

Hausdorff and Large Deviation Multifractal Spectra of Lévy Multistable Processes

Ronan Le Guével & Jacques Lévy Véhel

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Lévy multistable processes

Lévy multistable processes provide useful models in various applications where the studied phenomena display jumps with varying intensity, such as financial records, EEG or natural terrains:

Indeed, multistability is one practical way to deal with (increments-) non-stationarities observed in many real-world data.

The mathematical reason is that a multistable process X is tangent, at each time u , to a stable process Z_u in the following sense:

$$\lim_{r \rightarrow 0} \frac{X(u + rt) - X(u)}{r^h} = Z_u(t)$$

for a suitable h , where the limit is taken either in finite dimensional distributions or, when X has a version with càdlàg paths, in distribution.

Lévy multistable processes

We are given the following ingredients:

- $\alpha : [0, 1] \rightarrow (1, 2)$ a $\mathcal{C}^1$ function. Denote $\alpha_* = \inf_{u \in [0, 1]} \alpha(u)$ and $\alpha^* = \sup_{u \in [0, 1]} \alpha(u)$.
- $(\Gamma_i)_{i \geq 1}$ a sequence of arrival times of a Poisson process with unit arrival time.
- $(V_i)_{i \geq 1}$ a sequence of i.i.d. random variables with uniform distribution on $[0, 1]$.
- $(\gamma_i)_{i \geq 1}$ a sequence of i.i.d. random variables with distribution $\mathbb{P}(\gamma_i = 1) = \mathbb{P}(\gamma_i = -1) = 1/2$.

The three sequences $(\Gamma_i)_{i \geq 1}$, $(V_i)_{i \geq 1}$, and $(\gamma_i)_{i \geq 1}$ are independent.

Lévy multistable processes

We consider two versions of Lévy multistable processes:

Independent non stationary increments Lévy multistable process

This process admits the following representation:

$$B(t) = \sum_{i=1}^{\infty} \gamma_i C_{\alpha(V_i)}^{1/\alpha(V_i)} \Gamma_i^{-1/\alpha(V_i)} \mathbf{1}_{[0,t]}(V_i),$$

where $C_u = \left(\int_0^{\infty} x^{-u} \sin x \, dx \right)^{-1}$.

Lévy multistable processes

Correlated non stationary increments

This process admits the following representation:

$$D(t) = C_{\alpha(t)}^{1/\alpha(t)} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(t)} \mathbf{1}_{[0,t]}(V_i),$$

Both processes are semi-martingales and are tangent, at each time t , to $\alpha(t)$ —stable Lévy motion.

Multifractal analysis

Multifractal analysis gives a fairly complete description of the singularity structure of measures, functions or stochastic processes.

Multifractal analysis may be performed in geometric, probabilistic or statistical ways, by determining respectively the Hausdorff, large deviation, and Legendre multifractal spectra.

Hausdorff multifractal spectrum

Denote by $h_Y(t)$ the pointwise Hölder exponent of the stochastic process Y at t . The Hausdorff multifractal analysis of Y consists in measuring the Hausdorff dimension $\dim_H$ of the level sets of $h_Y(t)$, that is computing the dimension of $F_h = \{t \in [0, 1] : h_Y(t) = h\}$.

The Hausdorff multifractal spectrum is the random function $h \mapsto f_H(h) := \dim_H F_h$.

Large deviation multifractal spectrum

The (increments-based) large deviation multifractal spectrum of a process Y on $[0, 1]$ is the random function f_g defined on $\mathbb{R}$ by

$$f_g(\beta) = \lim_{\varepsilon \rightarrow 0^+} \liminf_{n \rightarrow +\infty} \frac{\log N_n^\varepsilon(\beta)}{\log n},$$

where, for a positive integer n and $\varepsilon > 0$,

$$N_n^\varepsilon(\beta) = \text{Card}\{j \in \{0, \dots, n-1\} : \beta - \varepsilon \leq \frac{\log |Y(\frac{j+1}{n}) - Y(\frac{j}{n})|}{-\log n} \leq \beta + \varepsilon\}.$$

Hausdorff multifractal spectrum of Lévy multistable processes

Theorem

With probability one, the common Hausdorff multifractal spectrum f_H of B and D satisfies:

$$f_H(h) = \begin{cases} -\infty & \text{for } h < 0; \\ h\alpha^* & \text{for } h \in [0, \frac{1}{\alpha^*}]; \\ 1 & \text{for } h \in (\frac{1}{\alpha^*}, \frac{1}{\alpha_*}); \\ \dim_H(\{t \in [0, 1] : \alpha(t) = \alpha_*\}) & \text{for } h = \frac{1}{\alpha_*}; \\ -\infty & \text{for } h > \frac{1}{\alpha_*}. \end{cases}$$

Note that, as soon as α is not a constant, Lévy multistable processes provide a decomposition of a subset of $[0, 1]$ into an uncountable disjoint union of sets with Hausdorff dimension one.

Large deviation multifractal spectrum of the Independent non stationary increments Lévy multistable process

Theorem

With probability one, the large deviation multifractal spectrum of B satisfies:

$$f_g(h) = \begin{cases} -\infty & \text{for } h < 0; \\ h\alpha^* & \text{for } h \in [0, \frac{1}{\alpha^*}]; \\ 1 & \text{for } \beta \in (\frac{1}{\alpha^*}, \frac{1}{\alpha_*}]; \\ 1 + \frac{1}{\alpha_*} - h & \text{for } h \in (\frac{1}{\alpha_*}, 1 + \frac{1}{\alpha_*}]; \\ -\infty & \text{for } h > \frac{1}{\alpha_*}. \end{cases}$$

B provides an example where the so-called *multifractal formalism* does not hold, that is, $f_H \leq f_g = f_l$ and $f_H \neq f_g$. The decreasing part with slope -1 for “large” exponents present in f_g but not in f_H is a common phenomenon when variations are measured with increments.

Ideas of the proof for the Hausdorff spectrum

The first step of the proof consists in estimating precisely the pointwise Hölder exponent of Y at any point t .

Let: $S = \cup_i \{V_i\}$, $\mathcal{S} = S^{\mathbf{N}}$ and $\mathcal{R}_t = \{(r_n)_{n \in \mathbf{N}} \in \mathcal{S} : r_n \rightarrow t\}$. If $(r_n)_n \in \mathcal{R}_t$, we put $V_{\phi(n)} = r_n$.

Define finally the non-negative function δ for $t \notin S$ by:

$$\delta(t) = \inf_{(V_{\phi(n)})_{n \in \mathbf{N}} \in \mathcal{R}_t} \liminf_{i \rightarrow \infty} - \frac{\log \phi(i)}{\log |V_{\phi(i)} - t|},$$

Ideas of the proof for the Hausdorff spectrum

Lemma

Almost surely, for all $t \in [0, 1] \setminus S$, the pointwise Hölder exponent of Y at t is given by

$$h_Y(t) = \frac{\delta(t)}{\alpha(t)}.$$

Ideas of the proof for the Hausdorff spectrum

Computation of $f_H(h)$ for $h \in (0, \frac{1}{\alpha^*}]$

Consider the system of points $\mathcal{P} = \{(V_i, \lambda_i)\}_{i \geq 1}$ where $\lambda_i = \frac{1}{1+\Gamma_i}$ and define the approximation rate of t by $\mathcal{P}$ as

$$\delta_t(\mathcal{P}) = \sup\{\delta \geq 1 : t \text{ belongs to an infinite number of balls } B(V_i, \lambda_i^\delta)\}.$$

Lemma

$$\delta_t(\mathcal{P}) = \frac{1}{\delta(t)}.$$

The continuity of the function α then allows us to conclude.

Ideas of the proof for the large deviation spectrum

In order to prove the result on the large deviation spectrum, we first show that the equality in each case holds true for any given β with probability one. Permuting “for all β ” and “almost surely” is then obtained thanks to two simple but useful general lemmas on the large deviation spectrum.

Ideas of the proof for the large deviation spectrum

Lemma

The large deviation spectrum of any real function is an upper semicontinuous function.

Lemma

Assume that there exist four functions $\underline{h}, \overline{h}, \underline{g}, \overline{g}$ with $\lim_{u \rightarrow 0} \underline{g}(u) = \lim_{u \rightarrow 0} \overline{g}(u) = 0$ such that, for all β in some interval I and all sufficiently small $\varepsilon > 0$, almost surely

$$\underline{h}(\beta) + \underline{g}(\varepsilon) \leq \liminf_{n \rightarrow +\infty} \frac{\log N_n^\varepsilon(\beta)}{\log n} \leq \overline{h}(\beta) + \overline{g}(\varepsilon).$$

Then, almost surely, for all β in I , $\underline{h}(\beta) \leq f_g(\beta) \leq \overline{h}(\beta)$.

Ideas of the proof for the large deviation spectrum

Denote

$$\begin{aligned} P_n^j &= \mathbb{P} \left(\beta - \varepsilon \leq \frac{\log |B(\frac{j+1}{n}) - B(\frac{j}{n})|}{-\log n} \leq \beta + \varepsilon \right) \\ &= \mathbb{P} \left(\frac{1}{n^{\beta+\varepsilon}} \leq |B(\frac{j+1}{n}) - B(\frac{j}{n})| \leq \frac{1}{n^{\beta-\varepsilon}} \right). \end{aligned}$$

Set also $X_j = \mathbf{1}_{\{\frac{1}{n^{\beta+\varepsilon}} \leq |B(\frac{j+1}{n}) - B(\frac{j}{n})| \leq \frac{1}{n^{\beta-\varepsilon}}\}}$ which follows a Bernoulli law with parameter P_n^j .

Then,

$$N_n^\varepsilon(\beta) = \sum_{j=1}^n X_j.$$

Ideas of the proof for the large deviation spectrum

For U an open interval of $(0, 1)$, write $J_n(U) = \{j : j/n \in U\}$.

Lemma

Assume there exist $b \in (-1, 0)$ and $K > 0$ such that for n large enough and $j \in \{1, 2, \dots, n\}$, $P_n^j \leq Kn^b$. Then, almost surely,

$$\liminf_{n \rightarrow +\infty} \frac{\log N_n^\varepsilon(\beta)}{\log n} \leq 1 + b.$$

Lemma

Assume there exist an open interval U , a real $b \in (-1, 0)$ and $K > 0$ such that for n large enough and $j \in J_n(U)$, $P_n^j \geq Kn^b$. Then, almost surely,

$$\liminf_{n \rightarrow +\infty} \frac{\log N_n^\varepsilon(\beta)}{\log n} \geq 1 + b.$$

Ideas of the proof for the large deviation spectrum

The previous results allows us to prove a series of lemma that deal with the different cases of interest for computing the large deviation spectrum.

Let $c_U = \inf_{t \in U} \alpha(t)$, $d_U = \sup_{t \in U} \alpha(t)$ and $t_j = j/n$.

Lemma

Assume $\beta < \frac{1}{\alpha^*}$. Then there exists a constant $K > 0$ such that for n large enough and $j = 1, \dots, n$,

$$P_n^j \leq K n^{\alpha(t_j)\beta + \alpha(t_j)\varepsilon - 1}.$$

Lemma

Assume $\beta < \frac{1}{d_U}$. Then there exists a constant $K > 0$ such that for n large enough and $j \in J_n(U)$,

$$P_n^j \geq K n^{\alpha(t_j)\beta - \alpha(t_j)\varepsilon - 1}.$$

Ideas of the proof for the large deviation spectrum

Lemma

Assume $\beta > \frac{1}{\alpha_*}$ and $\varepsilon \in (0, \beta - \frac{1}{\alpha_*})$. Then there exists a constant $K > 0$ such that for n large enough and $j = 1, \dots, n$,

$$P_n^j \leq K n^{\frac{1}{\alpha(t_j)} + \varepsilon - \beta}.$$

Lemma

Assume $\beta > \frac{1}{c_U}$ and $\varepsilon \in (0, \beta - \frac{1}{c_U})$. Then there exists a constant $K > 0$ such that for n large enough and $j \in J_n(U)$,

$$P_n^j \geq K n^{\frac{1}{\alpha(t_j)} + \varepsilon - \beta}.$$

Ideas of the proof for the large deviation spectrum

Upper bounds estimates are obtained using the fact that, by Parseval formula,

$$P_n^j = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\sin(\frac{\xi}{n^{\beta-\varepsilon}}) - \sin(\frac{\xi}{n^{\beta+\varepsilon}})}{\xi} e^{-\int_{t_j}^{t_j+1} |\xi|^{\alpha(x)} dx} d\xi.$$

Ideas of the proof for the large deviation spectrum

For lower bounds estimates, it is useful to replace the indicator function by a smooth (in our case, four times differentiable) even function $\varphi : \mathbb{R} \mapsto [0, 1]$ which is not identically zero and vanishes outside of $[-1, 1]$. Then, setting $\mu = \frac{1}{2} \left(\frac{1}{n^{\beta+\varepsilon}} + \frac{1}{n^{\beta-\varepsilon}} \right)$ and $\sigma = \frac{1}{2} \left(\frac{1}{n^{\beta-\varepsilon}} - \frac{1}{n^{\beta+\varepsilon}} \right)$, one has, for all $x \in \mathbb{R}$:

$$\varphi\left(-\frac{x+\mu}{\sigma}\right) + \varphi\left(\frac{x-\mu}{\sigma}\right) \leq \mathbf{1}_{[-n^{-\beta+\varepsilon}, -n^{-\beta-\varepsilon}]}(x) + \mathbf{1}_{[n^{-\beta-\varepsilon}, n^{-\beta+\varepsilon}]}(x),$$

and thus:

$$\begin{aligned} P_n^j &\geq \int_{\mathbb{R}} \left(\varphi\left(-\frac{x+\mu}{\sigma}\right) + \varphi\left(\frac{x-\mu}{\sigma}\right) \right) \mathbb{P}(B(t_{j+1}) - B(t_j) \in dx) \\ &\geq \frac{2}{\pi} \int_0^{+\infty} \cos\left(\frac{\mu}{\sigma}\eta\right) \hat{\varphi}(\eta) e^{-\int_{t_j}^{t_{j+1}} \left|\frac{\eta}{\sigma}\right|^{\alpha(x)} dx} d\eta. \end{aligned}$$