

The Fokker-Planck equation of the time-changed fractional Brownian motion and Ornstein-Uhlenbeck processes

Giacomo Ascione

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- 2 Weighted inverse subordination
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Outline

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Feller processes and abstract Cauchy problems

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It is well-known that for all $f \in C_0(\mathbb{R}^d)$ the abstract Cauchy problem

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Hence we can **represent the solution** by means of the conditional expectation $(\star)$.

How do we use the stochastic representation?

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- **From the PDE to the process:** e.g. the *not-feeling-the-boundary* principle.
- **From the process to the PDE:** e.g. determination stationary distribution.

Dzhrbashyan–Caputo fractional derivatives

Let $\alpha \in (0, 1)$ and consider the operator

$${}^c D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{\alpha-1} (f(\tau) - f(0)) d\tau,$$

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Denote by $\mathcal{L}$ the Laplace transform operator and assume that ${}^C D_{0+}^{\alpha} f$ admits abscissa of convergence $z_1 \in [-\infty, +\infty)$. Then

$$\mathcal{L}[{}^C D_{0+}^{\alpha} f](z) = z^{\alpha} \mathcal{L}[f](z) - z^{\alpha-1} f(0)$$

for $\Re(z) \geq \max\{0, z_1\}$.

Stable subordinators and their inverse

Let $\alpha \in (0, 1)$. An α -stable subordinator $S_\alpha = \{S_\alpha(t), t \geq 0\}$ is a Lévy process (i.e., a stochastically continuous process with stationary and independent increments) with characteristic function

$$\mathbb{E}[e^{-zS_\alpha(t)}] = e^{-tz^\alpha}, \quad z \geq 0.$$

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Since S_α is strictly increasing, L_α is increasing and continuous. Hence we can use it as a stochastic time. Furthermore, for all $t > 0$, $L_\alpha(t)$ admits density $f_\alpha(t, s)$ with

$$\mathcal{L}[f_\alpha(\cdot, s)](z) = z^{\alpha-1} e^{-sz^\alpha}$$

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Theorem [BM01]

Let $f \in \text{Dom}(\mathcal{G})$. Then the unique strong solution of

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A similar result holds for the *forward Kolmogorov equation*.

Gaussian processes and non-autonomous parabolic PDEs

Now consider $X = \{X(t), t \geq 0\}$ to be a 1-d Gaussian process in place of a Feller one. Hence, we drop the Markov property. We can still show that the density $p_X(t, x)$ of X satisfies a special PDE.

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We refer to this equation as the Fokker-Planck equation for X

Time-changing Gaussian processes

Let us consider the case of a Gaussian process X with constant mean. What happens to the Fokker-Planck equation if we apply the time-change to X ?

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$$\begin{aligned} & {}^c D_{0+}^\alpha p_X^\alpha(t, x) \\ &= I_{0+}^\alpha \left(\frac{\alpha}{2} \mathcal{L}^{-1} \left[\frac{1}{2\pi i} \int_{\gamma_\alpha} (\cdot^\alpha - z^\alpha) \mathcal{L}[V_X](\cdot^\alpha - z^\alpha) \mathcal{L}[p_X^\alpha(\cdot, x)](z^\alpha) dz \right] \right) (t), \end{aligned}$$

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where γ_α is a complex contour which is a transformation of another complex contour γ that leaves all the singularities of V_X on one side, provided all the involved quantities exist.

The considered equation involves the fractional integral of the inverse Laplace transform of a contour integral of a sort of convolution product of the Laplace transform of both the variance and the unknown itself (in Italian, we should end this sentence with *che al mercato mio padre compró*)...

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The equation is too complex to hope finding some useful information on p_X^α ! We need to better understand *what is the operator on the right-hand side*. That's what we are going to do here.

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- If u_α is the unique strong solution of (2), then it belongs to the range of IS^α and the function u such that $u_\alpha = IS^\alpha u$ is the unique strong solution of (1).

A class of admissible weights

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- We will work with Laplace transforms: we need $\mathcal{L}[\mathfrak{w}](z)$ to be well-defined for $\Re(z) > 0$
- We want the constant 1 to be in the domain of the *modified* inverse subordination operator: we need $\mathbb{E}[\mathfrak{w}[L_\alpha(t)]] < \infty$ for all $t > 0$.

We denote by $\mathcal{W}_\alpha$ the set of weights satisfying the previous assumptions. Observe that the constant 1 $\in \mathcal{W}_\alpha$.

Anticipation: we want the derivative of the variance of the parent Gaussian process to be an admissible weight!

Weighted inverse subordination

For a weight $\mathfrak{w} \in \mathcal{W}_\alpha$ we define the *weighted inverse subordination operator* $\text{IS}_\mathfrak{w}^\alpha : L^\infty(\mathbb{R}^+; \mathbb{B}) \rightarrow L^\infty(\mathbb{R}^+; \mathbb{B})$ as

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As a shorthand, we say that a function $v(t, x)$ defined for $t > 0$ and $x \in E \subset \mathbb{R}$ is dominated in E by a function $f(t)$ if $|v(t, x)| \leq f(t)$ for all $x \in E$, while it is locally dominated in E if for all compact subsets K of E there exists a function f_K such that $|v(t, x)| \leq f_K(t)$ for all $x \in K$.

Preservation of regularity

Now we consider only $\mathbb{B} = \mathbb{R}$. If we use a two variables function $v(t, x)$, the operator IS_w^α will be always assumed to work on t .

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If $\mathfrak{w} \in \mathcal{W}_{\alpha} \cap C(\mathbb{R}^+)$ and there exist $\mathfrak{w}_j \in \mathcal{W}_{\alpha}$ with $j = 1, 2$ such that $(-1)^j \mathfrak{w}_j$ is non-increasing and $\mathfrak{w} \leq \mathfrak{w}_1 + \mathfrak{w}_2$, then for all $v \in L^{\infty}(\mathbb{R}^+) \cap C(\mathbb{R}^+)$, $IS_{\mathfrak{w}}^{\alpha} v \in C(\mathbb{R}^+)$.

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- If $v(t, \cdot) \in C^n(a, b)$ is locally dominated, together with its partial derivatives in x , in (a, b) by functions $f_{[c, d]}$ with $\mathbb{E}[f_{[c, d]}(L_{\alpha}(t))\mathfrak{w}(L_{\alpha}(t))] < \infty$, then $IS_{\mathfrak{w}}^{\alpha} v(t, \cdot) \in C^n(a, b)$ with $\partial_x^n IS_{\mathfrak{w}}^{\alpha} v(t, \cdot) = IS_{\mathfrak{w}}^{\alpha} \partial_x^n v(t, \cdot)$.

Outline

- 1 Inverse subordinators and time-fractional equations
- 2 Weighted inverse subordination
- 3 The Fokker-Planck equation of the time-changed fractional Brownian motion**
- 4 The Fokker-Planck equation of the time-changed fractional Ornstein-Uhlenbeck processes
- 5 References

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Notation: we set $p^H(t, x) = p(t^{2H}, x)$.

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$$p^{H,f}(t, x) = \int_{\mathbb{R}} f(y) p^H(t, x - y) dy \quad (1)$$

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$$f \in L^\infty(\mathbb{R}) \Rightarrow p_\alpha^{H,f}(t, x) = \text{IS}^\alpha p^{H,f}(t, x)$$

An equation for $p^{H,f}$ and its Laplace transform

By adopting a simple change of variables, it is not difficult to check that $p^{H,f}$ satisfies the non-autonomous heat equation

$$\frac{\partial p^{H,f}(t,x)}{\partial t} = H t^{2H-1} \frac{\partial^2 p^{H,f}(t,x)}{\partial^2 x} \quad \text{with} \quad p^{H,f}(0,x) = f(x)$$

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Setting $\mathfrak{w}_H = Ht^{2H-1}$, we notice that it is a weight in $\mathcal{W}_\alpha$. Once we take the Laplace transform on both sides

$$z \mathcal{L}[p^{H,f}(\cdot, x)](z) - f(x) = \mathcal{L}_{\mathfrak{w}_H} \left[\frac{\partial^2 p^{H,f}(\cdot, x)}{\partial^2 x} \right] (z)$$

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This will be our starting point.

The Fokker-Planck equation for $p_{\alpha}^{H,f}$

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We evaluate the relation in z^{α}

$$z^{\alpha} \mathcal{L}[p^{H,f}(\cdot, x)](z^{\alpha}) - f(x) = \mathcal{L}_{\mathfrak{w}_H} \left[\frac{\partial^2 p^{H,f}(\cdot, x)}{\partial^2 x} \right] (z^{\alpha})$$

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We now manipulate the previous Laplace transform.

We multiply both sides by $z^{\alpha-1}$

$$z^{2\alpha-1} \mathcal{L}[p^{H,f}(\cdot, x)](z^{\alpha}) - z^{\alpha-1} f(x) = z^{\alpha-1} \mathcal{L}_{\mathfrak{w}_H} \left[\frac{\partial^2 p^{H,f}(\cdot, x)}{\partial^2 x} \right] (z^{\alpha})$$

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We use the formulae for the Laplace transform of IS^{α} and $IS_{\mathfrak{w}_H}^{\alpha}$

$$z^{\alpha} \mathcal{L}[p_{\alpha}^{H,f}(\cdot, x) - f(x)](z) = \mathcal{L} \left[IS_{\mathfrak{w}_H}^{\alpha} \frac{\partial^2 p^{H,f}(\cdot, x)}{\partial^2 x} \right] (z)$$

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We multiply everything by z^{-1}

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Once we have this, we can take the inverse Laplace transform to get

$$I_{0+}^{1-\alpha}(p_\alpha^{H,f}(\cdot, x) - f(x))(t) = \int_0^t \text{IS}_{\mathfrak{w}_H}^\alpha \frac{\partial^2 p^{H,f}(s, x)}{\partial^2 x} ds$$

where $I_{0+}^{1-\alpha}$ is the fractional integral of order $1 - \alpha$.

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$p^{H,f}(t, \cdot) \in C^2(\mathbb{R})$ and is locally dominated together with its derivative by a family of *good functions*. Hence

$$\text{IS}_{\mathfrak{w}_H}^{\alpha} \frac{\partial^2 p^{H,f}(s, x)}{\partial x^2} = \frac{\partial^2}{\partial x^2} \text{IS}_{\mathfrak{w}_H}^{\alpha} p^{H,f}(s, x) \text{ is continuous in } s$$

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Differentiating both sides of $(\star)$ we get the desired equation.

The Fokker-Planck equation for $p_\alpha^{H,f}$

$$I_{0+}^{1-\alpha}(p_\alpha^{H,f}(\cdot, x) - f(x))(t) = \int_0^t \text{IS}_{\mathfrak{w}_H}^\alpha \frac{\partial^2 p^{H,f}(s, x)}{\partial^2 x} ds \quad (\star)$$

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$$\text{IS}_{\mathfrak{w}_H}^\alpha \frac{\partial^2 p^{H,f}(s, x)}{\partial^2 x} = \frac{\partial^2}{\partial x^2} \text{IS}_{\mathfrak{w}_H}^\alpha p^{H,f}(s, x) \text{ is continuous in } s$$

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Theorem [APM24]

$p_\alpha^{H,f}$ satisfies for $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$

$${}^C D_{0+}^\alpha p_\alpha^{H,f}(t, x) = \frac{\partial^2}{\partial x^2} \text{IS}_{\mathfrak{w}_H}^\alpha p^{H,f}(t, x) \quad \text{with} \quad p_\alpha^{H,f}(0, x) = f(x)$$

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The operator $\frac{\partial^2}{\partial x^2} \text{IS}_{\mathfrak{w}_H}^\alpha$ substitutes the one obtained in [HKRU11]. It can be shown, with some cumbersome evaluations, that the two operators coincide.

Outline

- 1 Inverse subordinators and time-fractional equations
- 2 Weighted inverse subordination
- 3 The Fokker-Planck equation of the time-changed fractional Brownian motion
- 4 The Fokker-Planck equation of the time-changed fractional Ornstein-Uhlenbeck processes
- 5 References

The fractional Ornstein-Uhlenbeck process and its variance

Let $U^H = \{U^H(t), t \geq 0\}$ be a fractional Ornstein-Uhlenbeck process, solving for $\lambda > 0$ and $H \in (0, 1)$

$$dU_t^H = -\lambda U_t^H dt + dB_t^H \quad \text{with} \quad U_0^H = 0$$

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It is a centered Gaussian process whose variance is given by

$$V_H(t) = H \left(\int_0^t e^{-\lambda u} u^{2H-1} du + e^{-2\lambda t} \int_0^t e^{\lambda u} u^{2H-1} du \right).$$

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$$V_H'(t) \sim 2Ht^{2H-1} \quad \text{as } t \downarrow 0$$

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If $H > \frac{1}{2}$, then $V_H'/2$ is bounded and, in particular, belongs to $\mathcal{W}_\alpha$.

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We will argue as in [AMP22], without independent shift.

In what follows we have to exclude the case $x = 0$, since $q^H(t, 0)$ weakly converges towards the Dirac delta measure as $t \rightarrow 0^+$. Furthermore, we assume that $H > 1/2$.

An equation for q^H and its Laplace transform

For $t > 0$ and $x \in \mathbb{R} \setminus \{0\}$, the function $q^H(t, x)$ satisfies

$$\frac{\partial q^H(t, x)}{\partial t} = \frac{V'_H(t)}{2} \frac{\partial^2 q^H(t, x)}{\partial x^2} \quad \text{with} \quad q^H(0, x) = 0.$$

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Observe that $\mathfrak{w}_H := V'_H/2 \in \mathcal{W}_\alpha$. Taking the Laplace transform on both sides

$$z \mathcal{L}[q^H(\cdot, x)](z) = \mathcal{L}_{\mathfrak{w}_H} \left[\frac{\partial^2 q^H(\cdot, x)}{\partial x^2} \right] (z).$$

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For $x \neq 0$, the function $t > 0 \mapsto q^H(t, x)$ is bounded, hence we have $q_\alpha^H(t, x) = \text{IS}^\alpha q^H(t, x)$.

The Fokker-Planck equation for q_{α}^H

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We use the formulae for the Laplace transform of IS^{α} and $IS_{\mathfrak{w}_H}^{\alpha}$

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Once we have this, we can take the inverse Laplace transform to get

$$I_{0+}^{1-\alpha} q_\alpha^H(t, x) = \int_0^t \text{IS}_{\mathfrak{w}_H}^\alpha \frac{\partial^2 q^H(s, x)}{\partial^2 x} ds$$

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$q_\alpha^H(t, \cdot)$ weakly converges towards the Dirac delta measure $\delta(dx)$ as $t \rightarrow 0$.

This clearly implies that ${}^C D_{0+}^\alpha q_\alpha^H(t, 0)$ is not well-defined. It could be interesting to understand how to extend these equations to generalized functions.

Time-Regularity of q_α^H

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For fixed $x \in \mathbb{R} \setminus \{0\}$ the function $q^H(\cdot, x)$ is Lipschitz.

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For fixed $x \in \mathbb{R} \setminus \{0\}$ the function $q_\alpha^H(\cdot, x)$ belongs to $C^1(0, \infty)$ with derivative

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In particular, for $x \in \mathbb{R} \setminus \{0\}$, $q_\alpha^H(\cdot, x)$ is absolutely continuous.

As a consequence, we can evaluate ${}^C D_{0+}^\alpha q_\alpha^H(t, x)$ as

$${}^C D_{0+}^\alpha q_\alpha^H(t, x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-1} \mathbb{E} \left[L_\alpha(s) \frac{\partial q_\alpha^H(L_\alpha(s), x)}{\partial t} \right] ds.$$

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It can be verified that despite $q^H(t, \cdot) \in C^1(\mathbb{R})$, $q_\alpha^H(t, \cdot)$ admits an angular point in 0. The same holds for $IS_{w_H}^\alpha q^H$.

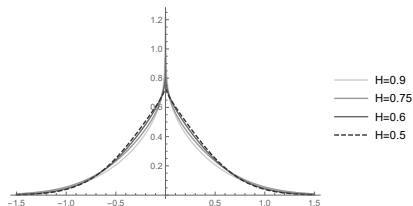


Figure: $q_\alpha^H(t, x)$ for $t = 1$ and different values of H , [AMP21]

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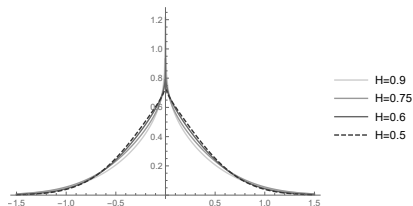


Figure: $q_\alpha^H(t, x)$ for $t = 1$ and different values of H , [AMP21]

Hence, even if we could find a *meaning* to ${}^C D_{0+}^\alpha q_\alpha^H(t, 0)$, the right-hand side of the Fokker-Planck equation cannot be defined in a classical way, since $\frac{\partial^2 IS_{w_H}^\alpha q^H(t, 0)}{\partial x^2}$ does not exist.

Possible future improvements

- **Independent shift for the fOU**: if we consider a random variable X independent of U^H and L_α and such that its density $f_X \in C_c^\infty(\mathbb{R})$, then we expect the density of the process $U^H + X$ to solve the same Fokker-Planck equation on $\mathbb{R}^+ \times \mathbb{R}$ with initial data f_X . This implies that q_α^H is the **fundamental solution** of the Fokker-Planck equation.

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- **Different initial value:** if we assume that $U^H(0) \neq 0$, then the equation could drastically change. If $\mathbb{E}[U^H(0)] = 0$ and $U^H(0)$ is absolutely continuous and independent of B^H , then we know that the density of U^H still solves the same Fokker-Planck equation, but with initial data equal to the density of $U^H(0)$ [APM24]. This is not true if $\mathbb{E}[U^H(0)] \neq 0$. This could be handled with two separate weights.

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Outline

- 1 Inverse subordinators and time-fractional equations
- 2 Weighted inverse subordination
- 3 The Fokker-Planck equation of the time-changed fractional Brownian motion
- 4 The Fokker-Planck equation of the time-changed fractional Ornstein-Uhlenbeck processes
- 5 References

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Thank you for the attention!!!