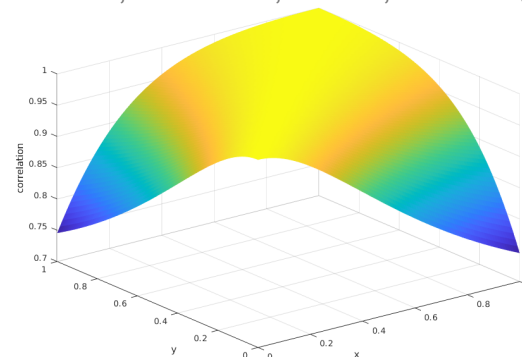
- **NO!**
 - integral operators
 - Correlation structure

- Ex: US Treasury forward rates
 - **2(!)** parameters

$$q(x, y) = \frac{c^2 + (1 - e^{-rx})(1 - e^{-ry})}{\sqrt{c^2 + (1 - e^{-rx})^2} \sqrt{c^2 + (1 - e^{-ry})^2}}$$



B., Paraschiv, Russo, Schurle (2023+)



3. Infinite dimensional models

- Some properties of the correlation term structure
 - “Flat” correlation structure for large maturities

$$\lim_{x \rightarrow \infty} k(x, x + \Delta) = 1$$

- Correlation decays to a positive constant

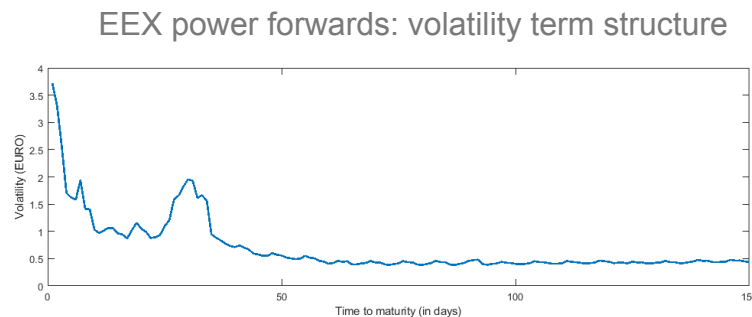
$$\lim_{\Delta \rightarrow \infty} k(x, x + \Delta) > 0$$

- Hilbert space H : Filipovic space
 - (weakly)-differentiable functions on the positive real line

$$\langle f, g \rangle = f(0)g(0) + \int_0^\infty e^{-\beta x} f'(x)g'(x)dx$$

3. Infinite dimensional models: Samuelson effect

- Define as $\mathcal{M} \in L(H)$
- Multiplication operator(s)
 - H is Banach algebra
 - Simple Samuelson effect
- Projections onto subspaces
 - “Multi-factor” analogue



B. Paraschiv (2018)

$$\mathcal{M}g(x) := e^{-\alpha x}g(x)$$

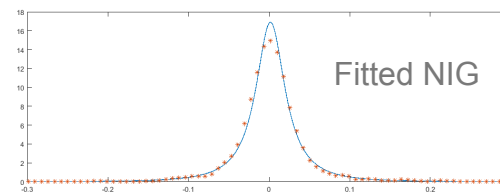
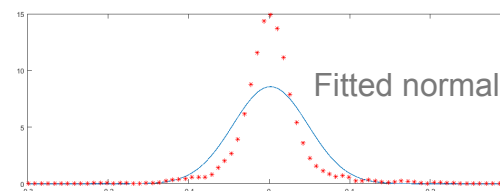
$$\mathcal{M}g(x) := e^{-\alpha_1 x}\langle g, \phi_1 \rangle \phi_1(x) + e^{-\alpha_2 x}\langle g, \phi_2 \rangle \phi_2(x)$$

3. Infinite dimensional models: non-Gaussianity

- Levy process $L(t)$ in H
 - Subordination of the Wiener process
- Subordinator $S(t)$
 - Increasing Levy process
- Covariance operator preserved
 - Marginals are Levy processes
 - Example NIG: IG-subordinator

$$L(t) := W(S(t))$$

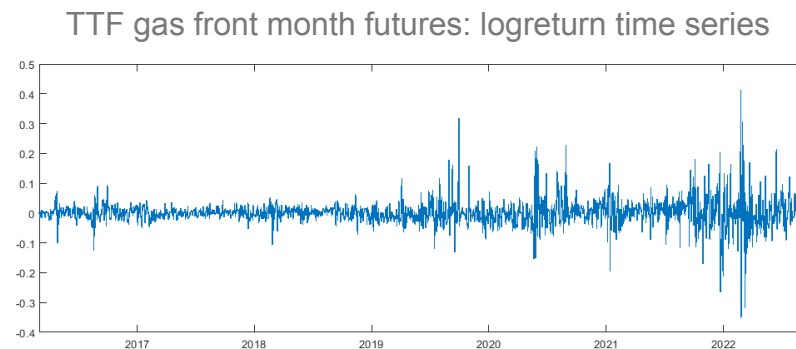
TTF gas front month futures: log returns
March 1 2016 to Sept 8 2022



$$t \mapsto L(t, x) \sim NIG$$

4. Stochastic volatility (SV) in ∞ -dimensions

- Volatility clustering in data
 - Sign of stochastic volatility
- SV across maturities
 - Linear operator on Hilbert space H
 - Positive definite Hilbert-Schmidt operator $L_{\text{HS}}(H)$
- **BNS-model** for SV
 - Barndorff-Nielsen & Shephard



$$\Sigma(t) := \mathcal{V}^{1/2}(t)$$

$$d\mathcal{V}(t) = \mathfrak{C}\mathcal{V}(t)dt + d\mathcal{L}(t)$$

B. Ruediger, Suess 2020
Cox, Karbach, Khedher 2022

4. SV in ∞ -dimensions

- SV with leverage and Samuelson effect
 - B. and Sgarra (2022)

$$df(t) = \partial_x f(t)dt + \mathcal{M}\Sigma(t)dW(t) + \rho d\mathcal{L}(t)$$

- Leverage: linear operator

$$\rho : L_{\text{HS}}(H) \rightarrow H$$

- Recall: $\mathcal{L}(t)$ noise in stochastic variance process

$$\mathcal{L}(t) = \sum_{k=1}^{N(t)} \mathcal{J}_k \quad \mathcal{J}_k \text{ iid positive definite } \in L_{\text{HS}}(H)$$
$$\mathcal{J}_k = Y_k^{\otimes 2} \quad Y_k \in H \text{ Gaussian}$$

5. Option pricing

- Options on fixed and delivery period forwards
 - European, exercise time at $\tau \leq T$ and payoff function p

$$p(F(\tau, T))$$

Fixed delivery

$$p(F(\tau, T_1, T_2))$$

Delivery period

$$p(x) = \max(x - K, 0)$$

Example: call

- Payoff represented by linear functionals

$$p(F(\tau, T)) = p(\delta_{T-\tau} f(\tau))$$

$$p(F(\tau, T_1, T_2)) = p\left(\frac{1}{T_2 - T_1} \int_{T_1-\tau}^{T_2-\tau} \delta_z f(\tau) dz\right)$$

$$p(\Lambda f(\tau)) \quad \Lambda \in H^*$$

5. Option pricing: Fourier

- Pricing in general using Fourier approach

$$p(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{p}(y) e^{ixy} dy \qquad \hat{p}(y) = \int_{\mathbb{R}} p(x) e^{-ixy} dx$$

- Option price

$$P(t, \tau) := \mathbb{E} [p(\Lambda f(\tau)) \mid \mathcal{F}_t] = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{p}(y) \mathbb{E} [\exp(iy \Lambda f(\tau))] dy$$

- Characteristic exponent known in leverage SV-model

$$\begin{aligned} \widehat{\mathbb{E}} [\exp(iy \Lambda f(\tau)) \mid \mathcal{F}_t] &= \exp \left(iy \Lambda \mathcal{S}(\tau - t) \underbrace{f(t)}_{\text{red circle}} - iy \Lambda \int_0^{\tau-t} \mathcal{S}(s) ds \rho \widehat{\mathbb{E}} [\mathcal{L}(1)] \right) \\ &\quad \times \exp \left(-\frac{1}{2} y^2 \Lambda \int_0^{\tau-t} \mathcal{S}(s) \mathcal{M} \mathcal{D}^{1/2} (\mathfrak{S}(\tau - t - s) \underbrace{\mathcal{Y}(t)}_{\text{red circle}}) \mathcal{D}^{1/2} \mathcal{M}^* \mathcal{S}^*(s) ds \Lambda^* 1 \right) \\ &\quad \times \exp \left(\int_0^{\tau-t} \widehat{\Psi}_{\mathcal{L}} \left(i \frac{y^2}{2} \int_0^s \mathfrak{S}(s-u)^* (\mathcal{D}^{1/2} \mathcal{M}^* \mathcal{S}(u)^* \Lambda^* 1)^{\otimes 2} du + y \rho^* \mathcal{S}(s)^* \Lambda^* 1 \right) ds \right) \end{aligned}$$

Springer Finance

Fred Espen Benth · Paul Krühner

Stochastic Models for Prices Dynamics in Energy and Commodity Markets

An Infinite-Dimensional Perspective

This monograph presents a theory for random field models in time and space, viewed as stochastic processes with values in a Hilbert space, to model the stochastic dynamics of forward and futures prices in energy, power, and commodity markets.

In this book, the well-known Heath–Jarrow–Morton approach from interest rate theory is adopted and extended into an infinite-dimensional framework, allowing for flexible modeling of price stochasticity across time and along the term structure curve. Various models are introduced based on stochastic partial differential equations with infinite-dimensional Lévy processes as noise drivers, emphasizing random fields described by low-dimensional parametric covariance functions instead of classical high-dimensional factor models. The Filipović space, a separable Hilbert space of Sobolev type, is found to be a convenient state space for the dynamics of forward and futures term structures. The monograph provides a classification of important operators in this space, covering covariance operators and the stochastic modeling of volatility term structures, including the Samuelson effect. Fourier methods are employed to price many derivatives of interest in energy, power, and commodity markets, and sensitivity 'delta' expressions can be derived. Additionally, the monograph covers forward curve smoothing, the connection between forwards with fixed delivery and delivery period, as well as the classical theory of forward and futures pricing.

This monograph will appeal to researchers and graduate students interested in mathematical finance and stochastic analysis applied in the challenging markets of energy, power, and commodities. Practitioners seeking sophisticated yet flexible and analytically tractable risk models will also find it valuable.

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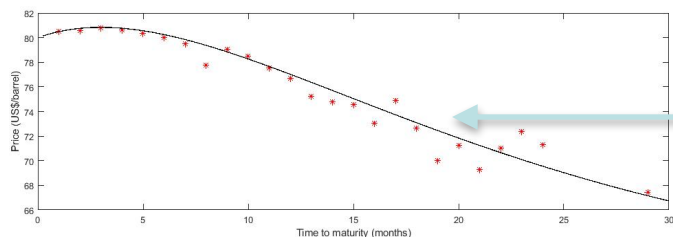
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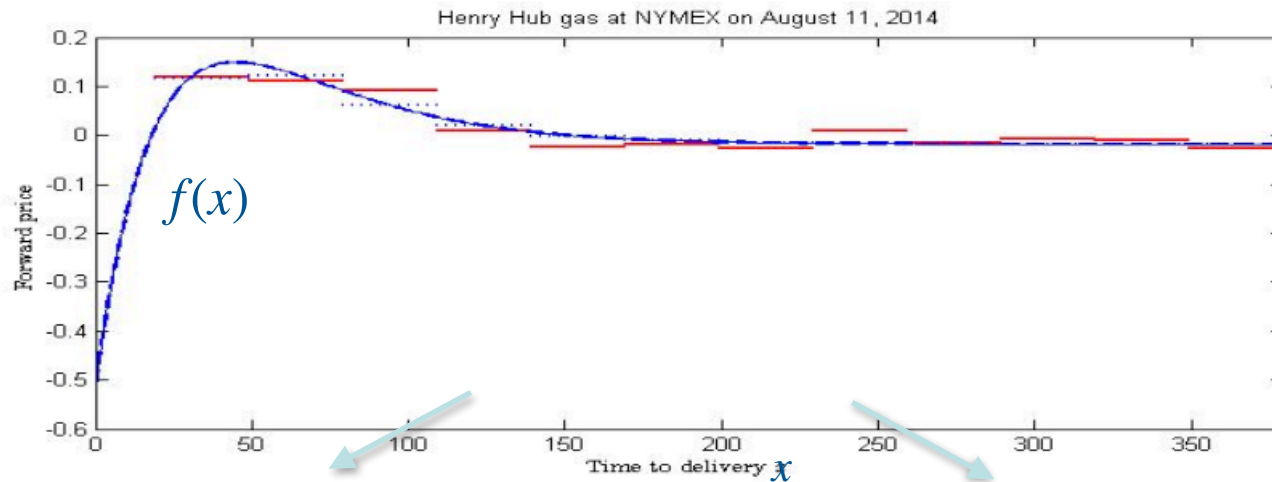
6. Neural nets (NN) and options

- Pricing via neural nets: Learning the map



$$f(t) \mapsto P(t, \tau; f(t))$$

- Price on delivery-period forward depends on the term structure!
 - ...and not on single contracts...
- Continuous map from H to the real line
 - Need NN on Hilbert space



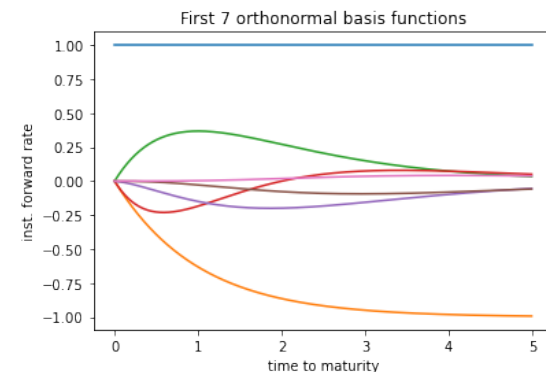
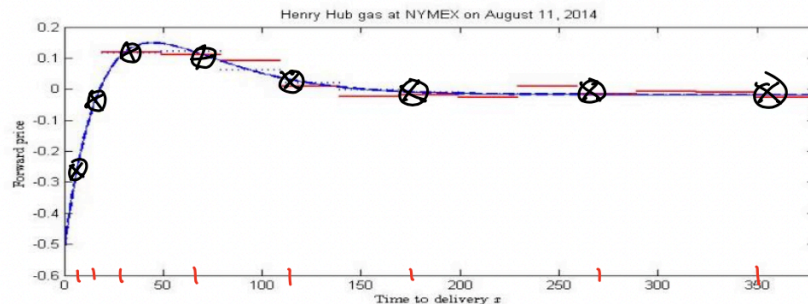
Classical neural network

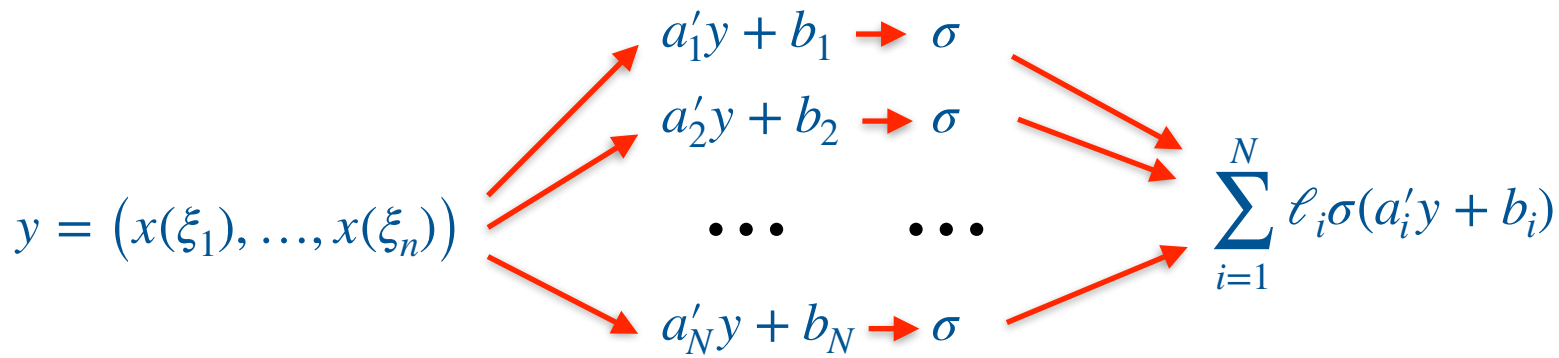
Train on sampled data
 $f(x_1), \dots, f(x_n)$ from
forward term structures

∞ -dimensional neural network

Train on samples of term
structure curves realized
as $f \in H$.

Adds structural information
from basis functions in H

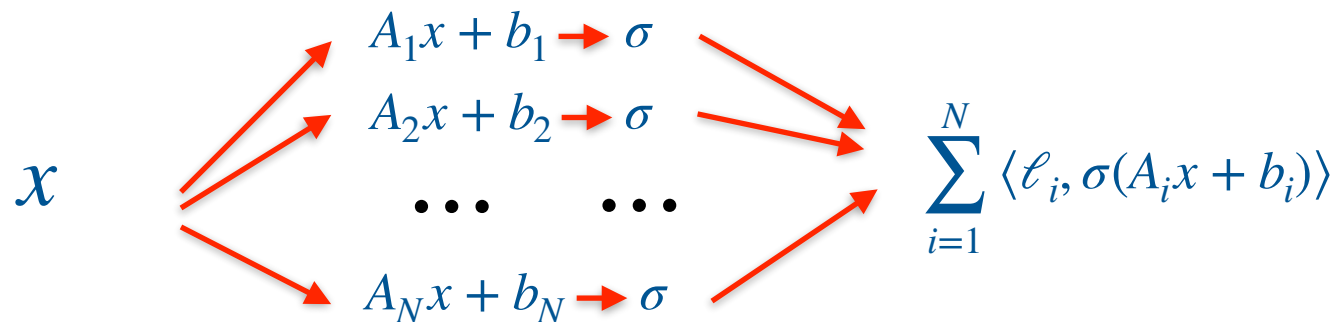




Input

Neurons

Output



$$A_i \in L(H) \quad b_i \in H \quad \sigma \in C(H) \quad \ell_i \in H$$

6. NN and options: mathematical idea

- Let μ be a measure on H supported on a compact
- Define for $g \in L^2(\mu)$

$$I(g) = \mathbb{E} \left[\int_H |p(\Lambda^{f^{t|\xi}}(\tau)) - g(\xi)|^2 \mu(d\xi) \right] \quad f^{t|\xi}(t) := \xi$$

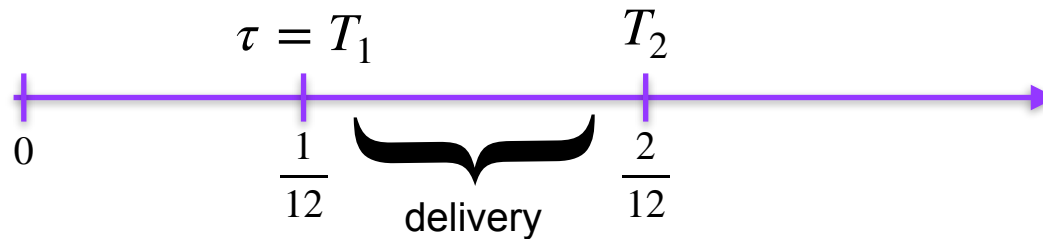
- **Proposition:** $I(P(t, \tau)) = \inf_{g \in L^2(\mu)} I(g)$

Proof: A direct calculation shows

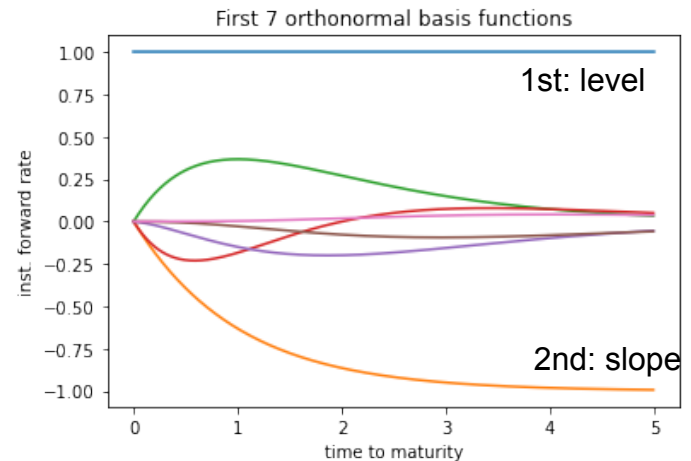
$$I(g) = \int_H \text{Var}(p(\Lambda^{f^{t|\xi}}(\tau))) \mu(d\xi) + \int_H |P(t, \tau; \xi) - g(\xi)|^2 \mu(d\xi) \geq I(V)$$

6. NN and options: example

- Delivery next month, and strike in one month at strike price 1
 - Exponential price model, constant volatility,
 - H is the Filipovic space



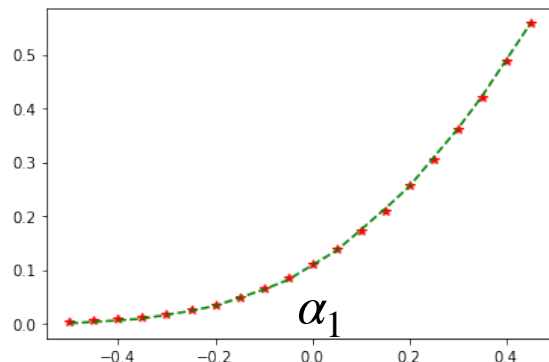
- Neural network may exploit information from basis functions



6. NN and options: example

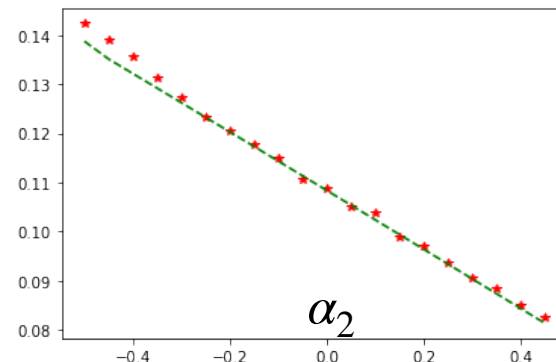
- **Trained** on one-layer NN, with 1 to 30 neurons.
 - Data are Monte Carlo-simulated prices
 - From a range of randomly drawn initial term structures ξ
- **Validated** on 10.000 initial curves ξ with corresponding “true” option prices
 - ξ ’s are randomly drawn from μ
 - “True” option prices from Monte Carlo simulations

Option price as a function of initial curve



Level, first coordinate

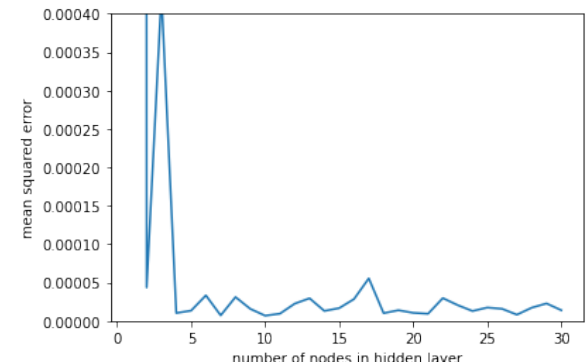
$$\alpha_2 = \alpha_3 = \dots = \alpha_7 = 0$$



Slope, second coordinate

$$\alpha_1 = \alpha_3 = \dots = \alpha_7 = 0$$

NN MSE



Nodes

THANK YOU FOR LISTENING!

References

- **Benth, Detering, Galimberti (2023)**. Neural networks in Frechet spaces. *Annals Math. Artificial Intellig*, 91:75–103
- **Benth, Detering, Galimberti (2024)**. Pricing options on flow forward by neural networks in Hilbert space. *Finance & Stochastics*, 21, pp 81–121, 2024.
- **Benth, Detering, Lavagnini (2021)**. Accuracy of deep learning in calibrating HJM forward curves. *Digital Finance*, 3(3-4):209–248
- **Benth, Kruhnert (2015)**. Derivatives pricing in energy markets: an infinite dimensional approach. *SIAM J. Financial Math.*, 6(1):825–869.
- **Benth, Paraschiv (2018)**. A structural model for electricity forward prices. *J. Banking & Finance*, 95: 203–216
- **Benth, Paraschiv, Russo, Schuerle (2023+)**. *In preparation*.
- **Benth, Ruediger, Suess (2018)**. Ornstein–Uhlenbeck processes in Hilbert space with non-Gaussian stochastic volatility. *Stoch. Proc. Applic.*, 128:461–486.
- **Benth, Schroers, Veraart (2020)**. A weak law of large numbers for realized covariation in a Hilbert space setting. *Stoch. Proc. Applic.*, 145:241–268
- **Benth, Sgarra (2021)**. A Barndorff-Nielsen and Shephard model with leverage in Hilbert space for commodity forward markets. *SSRN:abstract=3835053*
- **Cox, Karbach, Khedher (2022)**. An infinite-dimensional affine stochastic volatility model. *Math. Finance*, 32(3): 878–906
- **Feron, Gruet (2020)**. Estimation of the number of factors in a multi-factorial Heath-Jarrow-Morton model in electricity markets. *Preprint: hal-02880824*.
- **Geman (2005)**. *Commodities and Commodity Derivatives*. John Wiley & Sons, Chichester
- **Koekebakker, Ollmar (2005)**. The forward curve dynamics in Nordic electricity market. *Manag. Finance*, 31(6):74–95
- **Lucia, Schwartz (2002)**. Electricity prices and power derivatives: evidence from the Nordic power exchange. *Rev. Derivatives Res.*, 5(1):5–50
- **Samuelson (1965)**. Proof that properly anticipated prices fluctuate randomly. *Indust. Manag. Rev.*, 6:41–44