

Drift parameter estimation in Vasicek-type model driven by tempered fractional Brownian motion

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Introduction

Two broad classes of continuous stochastic Gaussian processes have been introduced recently:

- **Tempered fractional Brownian motion (TFBM)**



Mark M. Meerschaert and Farzad Sabzikar.

Tempered fractional Brownian motion.

Stat. Probab. Lett., 83(10):2269–2275, 2013.

- **Tempered fractional Brownian motion of the second kind (TFBMII)**



Farzad Sabzikar and Donatas Surgailis.

Tempered fractional Brownian and stable motions of second kind.

Stat. Probab. Lett., 132:17–27, 2018.

These processes modify the power-law kernel in the moving average representation of a fractional Brownian motion (FBM) by incorporating an exponential tempering.

Preliminaries of the tempered processes

Let $W = \{W_s, s \in \mathbb{R}\}$ be a two-sided Wiener process, $H > 0$, $\lambda > 0$.

Definition 1 ([Meerschaert and Sabzikar (2013)])

The stochastic process $B_{H,\lambda}^I = \{B_{H,\lambda}^I(t)\}_{t \in \mathbb{R}}$ defined by the Wiener integral

$$B_{H,\lambda}^I(t) := \int_{\mathbb{R}} [g_{H,\lambda,t}^I(s)] dW_s, \quad (1)$$

where

$$g_{H,\lambda,t}^I(s) := e^{-\lambda(t-s)+} (t-s)_+^{H-\frac{1}{2}} - e^{-\lambda(-s)+} (-s)_+^{H-\frac{1}{2}}, \quad s \in \mathbb{R},$$

is called a **tempered fractional Brownian motion** (TFBM).

Definition 2 ([Sabzikar and Surgailis (2018)])

The stochastic process $B_{H,\lambda}^{\parallel} = \{B_{H,\lambda}^{\parallel}(t)\}_{t \in \mathbb{R}}$ defined by the Wiener integral

$$B_{H,\lambda}^{\parallel}(t) := \int_{\mathbb{R}} g_{H,\lambda,t}^{\parallel}(s) dW_s, \quad (2)$$

where

$$\begin{aligned} g_{H,\lambda,t}^{\parallel}(s) := & e^{-\lambda(t-s)+} (t-s)_+^{H-\frac{1}{2}} - e^{-\lambda(-s)+} (-s)_+^{H-\frac{1}{2}} \\ & + \lambda \int_0^t (u-s)_+^{H-\frac{1}{2}} e^{-\lambda(u-s)+} du, \quad s \in \mathbb{R}, \end{aligned}$$

is called a **tempered fractional Brownian motion of the second kind (TFBMII)**.

Basic properties of TFBM and TFBMII

- 1 TFBM (1) and TFBMII (2) are Gaussian stochastic processes with stationary increments, having the following scaling property: for any scaling factor $c > 0$

$$\{X_{H,\lambda}(ct)\}_{t \in \mathbb{R}} \stackrel{\Delta}{=} \{c^H X_{H,c\lambda}(t)\}_{t \in \mathbb{R}}, \quad (3)$$

where $X_{H,\lambda}$ could be $B_{H,\lambda}^I$ or $B_{H,\lambda}^{II}$ (here $\stackrel{\Delta}{=}$ denotes the equality of all finite-dimensional distributions).

2 From (3)

$$\mathbb{E}[(X_{H,\lambda}(|t|))^2] = |t|^{2H} \mathbb{E}[(X_{H,\lambda|t|}(1))^2] := |t|^{2H} C_t^2, \quad (4)$$

where the function C_t^2 has the following explicit representation.

If $X_{H,\lambda} = B_{H,\lambda}'$, then

$$C_t^2 = (C_t')^2 = \mathbb{E}[(B_{H,\lambda|t|}'(1))^2] = \frac{2\Gamma(2H)}{(2\lambda|t|)^{2H}} - \frac{2\Gamma(H + \frac{1}{2})}{\sqrt{\pi}} \frac{1}{(2\lambda|t|)^H} K_H(\lambda|t|),$$

where $t \neq 0$ and $K_\nu(z)$ is the modified Bessel function of the second kind.

If $X_{H,\lambda} = B_{H,\lambda}''$, then

$$\begin{aligned} C_t^2 = (C_t'')^2 &= \mathbb{E}[(B_{H,\lambda|t|}''(|1|))^2] = \frac{(1 - 2H)\Gamma(H + \frac{1}{2})\Gamma(H)(\lambda t)^{-2H}}{\sqrt{\pi}} \\ &\times \left[1 - {}_2F_3\left(\{1, -1/2\}, \{1 - H, 1/2, 1\}, \lambda^2 t^2/4\right) \right] \\ &+ \frac{\Gamma(1 - H)\Gamma(H + \frac{1}{2})}{\sqrt{\pi} H 2^{2H}} {}_2F_3\left(\{1, H - 1/2\}, \{1, H + 1, H + 1/2\}, \lambda^2 t^2/4\right), \end{aligned}$$

where ${}_2F_3$ is the generalized hypergeometric function.

- ③ Covariance functions of tempered processes have a following form.
- TFBM (1) with parameters $H > 0$ and $\lambda > 0$ has the covariance function

$$\text{Cov} \left[B_{H,\lambda}'(t), B_{H,\lambda}'(s) \right] = \frac{1}{2} \left[C_t^2 |t|^{2H} + C_s^2 |s|^{2H} - C_{t-s}^2 |t-s|^{2H} \right],$$

for any $t, s \in \mathbb{R}$, where $C_t^2 = (C_t')^2$.

- TFBMII (2) with parameters $H > 0$ and $\lambda > 0$ has the covariance function

$$\text{Cov} \left[B_{H,\lambda}''(t), B_{H,\lambda}''(s) \right] = \frac{1}{2} \left[C_t^2 |t|^{2H} + C_s^2 |s|^{2H} - C_{t-s}^2 |t-s|^{2H} \right],$$

for any $t, s \in \mathbb{R}$, where $C_t^2 = (C_t'')^2$.

Proposition 1 ([Azmoodeh, Mishura, and Sabzikar (2022)])

(i) The TFBM (1) with parameters $H > 0$ and $\lambda > 0$ satisfies

$$\lim_{t \rightarrow +\infty} \mathbb{E} \left[B_{H,\lambda}^I(t) \right]^2 = \frac{2\Gamma(2H)}{(2\lambda)^{2H}}. \quad (5)$$

(ii) The TFBMII (2) with parameters $H > 0$ and $\lambda > 0$ satisfies

$$\lim_{t \rightarrow +\infty} \mathbb{E} \left[\frac{B_{H,\lambda}^{II}(t)}{\sqrt{t}} \right]^2 = \lambda^{1-2H} \Gamma^2 \left(H + \frac{1}{2} \right). \quad (6)$$

Theorem 3 ([Azmoodeh, Mishura, and Sabzikar (2022)])

Let X stand for a TFBM $B_{H,\lambda}^I$ from (1) or for a TFBMII $B_{H,\lambda}^{II}$ from (2) both with $0 < H < 1$ and $\lambda > 0$. Then, there exist positive constants C_1 and C_2 such that for any $s, t \in [0, 1]$

$$C_1 |t - s|^{2H} \leq \mathbb{E}[|X(t) - X(s)|^2] \leq C_2 |t - s|^{2H}, \quad (7)$$

Asymptotic behavior of variance of TFBM at zero

Lemma 4

① For any $H \in (0, 1)$

$$\mathbb{E} \left[B_{H,\lambda}^I(t)^2 \right] \sim \frac{\Gamma(H + \frac{1}{2})^2}{2H \sin(\pi H) \Gamma(2H)} t^{2H}, \quad \text{as } t \downarrow 0. \quad (8)$$

② For $H = 1$

$$\mathbb{E} \left[B_{1,\lambda}^I(t)^2 \right] \sim -\frac{t^2}{4} \log t, \quad \text{as } t \downarrow 0.$$

③ For any $H > 1$

$$\mathbb{E} \left[B_{H,\lambda}^I(t)^2 \right] \sim \frac{\Gamma(2H)}{2^{2H+1} \lambda^{2H-2} (H-1)} t^2, \quad \text{as } t \downarrow 0.$$

Remark 1

Let $H \in (0, 1)$. When $\lambda = 0$, we get (up to multiplicative constant) the conventional (untempered) fractional Brownian motion (FBM) through the Mandelbrodt–van Ness representation:

$$B_H(t) := \int_{\mathbb{R}} \left[(t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} \right] dW_s. \quad (9)$$

It is known (see, e.g., [Mishura (2008), Thm. 1.3.1]) that this process has the variance

$$\mathbb{E} [(B_H(t))^2] = \frac{\Gamma(H + \frac{1}{2})^2}{2H \sin(\pi H) \Gamma(2H)} \cdot t^{2H}, \quad t \geq 0.$$

Hence, the relation (8) means that the variance of TFBM is asymptotically equivalent to the variance of FBM as $t \downarrow 0$.

Note that in the case $H \geq 1$ the integral in (9) does not exist. The FBM is defined for $H = 1$ by different approach. This explains the difference in the behavior of TFBM and FBM in the case $H = 1$.

Lemma 5

- ① For any $H \in (0, 1)$ and for all $t, s \in \mathbb{R}_+$

$$\mathbb{E} \left[\left| B_{H,\lambda}'(t) - B_{H,\lambda}'(s) \right|^2 \right] \leq C \left(|t - s|^{2H} \wedge 1 \right).$$

- ② If $H = 1$, then for all $t, s \in \mathbb{R}_+$

$$\mathbb{E} \left[\left| B_{H,\lambda}'(t) - B_{H,\lambda}'(s) \right|^2 \right] \leq C \left(|t - s|^2 |\log |t - s|| \wedge 1 \right).$$

- ③ For any $H > 1$ and for all $t, s \in \mathbb{R}_+$

$$\mathbb{E} \left[\left| B_{H,\lambda}'(t) - B_{H,\lambda}'(s) \right|^2 \right] \leq C(|t - s|^2 \wedge 1).$$

Remark 2 (sample path properties of TFBM)

[Meerschaert and Sabzikar (2013)] have obtained the following results about sample path properties of TFBM:

- (a) If $H \in (0, 1)$, then TFBM satisfies Hölder condition of any order $\gamma \in (0, H)$ on any interval $[0, T]$;
- (b) For $H > 1$ TFBM has continuously differentiable sample paths and, moreover, they are p times continuously differentiable for $H > p$.

Note that (a) agrees with statement (i) of our Lemma 5, and follows from it via the Kolmogorov–Čentsov theorem.

- (c) for $H = 1$, the process $B'_{1,\lambda}(t)$ satisfies Hölder condition of any order $\gamma \in (0, 1)$; it is **not continuously differentiable** for any $t \geq 0$.

Asymptotic behavior of variance of TFBMII at zero

Lemma 6

(i) For any $H \in (0, 1)$

$$\mathbb{E} \left[B_{H,\lambda}''(t)^2 \right] \sim \frac{\Gamma \left(H + \frac{1}{2} \right)^2}{2H \sin(\pi H) \Gamma(2H)} t^{2H}, \quad \text{as } t \downarrow 0. \quad (10)$$

(ii) For $H = 1$

$$\mathbb{E} \left[B_{H,\lambda}''(t)^2 \right] \sim \frac{t^2}{4} \log t, \quad \text{as } t \downarrow 0.$$

(iii) For any $H > 1$

$$\mathbb{E} \left[B_{H,\lambda}''(t)^2 \right] \sim \frac{(2H-1) \Gamma(2H)}{2^{2H+1} \lambda^{2H-2} (H-1)} t^2, \quad \text{as } t \downarrow 0. \quad (11)$$

Remark 3

The comparison of Lemmas 4 and 6 shows that the variances of TFBM and TFBMII demonstrate the same behavior at zero for $H \in (0, 1]$. For $H > 1$, they also behave similarly, as Ct^2 , but the constants are different, so that the variance of TFBMII is bigger. Namely, the difference between constants is

$$\frac{(2H-1)\Gamma(2H)}{2^{2H+1}\lambda^{2H-2}(H-1)} - \frac{\Gamma(2H)}{2^{2H+1}\lambda^{2H-2}(H-1)} = \frac{\Gamma(2H)}{2^{2H}\lambda^{2H-2}}. \quad (12)$$

In order to explain this fact, we note that, for the same Wiener process W , TFBM in (1) and TFBMII in (2) are related as

$$B_{H,\lambda}''(t) = B_{H,\lambda}'(t) + \lambda Y_{H,\lambda}(t), \quad (13)$$

$$Y_{H,\lambda}(t) = \int_0^t B_{H,\lambda}'(s) ds + t\eta_{H,\lambda}, \quad \eta_{H,\lambda} = \int_{-\infty}^0 (-s)^{H-\frac{1}{2}} e^{-\lambda(-s)} dW_s. \quad (14)$$

The difference (12) corresponds to the variance of $\lambda\eta_{H,\lambda}$.

Lemma 7

❶ For any $H \in (0, \frac{1}{2})$ and for all $t, s \in \mathbb{R}_+$

$$\mathbb{E} \left[\left| B_{H,\lambda}''(t) - B_{H,\lambda}''(s) \right|^2 \right] \leq C \left(|t - s|^{2H} \vee |t - s| \right).$$

❷ For any $H \in [\frac{1}{2}, 1)$ and for all $t, s \in \mathbb{R}_+$

$$\mathbb{E} \left[\left| B_{H,\lambda}''(t) - B_{H,\lambda}''(s) \right|^2 \right] \leq C \left(|t - s|^{2H} \wedge |t - s| \right).$$

❸ If $H = 1$, then for all $t, s \in \mathbb{R}_+$

$$\mathbb{E} \left[\left| B_{H,\lambda}''(t) - B_{H,\lambda}''(s) \right|^2 \right] \leq C \left(|t - s|^2 |\log |t - s|| \wedge |t - s| \right).$$

❹ For any $H > 1$ and for all $t, s \in \mathbb{R}_+$

$$\mathbb{E} \left[\left| B_{H,\lambda}''(t) - B_{H,\lambda}''(s) \right|^2 \right] \leq C \left(|t - s|^2 \wedge |t - s| \right).$$

Remark 4

1. If $H \in (0, 1]$, then $B_{H,\lambda}''$ satisfies Hölder condition of any order $\gamma \in (0, H)$ on any interval $[0, T]$. Moreover, it fails to satisfy this condition for $\gamma > H$.
2. In order to investigate path regularity properties in more detail, we recall that, by representation (13),

$$B_{H,\lambda}''(t) = B_{H,\lambda}'(t) + \lambda Y_{H,\lambda}(t) = B_{H,\lambda}'(t) + \lambda \int_0^t B_{H,\lambda}'(s) ds + \lambda \eta_{H,\lambda} t,$$

where the random variable $\eta_{H,\lambda}$ is defined by (14). Then the process $B_{H,\lambda}''(t)$ has similar continuity and differentiability properties to that of $B_{H,\lambda}'(t)$, namely

- for $H = 1$, the process $B_{1,\lambda}''(t)$ is not continuously differentiable for any $t \geq 0$,
- for $H > 1$, the process $B_{H,\lambda}''(t)$ is continuously differentiable for any $t \geq 0$, moreover, it is $p \in \mathbb{N}$ times differentiable for $H > p$.

Asymptotic growth of the trajectories of TFBM and TFBMII

Let $G = \{G(t), t \geq 0\}$ be a centered Gaussian process and $a(t) > 0$ be a continuous strictly increasing function such that $a(t) \rightarrow \infty$ as $t \rightarrow \infty$. Introduce the sequence $b_0 = 0$, $b_{k+1} > b_k$, $b_k \rightarrow \infty$ as $k \rightarrow \infty$. Denote $a_k = a(b_k)$ and $m_k = \sup_{t \in [b_k, b_{k+1}]} \left(\mathbb{E} |G(t)|^2 \right)^{1/2}$.

Theorem 8 ([Dozzi, Kozachenko, Mishura, and R. (2018)])

Let the following conditions hold:

(i) There exist $c_k > 0$ and $0 < \beta \leq 1$ such that

$$\sup_{\substack{t, s \in [b_k, b_{k+1}] \\ |t-s| \leq h}} \left(\mathbb{E} |G(t) - G(s)|^2 \right)^{1/2} \leq c_k h^\beta.$$

(ii) $0 < m_k < \infty$ and $\alpha = \sum_{k=0}^{\infty} \frac{m_k}{a_k} < \infty$.

(iii) There exists $0 < \gamma \leq 1$ such that $\sum_{k=0}^{\infty} \frac{m_k^{1-\gamma/\beta} (b_{k+1} - b_k)^\gamma c_k^{\gamma/\beta}}{a_k} < \infty$.

Then with probability 1 we have for all $t > 0$ the following upper bound for the process G :

$$|G(t)| \leq a(t)\xi,$$

where ξ is non-negative random variable whose distribution has the tail admitting the following upper bound: for any $\theta \in (0, 1)$, any $u > 0$ and any $0 < \alpha < u$

$$\mathbb{P}\{\xi > u\} \leq 2^{\frac{2}{\beta}-1} \exp\left\{-\frac{u^2(1-\theta)^2}{2\alpha^2}\right\} A(\theta, \gamma),$$

where

$$A(\theta, \gamma) = \exp\left\{\frac{1}{\gamma\alpha} \left(\sum_{k=0}^{\infty} \frac{m_k^{1-\gamma/\beta}}{a_k} (b_{k+1} - b_k)^{\gamma} c_k^{\gamma/\beta}\right) \left(\frac{2^{2/\beta-1}}{\theta^{1/\beta}}\right)^{\gamma}\right\}.$$

Moreover, for any $u > 0$ and any $0 < \alpha < u$

$$\mathbb{P}\{\xi > u\} \leq 2^{\frac{2}{\beta}-1} \sqrt{e} \exp\left\{-\frac{u^2}{2\alpha^2}\right\} A\left(1 - \sqrt{1 - \frac{\alpha^2}{u^2}}, \gamma\right).$$

Asymptotic growth of the trajectories of TFBM

Theorem 9

For any $\delta > 0$ there exists a non-negative random variable $\xi = \xi(\delta)$ such that for all $t > 0$

$$\sup_{s \in [0, t]} \left| B_{H, \lambda}^I(s) \right| \leq (t^\delta \vee 1) \xi \quad a.s.,$$

and there exist positive constants $C_1 = C_1(\delta)$ and $C_2 = C_2(\delta)$ such that for all $u > 0$

$$P(\xi > u) \leq C_1 e^{-C_2 u^2}. \quad (15)$$

Let $\alpha = \frac{K_1}{1-e^{-\delta}}$. Then for any $\gamma > 0$ there exists a constant $C_3 > 0$ such that for all $u > \alpha$

$$P(\xi > u) \leq 2^{\frac{2}{\beta}-1} \sqrt{e} \exp \left\{ -\frac{u^2}{2\alpha^2} \right\} \exp \left\{ C_3 \left(1 - \sqrt{1 - \frac{\alpha^2}{u^2}} \right)^{-\gamma/\beta} \right\}.$$

Remark 5

For FBM $B^H = \{B_t^H, t \geq 0\}$, the following upper bound was established in [Kozachenko, Melnikov, and Mishura (2015)]

$$\sup_{s \in [0, t]} |B_s^H| \leq \left(1 \vee \left(t^H (\log^+ t)^p\right)\right) \xi(p) \quad (16)$$

for any $p > 0$. We see that TBFM demonstrates much slower asymptotic growth.

Asymptotic growth of the trajectories of TFBMII

Theorem 10

For any $p > 1$ there exists a non-negative random variable $\xi = \xi(p)$ such that for all $t > 0$

$$\sup_{s \in [0, t]} \left| B_{H, \lambda}''(s) \right| \leq \left(\left(t^{\frac{1}{2}} (\log^+ t)^p \right) \vee 1 \right) \xi \quad a.s., \quad (17)$$

and there exist positive constants $C_1 = C_1(p)$ and $C_2 = C_2(p)$ such that for all $u > 0$

$$P(\xi > u) \leq C_1 e^{-C_2 u^2}.$$

Let $\alpha = L_1 e^{1/2} \sum_{k=0}^{\infty} k^{-p}$, $\nu = H \wedge \frac{1}{2}$. Then for any $\gamma > 0$ there exists a constant $C_3 > 0$ such that for all $u > \alpha$

$$P(\xi > u) \leq 2^{\frac{2}{\nu}-1} \sqrt{e} \exp \left\{ -\frac{u^2}{2\alpha^2} \right\} \exp \left\{ C_3 \left(1 - \sqrt{1 - \frac{\alpha^2}{u^2}} \right)^{-\gamma/\nu} \right\}.$$

Remark 6

The result of Theorem 10 cannot be improved substantially in a sense that one cannot obtain the power of t less than $\frac{1}{2}$ in the right-hand side of (17). Indeed, the distribution of the random variable $B_{H,\lambda}''(t)$ is Gaussian with zero mean and with variance $(C_t'')^2 t^{2H}$. Then $B_{H,\lambda}''(t) = C_t'' t^H \xi$, where ξ is a standard normal random variable. But $C_t'' t^H \sim C t^{\frac{1}{2}}$ as $t \rightarrow \infty$, see (6).

Remark 7

Note that TFBMII demonstrates faster asymptotic growth at infinity than TFBM. Let us compare the result of Theorem 10 with the corresponding upper bound for FBM (16). We see that for $H > 1/2$, the asymptotic growth of TFBMII is slower compared to that of FBM. However, for $H < 1/2$, it is faster.

Strongly consistent drift estimation in the fractional tempered Vasicek model

Our focus in this section is on the statistical inference of the tempered fractional Vasicek model, which is described by the following stochastic differential equation:

$$Y_t = y_0 + \int_0^t (a + bY_s) ds + \sigma X_t, \quad t > 0, \quad Y_0 = y_0 \in \mathbb{R}, \quad (18)$$

where $a \in \mathbb{R}$, $b > 0$, $\sigma > 0$, $X = X_{H,\lambda}$ is TFBM $B_{H,\lambda}^I$ or TFBMII $B_{H,\lambda}^{II}$. Equivalently, $Y = \{Y_t, t \geq 0\}$ is the process given explicitly by

$$Y_t = y_0 e^{bt} - \frac{a}{b} (1 - e^{bt}) + \sigma \int_0^t e^{b(t-s)} dX_s. \quad (19)$$

Let us consider the estimation of unknown drift parameter $\theta = (a, b) \in \mathbb{R} \times (0, \infty)$ in the Vasicek model (18) driven by TFBM or TFBMII.

We define the estimator $\hat{\theta}_T = (\hat{a}_T, \hat{b}_T)$ as follows

$$\hat{a}_T = \frac{(Y_T - y_0) \left(\int_0^T Y_t^2 dt - \frac{1}{2}(Y_T + y_0) \int_0^T Y_t dt \right)}{T \int_0^T Y_t^2 dt - \left(\int_0^T Y_t dt \right)^2}, \quad (20)$$

$$\hat{b}_T = \frac{(Y_T - y_0) \left(\frac{1}{2} T(Y_T + y_0) - \int_0^T Y_t dt \right)}{T \int_0^T Y_t^2 dt - \left(\int_0^T Y_t dt \right)^2}. \quad (21)$$

It is worth noting that both $\hat{a}_T$ and $\hat{b}_T$ do not depend on the diffusion parameters $H > 0$, $\lambda > 0$, and $\sigma > 0$, which may be unknown.

The estimators (20)–(21) can be thought of as the least squares estimator (LSE), which minimizes

$$F(a, b) = \int_0^T \left(\dot{Y}_t - a - bY_t \right)^2 dt$$

It is not hard to see that the minimum is achieved at the following point:

$$\hat{a}_T = \frac{\int_0^T dY_t \int_0^T Y_t^2 dt - \int_0^T Y_t dt \int_0^T Y_t dY_t}{T \int_0^T Y_t^2 dt - \left(\int_0^T Y_t dt \right)^2}, \quad (22)$$

$$\hat{b}_T = \frac{T \int_0^T Y_t dY_t - \int_0^T dY_t \int_0^T Y_t dt}{T \int_0^T Y_t^2 dt - \left(\int_0^T Y_t dt \right)^2}. \quad (23)$$

If the process Y is γ -Hölder continuous on $[0, T]$ with any $\gamma > \frac{1}{2}$, then the integral $\int_0^T Y_t dY_t$ can be defined in Young's sense. In this case,

$$\int_0^T Y_t dY_t = \frac{1}{2} (Y_T^2 - Y_0^2). \quad (24)$$

Substitution of (24) into (22)–(23) leads to the estimators (20)–(21).

The following theorem is the main result of this section.

Theorem 11

The estimator $(\hat{a}_T, \hat{b}_T)$ is strongly consistent, i.e.,

$$\hat{a}_T \rightarrow a, \hat{b}_T \rightarrow b \quad \text{a.s. when } T \rightarrow \infty.$$

In order to prove the strong consistency of the estimator $(\hat{a}_T, \hat{b}_T)$, we need to study the asymptotic behavior of all processes and integrals involved into the right-hand sides of (20)–(21).

Let

$$a(t) = \begin{cases} t^\delta \vee 1, & \text{for } X = B'_{H,\lambda}, \\ \left(t^{\frac{1}{2}}(\log^+ t)^p\right) \vee 1, & \text{for } X = B''_{H,\lambda}, \end{cases}$$

where $\delta \in (0, 1)$ and $p > 1$ are arbitrarily chosen constants. Then according to Theorems 9 and 10, we have the following upper bound: for all $t > 0$

$$\sup_{0 \leq s \leq t} |X_s| \leq a(t)\xi \quad \text{a.s.}, \quad (25)$$

where ξ is a non-negative random variable satisfying (15).

Therefore

$$\int_0^\infty e^{-bs} |X_s| ds < \infty \quad \text{a.s.},$$

and the following random variable is well defined:

$$\zeta := y_0 + \frac{a}{b} + \sigma b \int_0^\infty e^{-bs} X_s ds.$$

Lemma 12

The following convergences hold a.s. when $T \rightarrow \infty$:

$$\begin{aligned} e^{-bT} Y_T &\rightarrow \zeta, \\ e^{-bT} \int_0^T Y_t dt &\rightarrow \frac{1}{b} \zeta, \quad e^{-bT} \int_0^T |Y_t| dt \rightarrow \frac{1}{b} |\zeta|, \\ e^{-2bT} \int_0^T Y_t^2 dt &\rightarrow \frac{1}{2b} \zeta^2, \quad T^{-1} e^{-bT} \int_0^T Y_t t dt \rightarrow \frac{1}{b} \zeta, \\ T^{-1} e^{-bT} \left(\int_0^T Y_t^2 dt - \frac{1}{2} (Y_T + y_0) \int_0^T Y_t dt \right) &\rightarrow \frac{a}{2b} \zeta. \end{aligned}$$

Possible generalization and related models

The analysis of the above proofs shows that Theorem 11 can be extended to the case of Vasicek model (18) with more general class of driving noises X . Namely, $X = \{X_t, t \geq 0\}$ can be any continuous centered Gaussian process with $X_0 = 0$, for which the condition (25) holds with some positive non-decreasing function a satisfying the assumptions:

$$\frac{a(t)}{t} \rightarrow 0 \quad \text{as } t \rightarrow \infty, \quad (26)$$

$$\int_0^\infty a(t)e^{-bt} dt < \infty \quad \text{for any } b > 0. \quad (27)$$

Remark 8

The consistency of $\hat{b}_T$ can be proved without assumption (26).

Let us apply these remarks to two examples of noise, namely multifractional Brownian motion and Gaussian Volterra process.

Example 13 (Multifractional Vasicek model)

Let $H: \mathbb{R}_+ \rightarrow [h_1, h_2] \subset (0, 1)$ be a κ -Hölder continuous function. We consider the model (18) driven by the (harmonizable) *multifractional Brownian motion* with functional parameter H , which is defined by (see [Benassi, Jaffard, and Roux (1997)])

$$X(t) = \int_{\mathbb{R}} \frac{e^{itu} - 1}{|u|^{H(t)+1/2}} d\widetilde{W}(u), \quad t \geq 0,$$

where $\widetilde{W}(du)$ is the “Fourier transform” of the white noise $W(du)$ that is a unique complex-valued random measure such that for all $f \in L^2(\mathbb{R})$

$$\int_{\mathbb{R}} f(u) W(du) = \int_{\mathbb{R}} \widehat{f}(u) \widetilde{W}(du) \quad \text{a.s.}$$

According to [Dozzi, Kozachenko, Mishura, and R. (2018)], the multifractional Brownian motion satisfies the condition (25) with

$$a(t) = t^{h^* + \delta} \vee 1,$$

where $h^* = \limsup_{t \rightarrow \infty} H(t)$ and $\delta > 0$ is arbitrary. Thus, the assumptions (26) and (27) hold; this implies the strong consistency of the estimator $(\hat{a}_T, \hat{b}_T)$ in this model.

Remark 9

The estimation of the parameter b for known $a = 0$ in the multifractional Vasicek model was studied in [Dozzi, Kozachenko, Mishura, and R. (2018)], where the strongly consistent estimator was constructed under additional assumption $\min \{h_1, \kappa\} > \frac{1}{2}$.

Example 14 (Vasicek model driven by a Gaussian Volterra process with power-type kernel)

Let $\alpha > -\frac{1}{2}$, $\beta \in \mathbb{R}$, $\gamma > -1$ be such numbers that $\alpha + \beta + \gamma > -\frac{3}{2}$. Let us consider the following Gaussian process

$$X_t = \int_0^t s^\alpha \int_s^t u^\beta (u-s)^\gamma du dW_s, \quad t \geq 0. \quad (28)$$

According to [Mishura, R., and Shklyar (2023)], it satisfies (25) with

$$a(t) = \left(t^{\alpha+\beta+\gamma+\frac{3}{2}} (\log(1 \vee \log t))^{1/2} \right) \vee 1.$$

Therefore, under additional condition

$$\alpha + \beta + \gamma < -\frac{1}{2}, \quad (29)$$

the assumptions (26) and (27) hold, hence the formulas (20)–(21) provide strongly consistent estimators for the parameters a and b of the Vasicek model (18) driven by the Gaussian Volterra process (28).



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