

Continuity of data-to-solution map for stochastic Camassa-Holm type equations

@ Workshop on Stochastics, Memory and Roughness

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Some known effects of noise

Generally, from a stochastic fluid model

$$dX = \overbrace{b(t, X(t))}^{\text{drift from fluid PDE}} dt + \text{noise},$$

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- **Prevent/delay singularities:** [Fedrizzi & Flandoli, *JFA*, 2013](linear transport equation/noise), [Glatt-Holtz & Vicol, *AoP*, 2014](Euler + linear noise), [Flandoli, Galeati, & Luo, *CPDE*, 2021](nonlinear equations + transport noise), [Tang & F-Y. Wang, *arXiv:2208.08312*, 2022](nonlinear equations + large noise), [Röckner, Zhu & Zhu, *SPA*, 2014] (SQG + linear noise).

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- **Enhance decay/dissipation:** [Tang, SIMA, 2018] (Camassa-Holm + multiplicative noise, decay of peakons) [Galeati, SPDE, 2020](linear transport equations + transport noise, enhance dissipation in scaling limit), [Li, Liu & Tang, SPA, 2021](MHD + multiplicative noise, enhance decay in H^s), [Tang & Z-A. Wang, CCM, 2023](Keller-Segel + multiplicative noise, enhance decay in L^∞), [Luo, JDDE, 2023](NS + transport noise, enhance dissipation).

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For example: the dependence on initial data for Euler equations is **at most** continuous [[Himonas & Misiotek, CMP, 2010](#)], but for the NS equations, it is **at least Lipschitz continuous**, see [[Henry, Book, 1981](#)].

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In **deterministic** case, results of **optimal dependence** on initial data can found in

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Schrödinger equation not continuous from certain H^s to H^{-1} [Tao, JFA, 2006]

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In **stochastic** case, there is one main difficulty, i.e., **lack of lifespan estimate**.

If we expect to see that X_n tends to X when $X_n(0)$ tends to $X(0)$, it seems that we implicitly require that:

Implicit requirement

there is a **common time interval** $[0, \tau]$ such that X_n exists on $[0, \tau]$ for all n .

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Even though we are certain that X_n exists on the interval $[0, \tau_n]$ with a probability $\mathbb{P}(\tau_n > 0) = 1$ for every $n \geq 1$, sometimes it remains ambiguous whether $\mathbb{P}(\inf_n \tau_n > 0) = 1$ or not.

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A new concept on the stability

In our recent works, we introduce a new notion about the **stability on the exiting time** and then consider the connections between this stability and dependence on initial data.

Stability of exiting time

For a given SPDE, assume we can solve it in certain Hilbert space $\mathbb{H}$ and $\mathbb{H} \hookrightarrow \mathbb{M}$ for a large Hilbert space $\mathbb{M}$.

Definition (Stability of exiting time)

Let $X_0, X_{0,n} \in \mathbb{H}$ ($n \geq 1$) be $\mathcal{F}_0$ -measurable random variables. Let $X, X_n \in \mathbb{H}$ be the corresponding (unique) solutions to a target SPDE with initial value X_0 and $X_{0,n}$, respectively. For any $R > 0$ and $n \in \mathbb{N}$, define the R -exiting time as

$$\tau_n^R := \inf \{t \geq 0 : \|X_n(t)\|_{\mathbb{H}} > R\}, \quad \tau^R := \inf \{t \geq 0 : \|X(t)\|_{\mathbb{H}} > R\},$$

where $\inf \emptyset := \infty$.

- ① Let $R > 0$. The R -exiting time is said to be stable at X , if $X_{0,n} \rightarrow X_0$ in $\mathbb{H}$ $\mathbb{P}$ -a.s. implies

$$\lim_{n \rightarrow \infty} \tau_n^R = \tau^R \quad \mathbb{P}\text{-a.s.} \quad (2.1)$$

- ② Let $R > 0$. If $X_{0,n} \rightarrow X_0$ in $\mathbb{M}$ $\mathbb{P}$ -a.s. **also implies** (2.1), then the R -exiting time is said to be **strongly** stable at X .

Stochastic Camassa-Holm type equations

Let $\{W_k, \widetilde{W}_k\}_{k \geq 1}$ is a family of independent 1-D Brownian motions. Consider the stochastic Camassa-Holm (CH) type system

$$du + \left[u^k \partial_x u + F(u) \right] dt = \sum_{k=1}^{\infty} \mathcal{A}_k u \circ dW_k + \sum_{k=1}^{\infty} h_k(t, u) d\widetilde{W}_k \quad (2.2)$$

where $\{\mathcal{A}_k\}$ is a sequence of differential operators, $\{h_k\}$ is a sequence of nonlinear functions, and

$$F(u) := F_1(u) + F_2(u) + F_3(u) \quad \begin{cases} F_1(u) := (1 - \partial_x^2)^{-1} \partial_x (u^{k+1}), \\ F_2(u) := \frac{2k-1}{2} (1 - \partial_x^2)^{-1} \partial_x (u^{k-1} u_x^2), \\ F_3(u) := \frac{k-1}{2} (1 - \partial_x^2)^{-1} (u^{k-2} u_x^3). \end{cases} \quad (2.3)$$

For the deterministic counterpart of (2.2), i.e., $\mathcal{A}_k = 0$ and $h = 0$,

- $k = 1$, CH equation.
- $k = 2$, Novikov equation.

We also note that, the case that the case $\mathcal{A}_k = \xi_k \partial_x$ is known as transport type noise.

Stochastic Camassa-Holm type equations

Some known results on stochastic **CH type** equations:

Local well-posedness [Chen, Gao, Guo, JDE, 2015] (CH + additive noise), [Tang, SIMA, 2018] (CH + multiplicative noise), [Alonso-Orán, Rohde & Tang, JNLS, 2021] (CH2 + transport noise), [Albeverio, Brzeźniak & Daletskii, JDE, 2021] (CH + transport noise), [Holden, Karlsen & Pang, DCDS, 2022] (CH + viscosity + transport noise), [Chen, Duan & Gao, AAP, 2023] (rotation CH2), [Tang & Yang, Annales de l'IHP, 2023] (CH with Coriolis effect + multiplicative noise) [Tang, JFA, 2023] (CH + **pseudo-differential** noise).

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Global existence [Rohde & Tang, JDDE, 2021] (CH with $k \geq 1$ + linear multiplicative noise, **sign condition** on initial data), [Ren, Tang & Wang, POTA, 2023] (CH + large **mean field** noise, **unconditional** global solution).

Weak instability

Theorem (Weak instability: general/rough statement)

If $\mathcal{A}_k = 0$ (we focus on pure multiplicative noise) and when h_k satisfies certain growth/decay conditions, then for (2.2), **at least one** of the following holds:

- 1 The exiting times of the zero solution is **not** strongly stable for all $R > 1$;
- 2 For any $T > 0$, the solution map $u_0 \mapsto u$ from $L^\infty(\Omega, H^s)$ to $L^1(\Omega; C([0, T]; H^s))$, is **not** uniformly continuous.

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(2.2) with $k \geq 2$ on $\mathbb{R}$: [\[Miao, Rohde & Tang. SPDE, 2023\]](#) (we also show the continuity of solution map from H^s to H^s so that it is nearly **optimal**).

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Non-local transport equation on $\mathbb{R}$: [Alonso-Orán, Miao & Tang. JDE, 2022] (non-local extension of uu_x).

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- Can we find such T_0 ?** If there is no further information, **so far, NOT good!** However, we can link this issue to the **stability of exiting time**. If the zero solution has strong stability in terms of exit time, then the problem can be handled.

Stabilization by pseudo-differential noise

Recall the stochastic Camassa-Holm (CH) type system (2.2):

$$du + \left[u^k \partial_x u + F(u) \right] dt = \sum_{k=1}^{\infty} \mathcal{A}_k u \circ dW_k + \sum_{k=1}^{\infty} h_k(t, u) d\widetilde{W}_k.$$

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By the following relation for a semi-martingale $f = f(t)$:

$$f \circ dW = f dW + \frac{1}{2} \langle f, W \rangle, \quad \langle \cdot, \cdot \rangle \text{ is the quadratic variation,}$$

one has

$$\sum_{k=1}^{\infty} \mathcal{A}_k u \circ dW_k = \sum_{k=1}^{\infty} \mathcal{A}_k u dW_k + \frac{1}{2} \sum_{k=1}^{\infty} \mathcal{A}_k^2 u dt.$$

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$$f \circ dW = f dW + \frac{1}{2} \langle f, W \rangle, \quad \langle \cdot, \cdot \rangle \text{ is the quadratic variation,}$$

one has

$$\sum_{k=1}^{\infty} \mathcal{A}_k u \circ dW_k = \sum_{k=1}^{\infty} \mathcal{A}_k u dW_k + \frac{1}{2} \sum_{k=1}^{\infty} \mathcal{A}_k^2 u dt.$$

So (2.2) becomes

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- $\mathcal{A}_k = \eta_k \mathbf{I} + (\xi_k \cdot \nabla)$: [Crisan, Flandoli & Holm, JNLS, 2018], [Goodair, Crisan & Lang, SPDE, 2023].
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Therefore, the **"correct"** question is: **in what sense** does the formal elliptic operator $\frac{1}{2} \sum_{k=1}^{\infty} \mathcal{A}_k^2$ contribute to stabilization?

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Therefore, the **"correct"** question is: **in what sense** does the formal elliptic operator $\frac{1}{2} \sum_{k=1}^{\infty} \mathcal{A}_k^2$ contribute to stabilization? **One of our ongoing work confirms this in the context of the scaling limit.**

Mean field case

Let $\mathbb{H}$ be a separable Hilbert space. The **mean-field** SDEs, or **McKean-Vlasov** SDEs, of the following type on $\mathbb{H}$:

$$dX(t) = b(t, X(t), \mathcal{L}_{X(t)}) dt + h(t, X(t), \mathcal{L}_{X(t)}) dW(t), \quad X(0) = X_0 \in \mathbb{H}, \quad t \in [0, T], \quad (3.2)$$

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In a similar spirit, we examine (3.1) in a certain random environment. If each path is subject to influences from all others, that is, each event is **shaped** by the **collective average behavior** of all events, it is reasonable to consider "**mean field noise**"

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We let $\mathbb{W}_{p,\mathbb{H}}(\cdot, \cdot)$ be L^p -Wasserstein distance on $\mathcal{P}(\mathbb{H})$ with $p \in [1, \infty)$. Since the fluid models are just locally Lipschitz, we assume that in (3.2), for any $N \geq 1$, there is a constant D_N such that for all $t \geq 0$:

$$\sup_{\|\xi\|_{\mathbb{H}}, \|\eta\|_{\mathbb{H}} \leq N, \mu, \nu \in \mathcal{P}(\mathbb{H})} \|b(t, \xi, \mu) - b(t, \eta, \nu)\|_{\mathbb{H}} + \|h(t, \xi, \mu) - h(t, \eta, \nu)\|_{\mathcal{L}_2(\mathbb{U}; \mathbb{H})} \leq D_N \{\|\xi - \eta\|_{\mathbb{H}} + \mathbb{W}_{2,\mathbb{H}}(\mu, \nu)\}$$

Difficulty

Estimate on two solutions?

For two solutions X, Y and $\tau := \inf\{t \geq 0 : \|X(t)\|_{\mathbb{H}} \vee \|Y(t)\|_{\mathbb{H}} > N\}$, we have the following question:

$$\begin{aligned} & \mathbb{E} \|X(t \wedge \tau) - Y(t \wedge \tau)\|_{\mathbb{H}}^2 \\ & \lesssim \cdots + \int_0^{t \wedge \tau} \left(\mathbb{W}_{2, \mathbb{H}}(\mathcal{L}_{X(s)}, \mathcal{L}_{Y(s)}) \right)^2 ds \stackrel{?}{\lesssim} \int_0^t \mathbb{E} \|X(s \wedge \tau) - Y(s \wedge \tau)\|_{\mathbb{H}}^2 ds? \end{aligned}$$

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Difficulty

As a consequence, locally Lipschitz condition is **no** longer sufficient for estimating difference of two solutions (hence not sufficient for pathwise uniqueness) in mean-field cases because we do **not** know

$$\left(\mathbb{W}_{2, \mathbb{H}}(\mathcal{L}_{X(s)}, \mathcal{L}_{Y(s)}) \right)^2 \stackrel{?}{\leq} C \mathbb{E} \|X(s \wedge \tau) - Y(s \wedge \tau)\|_{\mathbb{H}}^2 \text{ for some } C > 0?$$

Intuitively, do we have

$$(\mathbb{E} \|X(t)\|_{\mathbb{H}} - \mathbb{E} \|Y(t)\|_{\mathbb{H}})^2 \stackrel{?}{\leq} C \mathbb{E} \|X(s \wedge \tau) - Y(s \wedge \tau)\|_{\mathbb{H}}^2 \text{ for some } C > 0?$$

Difficulties

Usually, in deterministic case, we say that $(X(t))_{t \in [0, \infty)}$ is a **global-in-time** (classical) solution to some PDE if

for any $T > 0$, $(X(t))_{t \in [0, T]} = (X(t))_{t \in [0, \infty)} \Big|_{[0, T]}$ satisfies the PDE on $t \in [0, T]$.

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This **conflicts** to the natural belief that a **global** solution should be a **local** solution.

Distribution-path dependent SDEs

For a Banach space $\mathbb{B}$, we consider the [path space](#):

$$\mathcal{C}_{T,\mathbb{B}} := C([0, T]; \mathbb{B}), \quad \mathcal{C}_{T,\mathbb{B}}^w := \{\xi : [0, T] \rightarrow \mathbb{B} \text{ is weak continuous}\}.$$

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We state our **target problem** involving distribution and path:

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Local Wasserstein distance

Let $\xi \in \mathcal{C}_{T,\mathbb{B}}^w$ and define

$$t_n^\xi := T \wedge \inf\{t \geq 0 : \|\xi(t)\|_{\mathbb{B}} \geq n\}, \quad \xi \in \mathcal{C}_{T,\mathbb{B}}^w. \quad (3.5)$$

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For $\mu, \nu \in \mathcal{P}_{T,\mathbb{B}}$, we introduce the following "local" L^2 -Wasserstein distance

$$\mathbb{W}_{2,\mathbb{B},N}(\mu, \nu) = \inf_{\pi \in \mathcal{C}(\mu, \nu)} \left(\int_{\mathcal{C}_{T,\mathbb{B}} \times \mathcal{C}_{T,\mathbb{B}}} \|\xi_{t \wedge t_N^\xi \wedge t_N^\eta} - \eta_{t \wedge t_N^\xi \wedge t_N^\eta}\|_{T,\mathbb{B}}^2 \pi(d\xi, d\eta) \right)^{\frac{1}{2}}.$$

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We note that

- t_n^ξ is a continuous (hence measurable) function in ξ ;
- If X is an adapted random variable on $\mathcal{C}_{T,\mathbb{B}}^W$, then t_n^X is a stopping time.

Local Wasserstein distance

If $X, Y \in \mathcal{C}_{T, \mathbb{B}}$ are two adapted random variables, then $\pi_t := \mathcal{L}_{(X_t, Y_t)} \in \mathfrak{C}(\mathcal{L}_{X_t}, \mathcal{L}_{Y_t})$ is a probability measure on $\mathcal{C}_{T, \mathbb{B}} \times \mathcal{C}_{T, \mathbb{B}}$.

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$$\tau_n = t_n^X \wedge t_n^Y = T \wedge \inf\{t \geq 0 : \|X(t)\|_{\mathbb{H}} \vee \|Y(t)\|_{\mathbb{B}} \geq n\}, \quad n \geq 1,$$
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Estimate involving $\mathbb{W}_{2, \mathbb{B}, n}$

By definition, we arrive at

$$\mathbb{W}_{2, \mathbb{B}, n}(\mathcal{L}_{X_t}, \mathcal{L}_{Y_t})^2 \leq \int_{\mathcal{C}_{T, \mathbb{B}} \times \mathcal{C}_{T, \mathbb{B}}} F(\xi, \eta) \pi_t(d\xi, d\eta) = \mathbb{E} \|X_{\tau_n \wedge t} - Y_{\tau_n \wedge t}\|_{T, \mathbb{B}}^2.$$

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If $X, Y \in \mathcal{C}_{T, \mathbb{B}}$ are two adapted random variables, then $\pi_t := \mathcal{L}_{(X_t, Y_t)} \in \mathfrak{C}(\mathcal{L}_{X_t}, \mathcal{L}_{Y_t})$ is a probability measure on $\mathcal{C}_{T, \mathbb{B}} \times \mathcal{C}_{T, \mathbb{B}}$.

Define

$$\tau_n = t_n^X \wedge t_n^Y = T \wedge \inf\{t \geq 0 : \|X(t)\|_{\mathbb{H}} \vee \|Y(t)\|_{\mathbb{B}} \geq n\}, \quad n \geq 1,$$
$$F(\xi, \eta) := \|\xi_{t \wedge t_n^{\xi} \wedge t_n^{\eta}} - \eta_{t \wedge t_n^{\xi} \wedge t_n^{\eta}}\|_{T, \mathbb{B}}^2 = \sup_{s \in [0, t \wedge t_n^{\xi} \wedge t_n^{\eta}]} \|\xi(s) - \eta(s)\|_{\mathbb{B}}^2, \quad \xi, \eta \in \mathcal{C}_{T, \mathbb{B}}.$$

Estimate involving $\mathbb{W}_{2, \mathbb{B}, n}$

By definition, we arrive at

$$\mathbb{W}_{2, \mathbb{B}, n}(\mathcal{L}_{X_t}, \mathcal{L}_{Y_t})^2 \leq \int_{\mathcal{C}_{T, \mathbb{B}} \times \mathcal{C}_{T, \mathbb{B}}} F(\xi, \eta) \pi_t(d\xi, d\eta) = \mathbb{E} \|X_{\tau_{n \wedge t}} - Y_{\tau_{n \wedge t}}\|_{T, \mathbb{B}}^2.$$

With this, we can extend previous analysis to SPDEs with **path-distribution dependent noise**.

Local Wasserstein distance

If $X, Y \in \mathcal{C}_{T, \mathbb{B}}$ are two adapted random variables, then $\pi_t := \mathcal{L}_{(X_t, Y_t)} \in \mathfrak{C}(\mathcal{L}_{X_t}, \mathcal{L}_{Y_t})$ is a probability measure on $\mathcal{C}_{T, \mathbb{B}} \times \mathcal{C}_{T, \mathbb{B}}$.

Define

$$\tau_n = t_n^X \wedge t_n^Y = T \wedge \inf\{t \geq 0 : \|X(t)\|_{\mathbb{H}} \vee \|Y(t)\|_{\mathbb{B}} \geq n\}, \quad n \geq 1,$$
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Estimate involving $\mathbb{W}_{2, \mathbb{B}, n}$

By definition, we arrive at

$$\mathbb{W}_{2, \mathbb{B}, n}(\mathcal{L}_{X_t}, \mathcal{L}_{Y_t})^2 \leq \int_{\mathcal{C}_{T, \mathbb{B}} \times \mathcal{C}_{T, \mathbb{B}}} F(\xi, \eta) \pi_t(d\xi, d\eta) = \mathbb{E} \|X_{\tau_{n \wedge t}} - Y_{\tau_{n \wedge t}}\|_{T, \mathbb{B}}^2.$$

With this, we can extend previous analysis to SPDEs with **path-distribution dependent noise**.

- The **local** Wasserstein distance was **initially** proposed in [Ren, Tang & Wang, POTA, 2023] to handle uniqueness.
- In our **ongoing work**, it will be used to prove existence, uniqueness and stability of **global** solutions to stochastic CH equation with path-distribution dependent noise on $\mathbb{R}$ and $\mathbb{T}$.

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Summary

- Continuity of solution map is not well-studied for SPDEs;
- It requires some analysis on exiting times, so the stability on exiting times are proposed;
- It can be expected that pseudo-differential noise (including transport noise) has some stabilization effect;
- The essential part of continuity of solution map is estimating differences between two solutions. In mean field noise, we give a new metric to describe it.

THANK YOU!

HAO TANG

**Continuity of data-to-solution map for stochastic
Camassa-Holm type equations**

@ Workshop on Stochastics, Memory and Roughness

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