

Effective fractional wave equations in random multiscale media

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↔ Experimental observations for body waves in the earth's crust gives **frequency dependent attenuation**.

- Empirically observed intensity 'apparent' power law attenuation for harmonic wave:

$$\frac{T_z}{T_0} = \exp(-C|\omega|^{(\alpha+1)}z), \quad \alpha \in (-1, 1).$$

How can we relate medium properties to a power law attenuation?

Geophysics and power laws

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$\alpha \simeq 0.4$ for low frequencies $< 10^{-3}\text{Hz}$ [Lekic 2009],

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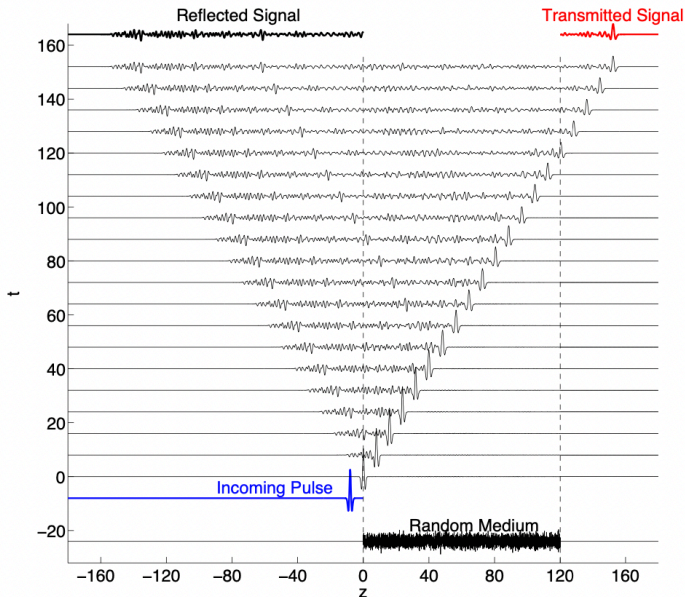
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- Recent applications to medical imaging; Holm & Sinkus.

Numerical experiment in one-dimensional random medium



Scalar wave equation in one-dimensional random medium

Scalar wave equation:

$$\frac{\partial^2 p^\varepsilon}{\partial z^2} - \frac{1}{c^\varepsilon(z)^2} \frac{\partial^2 p^\varepsilon}{\partial t^2} = 0$$

A section of random medium is sandwiched in between two homogeneous half-spaces:

$$\frac{1}{c^\varepsilon(z)^2} = \begin{cases} \frac{1}{c_o^2} \left(1 + \varepsilon v\left(\frac{z}{\varepsilon^2}\right) \right) & \text{for } z \in [0, L] \\ \frac{1}{c_o^2} & \text{for } z \in (-\infty, 0) \cup (L, \infty) \end{cases}$$

The background velocity is c_o .

$v(z)$ is a zero-mean, stationary, random process with covariance function:

$$\phi(z) := \mathbb{E}[v(y)v(y+z)]$$

Pulse wave incoming from the left with pulse profile

$$p^\varepsilon(t, z) = p_0\left(\frac{t - z/c_o}{\varepsilon^2}\right), \quad t \ll 0$$

Long-range and short-range correlations

→ Medium correlations $\phi(z) = \mathbb{E}[\mathbf{v}(y)\mathbf{v}(y+z)]$

- Long-range correlations: ϕ is not integrable and has a power decay at infinity:

$$\phi(z) \stackrel{|z| \rightarrow \infty}{\simeq} r_{H'} \left| \frac{z}{\ell_c} \right|^{2H'-2}$$

where $r_{H'} > 0$ and $H' \in (1/2, 1)$ (i.e., $2H' - 2 \in (-1, 0)$).

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$$\phi(z) \stackrel{|z| \rightarrow 0}{\simeq} \phi(0) \left(1 - d_H \left| \frac{z}{\ell_c} \right|^{2H} + O\left(\left| \frac{z}{\ell_c} \right|\right) \right)$$

where $d_H > 0$ and $H \in (0, 1/2)$ (i.e., $2H \in (0, 1)$).

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↔ Example media are fractional OU & fractional noise.

First - mixing case (1/3)

- $\phi(z) = \mathbb{E}[v(y)v(y+z)]$ is integrable and regular at zero (Lipschitz).

↪ The correlation length $\ell_c = \frac{1}{\phi(0)} \int_0^\infty \phi(z) dz$.

↪ The covariance function can be expanded as

$$\phi(z) = \phi(0) \left(1 - d_c \frac{|z|}{\ell_c} + o\left(\frac{|z|}{\ell_c}\right) \right), \quad |z| \ll \ell_c$$

- Example: Binary medium. The process v is stepwise constant.

→ $(l_j)_{j \geq 0}$: lengths of the elementary intervals.

→ $(n_j)_{j \geq 0}$: values $\in \{-\sigma, +\sigma\}$ taken by the process over each elementary interval, n_j are independent and identically distributed (i.i.d.):

$$\mathbb{P}(n_j = \pm\sigma) = \frac{1}{2}$$

The lengths l_j are i.i.d. with the probability density function (pdf):

$$p_h(z) = \frac{1}{\ell_c} \exp\left(-\frac{z}{\ell_c}\right) \mathbf{1}_{[0,\infty)}(z)$$

↪ The covariance function is: $\phi(z) = \sigma^2 \exp\left(-\frac{|z|}{\ell_c}\right)$

Effective pulse propagation - mixing case (2/3)

Introduce the random travel time

$$\tau^\varepsilon(L) = \frac{L}{c_0} + \varepsilon^2 \tau_c^\varepsilon(L), \quad \tau_c^\varepsilon(L) = \frac{1}{2c_0\varepsilon} \int_0^L v\left(\frac{z}{\varepsilon^2}\right) dz$$

In the limit $\varepsilon \rightarrow 0$, at $z = L$,

1) the *random time shift* $\tau_c^\varepsilon(L)$ converges in distribution to a Gaussian random variable $\tau_c(L)$ ($\sim$ Brownian motion) with mean zero and variance:

$$\mathbb{E}[\tau_c(L)^2] = \frac{\hat{\phi}_c(0)\ell_c}{8c_0^2} L$$

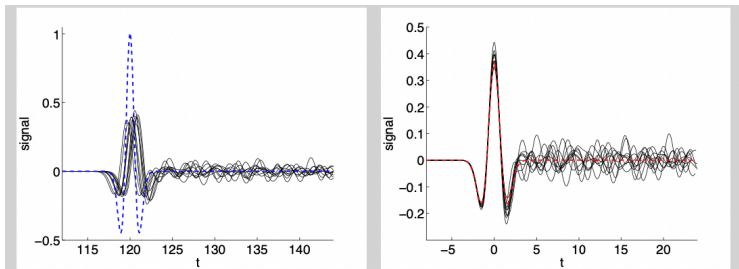
2) the pulse profile converges in probability to a *deterministic profile*:

$$p^\varepsilon(z = L, t = \tau^\varepsilon(L) + \varepsilon^2 \tau) \xrightarrow{\varepsilon \rightarrow 0} p_L(\tau)$$

$$p_L(\tau) = \frac{1}{2\pi} \int \hat{p}_0(\omega) T_L(\omega) e^{-i\omega\tau} d\omega, \quad T_L(\omega) = \exp\left(-\frac{\omega^2[\hat{\phi}_c(\omega) + i\hat{\phi}_s(\omega)]}{8c_0^2} L\right)$$

$$\hat{\phi}_c(\omega) := 2 \int_0^\infty \phi(z) \cos\left(\frac{2\omega z}{c_0}\right) dz, \quad \hat{\phi}_s(\omega) := 2 \int_0^\infty \phi(z) \sin\left(\frac{2\omega z}{c_0}\right) dz.$$

Effective pulse propagation - mixing case (2/3)



Time profiles of the transmitted wave at $z = L$

Without time shift

Black: ten numerical results
obtained with ten realizations
of the random medium

Blue: original pulse
 $p_0(t)$

With time shift

Red: theoretical (deterministic)
transmitted pulse shape

$p_L(t)$

Remarks - mixing case (3/3)

- Remark 1: Known as O'Doherty-Anstey theory in geophysics (70's). Proved in mathematics by Papanicolaou and co (90's).

- If $\frac{\omega \ell_c}{c_0} \ll 1$ (the wavelength $\gg \ell_c$ probes the tail of $\phi(z)$), then:

$$T_L(\omega) = \exp\left(-\frac{\hat{\phi}_c(0)}{8} \frac{\omega^2 \ell_c^2}{c_0^2} \frac{L}{\ell_c}\right)$$

$\hookrightarrow$ Effective second-order diffusion (attenuation), no effective dispersion.

Here $\alpha = 1$.

- Remark 2: What happens if $\int_0^\infty \phi(z) dz = \infty$?

- If $\frac{\omega \ell_c}{c_0} \gg 1$ (the wavelength $\ll \ell_c$ probes the small- z behavior of $\phi(z)$), then:

$$T_L(\omega) = \exp\left(-\frac{\phi(0)d_c}{16} \frac{L}{\ell_c} - i \frac{\phi(0)}{8} \frac{\omega \ell_c}{c_0} \frac{L}{\ell_c}\right)$$

$\hookrightarrow$ Effective constant attenuation, no effective dispersion, additional time shift.

Here $\alpha = -1$.

- Remark 3: What happens if $\phi(z)$ is not regular at zero ?

Binary medium - short- and long-range correlations

The process v is **stepwise constant**.

→ $(l_j)_{j \geq 0}$: lengths of the elementary intervals.

→ $(n_j)_{j \geq 0}$: values $\in \{-\sigma, +\sigma\}$ taken by the process over each elementary interval.

The values n_j are i.i.d. with the distribution

$$\mathbb{P}(n_j = \pm\sigma) = \frac{1}{2}$$

The lengths l_j are i.i.d. with the pdf:

$$\begin{aligned} p_{l_i}(z) = & \left(1 - q\right) \frac{1 - 2H}{\delta^{2H-1} - 1} \frac{\ell_c^{1-2H}}{z^{2-2H}} \mathbf{1}_{[\ell_i, \ell_c]}(z) \\ & + q(3 - 2H') \frac{\ell_c^{3-2H'}}{z^{4-2H'}} \mathbf{1}_{[\ell_c, \infty)}(z), \end{aligned}$$

where $H \in (0, 1/2)$, $H' \in (1/2, 1)$, $\delta = \ell_i/\ell_c$, and $0 < \ell_i \ll \ell_c$.

Binary medium - short- and long-range correlations, Continued

The covariance function has both H -short-range and H' -long-range properties:

- if $|z| \geq \ell_c$, then

$$\phi(z) \simeq \frac{a\sigma^2}{(2-2H')\left(\frac{1-2H}{2H} + a\frac{3-2H'}{2-2H'}\right)} \frac{\ell_c^{2-2H'}}{|z|^{2-2H'}}$$

which shows it has the H' -long-range property in the range $|z| \in [\ell_c, \infty)$.

- if $\ell_i \ll |z| \ll \ell_c$, then

$$\phi(z) \simeq \sigma^2 \left\{ 1 - \frac{1}{1-2H + a2H\frac{3-2H'}{2-2H'}} \frac{|z|^{2H}}{\ell_c^{2H}} \right\}$$

which shows it has the H -short-range property in the range $|z| \in (\ell_i, \ell_c)$.

- if $|z| \leq \ell_i$, then

$$\phi(z) \simeq \sigma^2 \left\{ 1 - \frac{1}{\left(\frac{1-2H}{2H} + a\frac{3-2H'}{2-2H'}\right)\delta^{1-2H}} \frac{|z|}{\ell_c} \right\}$$

which shows it has the 'regular' property in the range $|z| \in [0, \ell_i]$.

Analysis of pulse propagation (1/3)

Write wave equation as a hyperbolic system:

$$\begin{cases} \frac{\partial p^\varepsilon}{\partial z} + \frac{\partial u^\varepsilon}{\partial t} = 0 \\ \frac{\partial u^\varepsilon}{\partial z} + \frac{1}{c^\varepsilon(z)^2} \frac{\partial p^\varepsilon}{\partial t} = 0 \end{cases}$$

Introduce the right- and left-going wave amplitudes:

$$\begin{bmatrix} A^\varepsilon(t, z) \\ B^\varepsilon(t, z) \end{bmatrix} = \begin{bmatrix} c^\varepsilon(z)^{-1/2} p^\varepsilon(t, z) + c^\varepsilon(z)^{1/2} u^\varepsilon(t, z) \\ -c^\varepsilon(z)^{-1/2} p^\varepsilon(t, z) + c^\varepsilon(z)^{1/2} u^\varepsilon(t, z) \end{bmatrix}$$

$$\frac{\partial}{\partial z} \begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix} = -\frac{1}{c^\varepsilon(z)} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix} + \frac{\partial_z c^\varepsilon(z)}{2c^\varepsilon(z)} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix}$$

Incoming wave:

$$A^\varepsilon(t, z) = 2c_0^{-1/2} p_0 \left(\frac{t - z/c_0}{\varepsilon^2} \right), \quad B^\varepsilon(t, z) = 0, \quad t < 0$$

- transport along the random characteristics with the local sound speed $c^\varepsilon(z)$.
- coupling between the right- and left-going modes.

$$\frac{\partial}{\partial z} \begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix} = -\frac{1}{c^\varepsilon(z)} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix} + \frac{\partial_z c^\varepsilon(z)}{2c^\varepsilon(z)} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix}$$

Main steps of the analysis:

- rewrite the evolution equations of the modes by centering along the characteristic of the right-going mode: we obtain an lower-triangular system
- apply limit theorems to this system in order to establish an effective equation for the wave front.

Analysis of pulse propagation (2/3)

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$$P_{k+1} = \text{diag}[Q_k, Q_{k+1}, \dots] P_k, \quad \text{with Gouipillaud medium}$$

S SIAP 2003, Fouque et al 2007, Garnier and S JASA 2010

Second Effective pulse propagation - long-range correlations (1/4)

- Covariance function:

$$\phi(z) \stackrel{|z| \rightarrow \infty}{\simeq} r_{H'} \left| \frac{z}{\ell_c} \right|^{2H'-2}, \quad H' \in (1/2, 1)$$

- Introduce the random travel time

$$\tau^\varepsilon(L) = \frac{L}{c_0} + \varepsilon^{3-2H'} \tau_c^\varepsilon(L), \quad \tau_c^\varepsilon(L) = \frac{1}{2c_0 \varepsilon^{2-2H'}} \int_0^L v\left(\frac{z}{\varepsilon^2}\right) dz$$

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1) the *random time shift* $\tau_c^\varepsilon(L)$ converges in distribution to a zero-mean random variable with variance of order one,

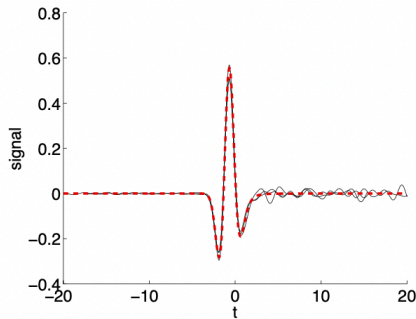
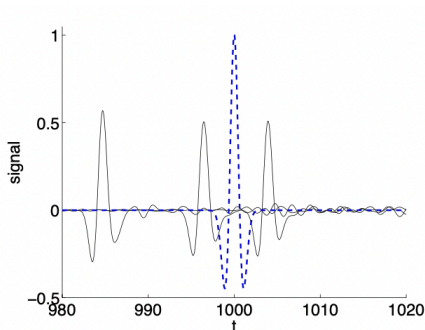
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Note that the time shift is of order $\varepsilon^{3-2H'} \gg \varepsilon^2$ (here $H' > 1/2$, i.e.

$3 - 2H' < 2$): **the random time shift is larger than the deterministic deformation.**

Effective pulse propagation - long-range correlations (2/4)



Blue: original pulse profile $p_0(t)$

Red: theoretical transmitted profile $p_L(t)$

Black: results of three numerical simulations (with centering in the right picture).

Here $H' = 0.875$.

Effective pulse propagation - long-range correlations (3/4)

More precisely:

1) the random time shift $\tau_c^\varepsilon(L)$ converges in distribution to a Gaussian random variable $\tau_c(L)$ with mean zero and variance ($\sim$ fractional Brownian motion):

$$\mathbb{E}[\tau_c(L)^2] = \frac{\ell_c^{2-2H'}}{c_o^2} \frac{r_{H'}}{4H'(2H'-1)} L^{2H'}$$

2) If $\frac{\omega \ell_c}{c_o} \ll 1$, then

$$T_L(\omega) = \exp\left(-\frac{r_{H'}}{2} \frac{\Gamma(2H'-1)}{2^{2H'}} \cos\left(\left(H'-\frac{1}{2}\right)\pi\right) \left(\frac{|\omega| \ell_c}{c_o}\right)^{3-2H'} \frac{L}{\ell_c} \right. \\ \left. - i \frac{r_{H'}}{2} \frac{\Gamma(2H'-1)}{2^{2H'}} \sin\left(\left(H'-\frac{1}{2}\right)\pi\right) \left(\frac{|\omega| \ell_c}{c_o}\right)^{2-2H'} \frac{\omega \ell_c}{c_o} \frac{L}{\ell_c} \right)$$

$\hookrightarrow$ **Effective fractional diffusion (attenuation)** $\sim |\omega|^{3-2H'}$, $3-2H' \in (1,2)$.

$\hookrightarrow$ **Effective fractional dispersion** $\sim |\omega|^{2-2H'} \omega$.

Here $\alpha = 2-2H' \in (0,1)$.

Effective pulse propagation - long-range correlations (4/4)

Effective fractional wave equation (up to random time correction):

$$\frac{\partial^2 p}{\partial z^2} - \frac{1}{c_o^2} \frac{\partial^2 p}{\partial t^2} = \frac{r_{H'} \ell_c^{2-2H'}}{2^{2H'} c_o^{4-2H'}} \int_0^\infty \frac{1}{s^{2-2H'}} \frac{\partial^3 p}{\partial t^3}(t-s) ds$$

Same form as wave equations used in sound propagation in lossy media;

→ slightly different from the standard models [M. Caputo, Geophys. J. R.

Astron. Soc. 1, 529 (1967), T. L. Szabo, J. Acoust. Soc. Am. 96, 491 (1994)];

using Riemann-Liouville fractional derivative (that does not respect causality);

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→ Finally the long- and short-range correlation case.

$$\frac{\ln |T_L(\omega)|}{L} \approx \begin{cases} -\left(\frac{\omega \ell_c}{c_o}\right)^{3-2H'} \frac{\Gamma(2H'-1) \cos((H'-1/2)\pi)}{2^{2H'}(3-2H')(1+C)} & , \frac{\omega}{c_o} \ll \frac{1}{\ell_c} , \\ -\left(\frac{\omega \ell_c}{c_o}\right)^{1-2H} \frac{\Gamma(1+2H) \sin(H\pi)}{2^{2+2H}(1-2H)(1+C^{-1})} & , \frac{1}{\ell_c} \ll \frac{\omega}{c_o} \ll \frac{1}{\ell_i} , \\ -\frac{H\delta^{2H-1}}{4(1-2H)(1+C^{-1})} & , \frac{\omega}{c_o} \gg \frac{1}{\ell_i} , \end{cases}$$

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• Summary:

- For low frequencies $\frac{\omega}{c_o} \ll \frac{1}{\ell_c}$, the H' -long-range correlation property gives an exponent $\alpha = 2 - 2H' \in (0, 1)$.
- For mid frequencies $\frac{1}{\ell_c} \ll \frac{\omega}{c_o} \ll \frac{1}{\ell_i}$, the H -short-range correlation property gives an exponent $\alpha = -2H \in (-1, 0)$.
- For high frequencies $\frac{\omega}{c_o} \gg \frac{1}{\ell_i}$, the regular property gives an exponent $\alpha = -1$.

• Remember the experimental findings:

- $\alpha \simeq 0.4$ for low frequencies $< 10^{-3}\text{Hz}$ [Lekic 2009],
- $\alpha \in (-0.4, 0)$ for mid frequencies $(10^{-3}, 1)\text{Hz}$ [Choy 1986, Shito 2004],
- $\alpha \simeq -1$ for high frequencies $> 1\text{Hz}$ [Choy 1986, Cormier 2011].

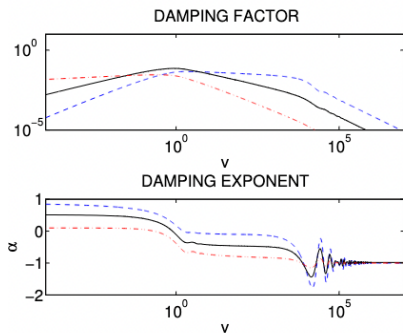
Conclusions

- In a random medium with a covariance function that decays at infinity as $|z|^{2H'-2}$, $H' \in (1/2, 1)$, the attenuation factor at low frequency corresponds to $\alpha = 2 - 2H' \in (0, 1)$.
- In a random medium with a covariance function that behaves at zero like $1 - d_H|z|^{2H}$, $H \in (0, 1/2)$, the attenuation factor at high frequency corresponds to $\alpha = -2H \in (-1, 0)$.

Conclusions

- In a random medium with a covariance function that decays at infinity as $|z|^{2H'-2}$, $H' \in (1/2, 1)$, the attenuation factor at low frequency corresponds to $\alpha = 2 - 2H' \in (0, 1)$.
- In a random medium with a covariance function that behaves at zero like $1 - d_H|z|^{2H}$, $H \in (0, 1/2)$, the attenuation factor at high frequency corresponds to $\alpha = -2H \in (-1, 0)$.
- A special frequency-dependent phase is associated with the frequency-dependent attenuation and it ensures that causality and Kramers-Kronig relations are respected.
- Effective fractional wave equations can be written that have the form of equations studied in the literature in the context of wave propagation in lossy media.
- A simple binary medium with short- and long-range correlation properties can explain the frequency dependence of the apparent attenuation observed for seismic body waves.

Illustration frequency dependent damping



Here (H, H') are $(.05, .55)$, $(.25, .75)$, and $(.45, .95)$ for respectively the dashed blue, solid black, and dash-dotted red lines, and in all cases $\delta = 10^{-4}$.

Remark: Atmospheric turbulence

- Modeling in terms of a random medium:

$$n^2(\mathbf{x}) = \frac{c_o^2}{c^2(\mathbf{x})} = 1 + \mu(\mathbf{x}), \quad \mathbf{x} = (\mathbf{x}, z) \in \mathbb{R}^{d+1},$$

- The power spectral density of μ :

$$\begin{aligned} \mathbb{S}(\vec{\mathbf{k}}) &= \chi_\beta \mathbf{1}_{(L_o^{-1}, l_o^{-1})}(|\vec{\mathbf{k}}|) |\vec{\mathbf{k}}|^{-d-\beta}, \\ \text{Cov}_\mu(\mathbf{x}) &= \mathbb{E} [\mu(\mathbf{x}') \mu(\mathbf{x}' + \mathbf{x})] \end{aligned}$$

with χ_β is a constant, $\beta \in (0, 1) \cup (1, 2)$ ¹ classic Kolmogorov turbulence $\sim \beta = 5/3$, clouds $\sim \beta < 1$ and (l_o, L_o) inertial range.

¹Garnier, Borcea & S SIAP '23.

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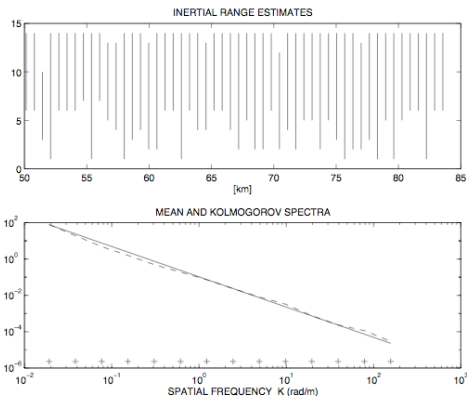
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→ Centering via $\sim$ fBm with $H = (1 + \beta)/2$. → Wave in random frame characterized via Itô-Schrödinger:

$$d_z a(z, \mathbf{x}) = \frac{i}{2k_o} \Delta_{\mathbf{x}} a(z, \mathbf{x}) dz + \frac{ik_o}{2} a(z, \mathbf{x}) \circ d_z B(z, \mathbf{x})$$

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Remark: Atmospheric data



Correlation range of slope fluctuations $\sim 1\text{km}$.

Solid line is Kolmogorov slope ($H=1/3$) dashed line is mean slope[†].

[†] *Papanicolaou & S, Wavelet based estimation of Kolmogorov turbulence.*

Long-range Dependence: Theory and Applications, Eds Taqqu et al '02.