

Long run stochastic control problems
with general discounting
How to overcome time inconsistency

Łukasz Stettner

Institute of Mathematics PAS

Discounting - constant vs. hyperbolic or general

what is a current value of contingent claim of 100 USD which we receive after two years ? - $\phi(2)100$ USD

general discount rate:

$\phi : \mathbb{N} \rightarrow [0, 1] : \phi(0) = 1$, $n \rightarrow \phi(n)$ nonincreasing function and $\phi(n+k) \geq \phi(n)\phi(k)$, for $n, k \in \mathbb{N}$,

we assume that $\sum_{k=0}^{\infty} \phi(k) = \infty$.

general (strict) discount rate: $n \rightarrow \phi(n)$ strictly decreasing and $\phi(n+k) > \phi(n)\phi(k)$, for $n, k \in \mathbb{N}$.

Example: **hyperbolic discounting:** $\phi(i) = (1 + hi)^{-\frac{r}{h}}$, where $h > 0, r > 0$ and $\frac{r}{h} \leq 1$.

constant discounting: $\phi(i) = e^{-\alpha i}$

Loewenstein, G. and Prelec, D. 1992, Rohde, K.I.M. 2010, time inconsistent problems (Björk, T., and Khapko, M., and Murgoci, A., Springer 2021)

Problems with general discounting

optimal stopping $\max_{\tau} E_x [\phi(\tau)g(X_{\tau})]$

we don't have Bellman equation unless we add time variable

various concepts of equilibrium Y. I. Huang (Z. Zhou and collaborators), S. Christensen K. Lindensj6, E. Bayraktar, J. Zhang, Z. Zhou

problems in discrete and in continuous time

discrete time long run control problems average reward per unit time and other functionals (Ł. Stettner 2023),

impulse control with generalized discounting (D. Jelito, Ł. Stettner 2023)

in certain cases time inconsistent control problems can be replaced by time consistent problems without extending the dimension of the problem

Long run stochastic control problems

(X_n) discrete Markov process on $(\Omega, F, (F_n))$ taking values in a locally compact space E endowed with Borel σ field $\mathcal{E}$. The process has transition operator (kernel) $\mathbb{P}^a(x, \cdot)$, where $x \in E$ and $a \in U$ a fixed compact set of control parameters.

Markov process is controlled using sequence $V = (a_0, a_1, \dots, a_n \dots)$ such that $a_n \in U$ is F_n adapted. We denote by $\mathbb{E}_{k,x}^V$ expected value corresponding to controlled Markov process with the use of sequence V and starting at time k from x . When $k = 0$ we simply write $\mathbb{E}_x^V$.

We want to maximize the following **long run average generally discounted** functional

$$J_{k,x}^g(V) = \liminf_{n \rightarrow \infty} \left(\sum_{i=k}^{n+k-1} \phi(i) \right)^{-1} \mathbb{E}_{k,x}^V \left[\sum_{i=k}^{n+k-1} \phi(i) c(X_{i-k}, a_{i-k}) \right] \quad (1)$$

Further functionals

long run risk sensitive generally discounted functional

$$I_{k,x}^{\gamma,g}(V) = \liminf_{n \rightarrow \infty} \left(\gamma \sum_{i=k}^{n+k-1} \phi(i) \right)^{-1} \ln \left(\mathbb{E}_{k,x}^V \left[\exp \left\{ \gamma \sum_{i=k}^{n+k-1} \phi(i) c(X_{i-k}, a_{i-k}) \right\} \right] \right) \quad (2)$$

with $\gamma \neq 0$.

average reward per unit time functional ($\phi \equiv 1$)

$$J_x(V) = \liminf_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_x^V \left[\sum_{i=0}^{n-1} c(X_i, a_i) \right] \quad (3)$$

long run risk sensitive functional ($\phi \equiv 1$)

$$I_x^\gamma(V) = \liminf_{n \rightarrow \infty} \frac{1}{n\gamma} \ln \left(\mathbb{E}_x^V \left[\exp \left\{ \gamma \sum_{i=0}^{n-1} c(X_i, a_i) \right\} \right] \right). \quad (4)$$

Why risk sensitive problems are important?

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{E} [W_t^\gamma] \quad \text{or equivalently as} \quad \liminf_{t \rightarrow \infty} \frac{1}{t} \mu^\gamma(\ln W_t), \quad (5)$$

where $\mu^\gamma(\cdot) := 1/\gamma \cdot \ln \mathbb{E}[\exp(\gamma(\cdot))]$ is the entropic utility

From the second order Taylor's expansion of the entropic utility, around $\gamma = 0$, we get

$$\mu^\gamma(\ln W_t) = \mathbb{E}[\ln W_t] + \frac{\gamma}{2} \text{Var}[\ln W_t] + O(\gamma^2, t), \quad t > 0,$$

Main results I:

1. optimal control for the functional

$$J_x(V) = \liminf_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_x^V \left[\sum_{i=0}^{n-1} c(X_i, a_i) \right] \quad (6)$$

is also optimal for the functional

$$J_{k,x}^g(V) = \liminf_{n \rightarrow \infty} \left(\sum_{i=k}^{n+k-1} \phi(i) \right)^{-1} \mathbb{E}_{k,x}^V \left[\sum_{i=k}^{n+k-1} \phi(i) c(X_{i-k}, a_{i-k}) \right] \quad (7)$$

Main results II:

2. optimal control for the functional

$$J_x(V) = \liminf_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_x^V \left[\sum_{i=0}^{n-1} c(X_i, a_i) \right] \quad (8)$$

is also nearly for

$$I_x^\gamma(V) = \liminf_{n \rightarrow \infty} \frac{1}{n\gamma} \ln \left(\mathbb{E}_x^V \left[\exp \left\{ \gamma \sum_{i=0}^{n-1} c(X_i, a_i) \right\} \right] \right). \quad (9)$$

with $\gamma < 0$ and for

$$I_{k,x}^{\gamma,g}(V) = \liminf_{n \rightarrow \infty} \left(\gamma \sum_{i=k}^{n+k-1} \phi(i) \right)^{-1} \ln \left(\mathbb{E}_{k,x}^V \left[\exp \left\{ \gamma \sum_{i=k}^{n+k-1} \phi(i) c(X_{i-k}, a_{i-k}) \right\} \right] \right) \quad (10)$$

provided that $0 > \gamma$ is sufficiently close to 0.

Fundamental assumptions:

uniform ergodicity:

$$[A.1] \quad \sup_{x, x' \in E} \sup_{a, a' \in U} \sup_{D \in \mathcal{E}} |\mathbb{P}^a(x, D) - \mathbb{P}^{a'}(x', D)| = \Delta < 1 \quad (11)$$

Feller property:

$$[A.2] \quad \text{the mapping } E \times U \ni (x, a) \mapsto \mathbb{P}^a f(x) \text{ is} \quad (12)$$

continuous for $f \in C(E)$.

Long run average discounted functional problem.

Bellman equation: we look for $(w(i, x))$ and $(\lambda(i))$ such that

$$w(i, x) = \sup_{a \in U} \left[\phi(i)(c(x, a) - \lambda(i)) + \int_E w(i+1, y) \mathbb{P}^a(x, dy) \right] \quad (13)$$

for $i \geq k$. The problem is time inconsistent. Let $\|f\|_{sp} := \sup_{x \in E} f(x) - \inf_{x' \in E} f(x')$, for $f \in C(E)$ and the space $C_{sp} = C(\mathbb{N} \times E) / \sim$, where for $f, g \in C(\mathbb{N} \times E)$ we have $f \sim g \equiv \forall_{i \in \mathbb{N}} \|f(i, \cdot) - g(i, \cdot)\|_{sp} = 0$. We consider in C_{sp} the norm $\|f\| := \sup_{i \in \mathbb{N}} \|f(i, \cdot) - g(i, \cdot)\|_{sp}$.

Theorem 1 *Under [A.1] and [A.2] there exist solutions $w \in C_{sp}$ and $(\lambda(i))$ to the equation (13). Moreover*

$$\sup_V J_{k,x}^g(V) = \liminf_{n \rightarrow \infty} \frac{\sum_{i=k}^{n+k-1} \lambda(i) \phi(i)}{\sum_{i=k}^{n+k-1} \phi(i)} \quad (14)$$

and the limit do not depend on $k \in \mathbb{N}$ and $x \in E$. Optimal strategy $\hat{V}$ for the functional $J_{k,x}^g$ is of the form $\hat{V} = (\hat{u}_k(X_0), \hat{u}_{k+1}(X_1), \dots, \hat{u}_{k+i}(X_i), \dots)$, where $\hat{u}_i$, dla $i \geq k$ is a Borel selector of the right hand side of the equation (13).

Average reward per unit time

When $\phi \equiv 1$ (no discount rate) we look for w and λ such, that

$$w(x) = \sup_{a \in U} \left[c(x, a) - \lambda + \int_E w(y) \mathbb{P}^a(x, dy) \right] \quad (15)$$

holds for $x \in E$.

Corollary 1 *Under [A.1] and [A.2] there is $w \in C(E)$ and a constant λ for which the equation (15) is satisfied. Moreover $\|w\|_{sp} \leq \frac{1}{1-\Delta}$,*

$$\sup_V J_x(V) = \lambda, \quad (16)$$

and control $\hat{V} = (\hat{u}(X_0), \hat{u}(X_1), \dots, \hat{u}(X_n), \dots)$, where $\hat{u}$ is a Borel selector of the right hand side of (15), is optimal for (3).

Theorem 2 *Under assumptions [A.1] i [A.2] the strategy $\hat{V}$ defined in Corollary 1 is also optimal (1) and for $k \in \mathbb{N}$ and $x \in E$ we have $\sup_V J_{k,x}^g(V) = \lambda$.*

Further properties of the average reward per unit time

Corollary 2 Under [A.1] for stationary Markov strategy u there is a bounded function $w^u \in B(E)$ and a constant λ^u such that *additive Poisson equation*

$$w^u(x) = \left[(c(x, u(x)) - \lambda^u) + \int_E w^u(y) \mathbb{P}^{u(x)}(x, dy) \right]. \quad (17)$$

is satisfied. Moreover $\|w^u\|_{sp} \leq \frac{1}{1-\Delta}$ and

$$\lambda^u = J_x(u) = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_x^u \left[\sum_{i=0}^{n-1} c(X_i, u(X_i)) \right]. \quad (18)$$

Corollary 3 Under [A.1] for stationary Markov strategy $V_m = (u(X_0), u(X_1), \dots, u(X_i), \dots) = u$ we have

$$J_{k,x}^g(V_m) = \lambda^u. \quad (19)$$

The role of continuous controls

Assumption

[A.3] there is $\eta \in \mathcal{P}(E)$ such that

$$\mathbb{P}^a(x, B) = \int_B p(x, a, y) \eta(dy) \quad \text{for } B \in \mathcal{E} \text{ and } x \in E, \quad (20)$$

the mapping $E \times U \ni (x, a) \rightarrow p(x, a, y)$ is continuous for $y \in E$ and there is a constant $M > 0$ such that $\frac{1}{M} \leq p(x, a, y) \leq M$ for $x, y \in E$ and $a \in U$.

Proposition 1 *Under [A.3] we have*

$$\lambda = \sup_{u \text{ continuous}} \lambda^u. \quad (21)$$

Problems with risk sensitive functionals

Bellman equations: $\gamma < 0$ for $i \geq k$ we have

$$e^{w^\gamma(i,x)} = \inf_{a \in U} \left[e^{(\phi(i)(c(x,a) - \lambda^\gamma(i)))\gamma} \int_E e^{w^\gamma(i+1,y)} \mathbb{P}^a(x, dy) \right] \quad (22)$$

while for $\gamma > 0$

$$e^{w^\gamma(i,x)} = \sup_{a \in U} \left[e^{(\phi(i)(c(x,a) - \lambda^\gamma(i)))\gamma} \int_E e^{w^\gamma(i+1,y)} \mathbb{P}^a(x, dy) \right] \quad (23)$$

Further assumptions:

$$[B.1] \quad \sup_{w \in C(E)} \sup_{x, x' \in E} \sup_{a, a' \in U} \frac{\int_E e^{w(y)} \mathbb{P}^a(x, dy)}{\int_E e^{w(y)} \mathbb{P}^{a'}(x', dy)} = K < \infty. \quad (24)$$

Lemma 1 *Assumption [B.1] says that measures $\mathbb{P}^a(x, dy)$ and $\mathbb{P}^{a'}(x', dy)$ are equivalent with densities uniformly (in $a, a' \in U$, $x, x' \in E$) bounded from below and from above.*

Solutions to the risk sensitive Bellman equations

Alternative assumption to [B.1]:

$$[B.2] \quad e^{\|\gamma c\|_{sp}} \Delta < 1 \quad (25)$$

Let $r(\gamma) := \|\gamma c\|_{sp} - \ln(1 - \Delta e^{\|\gamma c\|_{sp}})$.

Theorem 3 *Under [A.1] and [A.2] as well as [B.1] or [B.2] for $\gamma \in \mathbb{R}$ or for γ for which [B.2] is satisfied there is a function $w^\gamma \in C_{sp}$ such that $\inf_y w^\gamma(i, y) = 0$ for $i \geq k$ and a sequence $(\lambda^\gamma(i))$, for which equations (22) or (23) are satisfied. Moreover under [B.1] we have $\|w^\gamma\| := \sup_{x \in E} \sup_{i \geq k} |w^\gamma(i, x)| \leq \|\gamma c\|_{sp} + K$, while under [B.2] we have $\|w^\gamma\| \leq r(\gamma)$. Moreover*

$$\sup_V I_{k,x}^{\gamma,g}(\hat{V}) = I_{k,x}^{\gamma,g}(\hat{V}) = \liminf_{n \rightarrow \infty} \left(\sum_{i=k}^{n+k-1} \phi(i) \right)^{-1} \left(\sum_{i=k}^{n+k-1} \phi(i) \lambda^\gamma(i) \right), \quad (26)$$

where $\hat{V} = (\hat{u}_k(X_0), \hat{u}_{k+1}(X_1), \dots, \hat{u}_i(X_i), \dots)$ is a Markov strategy consisting of the selectors to the equations (22) or (23) respectively.

Risk sensitive functional without discounting

Bellman equations for $\gamma < 0$

$$e^{w^\gamma(x)} = \inf_{a \in U} \left[e^{(c(x,a) - \lambda^\gamma)\gamma} \int_E e^{w^\gamma(y)} \mathbb{P}^a(x, dy) \right] \quad (27)$$

for $\gamma > 0$

$$e^{w^\gamma(x)} = \sup_{a \in U} \left[e^{(c(x,a) - \lambda^\gamma)\gamma} \int_E e^{w^\gamma(y)} \mathbb{P}^a(x, dy) \right]. \quad (28)$$

Corollary 4 *Under [A.1] and [A.2] as well as [B.1] or [B.2] for $\gamma \in \mathbb{R}$ or for γ such that [B.2] is satisfied there is a function $w^\gamma \in C(E)$ such that $\inf_y w^\gamma(y) = 0$ and a constant λ^γ , for which equations (27) or (28) are satisfied. Moreover under [B.1] we have $\|w^\gamma\| := \sup_{x \in E} |w^\gamma(x)| \leq \|\gamma c\|_{sp} + K$, while under [B.2] we have $\|w^\gamma\| \leq r(\gamma)$. In addition optimal value of (4) is λ^γ , while optimal strategy is stationary of the form $a_i = \hat{u}^\gamma(X_i)$, where $\hat{u}^\gamma$ is a selector of the right hand side of (27) for negative γ or (28) for positive γ .*

Multiplicative Poisson equations and continuous controls

Corollary 5 *Under assumptions [A.1] and [A.2] as well as [B.1] or [B.2] for $\gamma \in \mathbb{R}$ or for γ for which [B.2] is satisfied and stationary Markov strategy $V_m = u$ there is a bounded function $w^{u,\gamma} \in B(E)$ such that $\inf_{y \in E} w^{u,\gamma}(y) = 0$ and a constant $\lambda^{u,\gamma}$ for which *multiplicative Poisson equation* is satisfied.*

$$e^{w^{u,\gamma}(x)} = e^{(c(x,u(x)) - \lambda^{u,\gamma})\gamma} \int_E e^{w^{u,\gamma}(y)} \mathbb{P}^{u(x)}(x, dy) \quad (29)$$

Moreover under [B.1] we have $\|w^{u,\gamma}\| := \sup_{x \in E} |w^{u,\gamma}(x)| \leq \|\gamma c\|_{sp}$ while under [B.2] $\|w^{u,\gamma}\| \leq r(\gamma)$ and

$$\lambda^{u,\gamma} = I_x^\gamma(u) = \lim_{n \rightarrow \infty} \frac{1}{n\gamma} \ln \left(\mathbb{E}_x^u \left[\exp \left\{ \gamma \sum_{i=0}^{n-1} c(X_i, u(X_i)) \right\} \right] \right). \quad (30)$$

Proposition 2 *Under [A.3] and [B.1] or [B.2] we have*

$$\lambda^\gamma = \sup_{u \text{ continuous}} \lambda^{u,\gamma}. \quad (31)$$

Risk sensitive functionals with or without discounting

Theorem 4 *For $\gamma > 0$ we have*

$$\sup_V I_{k,x}^{\gamma,g}(V) \leq \lambda^\gamma \quad (32)$$

Asymptotics when γ is close to 0: for $\gamma > 0$ we have

$$\frac{1}{-\gamma} \ln \left(\mathbb{E} \left[e^{-\gamma Y} \right] \right) \leq \mathbb{E}[Y] \leq \frac{1}{\gamma} \ln \left(\mathbb{E} \left[e^{\gamma Y} \right] \right). \quad (33)$$

and therefore

$$I_{k,x}^{-\gamma,g}(V) \leq J_{k,x}^g(V) \leq I_{k,x}^{\gamma,g}(V). \quad (34)$$

Theorem 5 *Under [A.3] we have $\lim_{\gamma \rightarrow 0^-} v^{\gamma,g}(k, x) = \lambda$, for $k \in \mathbb{N}$ and $x \in E$. Stationary optimal control $\hat{u}$ for the functional (3) is nearly optimal for the functional (2) for $\gamma < 0$ sufficiently close to 0. In the case when $\gamma > 0$ for each continuous stationary Markov control $u : E \rightarrow U$ we have*

$$\lim_{\gamma \rightarrow 0} I_{k,x}^{\gamma,g}(u) = \lim_{\gamma \rightarrow 0} I_x^\gamma(u) = \lambda^u. \quad (35)$$

Large deviations of weighted empirical measures

Feller Markov process (X_n)

$$S_k^n(B) := \left(\sum_{i=k}^{n+k-1} \phi(i) \right)^{-1} \sum_{i=k}^{n+k-1} \phi(i) 1_B(X_{i-k}) \quad (36)$$

$$Q_{k,x}^n(D) := \mathbb{P}_{k,x} \{S_k^n \in D\} \quad (37)$$

Let $F := \{f \in C(E) : \exists_{a>0} f(x) \geq a, \text{ for } x \in E\}$ and for $\nu \in \mathcal{P}(E)$

$$I(\nu) := \sup_{f \in F} \int_E \ln \frac{f(x)}{\mathbb{P}f(x)} \nu(dx) \quad (38)$$

Assumption [D.1] there is $\eta \in \mathcal{P}(E)$ such that

$$\mathbb{P}(x, B) = \int_B p(x, y) \eta(dy) \quad \text{dla } B \in \mathcal{E} \text{ i } x \in E, \quad (39)$$

the mapping $E \ni x \rightarrow p(x, y)$ is continuous for $y \in E$ and there is $M > 0$ such that $\frac{1}{M} \leq p(x, y) \leq M$ for $x, y \in E$.

Main result

Theorem 6 *For compact set $C \subset \mathcal{P}(E)$ or closed subset $C \subset \mathcal{P}(E)$ under additional assumption [D.1] we have*

$$\limsup_{n \rightarrow \infty} \left(\sum_{i=k}^{n+k-1} \phi(i) \right)^{-1} \sup_{x \in E} \ln Q_{k,x}^n(C) \leq - \inf_{\nu \in C} I(\nu). \quad (40)$$

Lemma 2 *If there is a unique invariant measure μ for the operator $\mathbb{P}(x, \cdot)$ and $\mu \notin C$, where $C \subset \mathcal{P}(E)$ is closed and E is compact or E is complete separable and [D.1] is satisfied then*

$$\inf_{\zeta \in C} I(\zeta) > 0. \quad (41)$$

Long run impulse control

$X = (X_t)_{t \geq 0}$ be a continuous time standard Feller–Markov process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ taking values in a compact space E .

impulse control strategy, i.e. a sequence $V := (\tau_i, \xi_i)_{i=1}^{\infty}$, where (τ_i) is an increasing sequence of stopping times and (ξ_i) is a sequence of $\mathcal{F}_{\tau_i}$ -measurable

$$J^d(x, V) := \limsup_{T \rightarrow \infty} \frac{\mathbb{E}_x^V \left[\int_0^T \phi(s) g(Y_s) ds + \sum_{i=1}^{\infty} 1_{\{\tau_i \leq T\}} \phi(\tau_i) c(X_{\tau_i}^i, X_{\tau_i}^{i+1}) \right]}{\int_0^T \phi(s) ds}, \quad (42)$$

$$J(x, V) := \limsup_{T \rightarrow \infty} \frac{\mathbb{E}_x^V \left[\int_0^T g(Y_s) ds + \sum_{i=1}^{\infty} 1_{\{\tau_i \leq T\}} c(X_{\tau_i}^i, X_{\tau_i}^{i+1}) \right]}{T}, \quad (43)$$

$\int_0^{\infty} \phi(s) ds = \infty$, $c : E \times U \rightarrow \mathbb{R}_+$ is continuous, bounded, strictly positive,

$$0 < c_0 \leq c(x, \xi) \leq c(x, \eta) + c(\eta, \xi), \quad x, \xi, \eta \in E. \quad (44)$$

Discrete time problem approximation

$$\varphi_h(i) := \frac{1}{h} \int_{ih}^{(i+1)h} \phi(s) ds, \quad i \in \mathbb{N}. \quad (45)$$

$\varphi_h(0) \leq 1$ and the map $i \mapsto \varphi_h(i)$ is decreasing. Also, we get $\sum_{i=1}^{\infty} \varphi_h(i) = \infty$ and $\varphi_h(i+k) \geq \varphi_h(i)\varphi_h(k)$, $i, k \in \mathbb{N}$.

$\mathbb{V}_h \subset \mathbb{V}$ the subfamily of impulse control strategies with impulse times restricted to $\{0, h, 2h, \dots\}$

$$J_h^d(x, V) := \limsup_{n \rightarrow \infty} \frac{1}{\sum_{i=0}^{n-1} \varphi_h(i)} \mathbb{E}_x^V \left[\sum_{i=0}^{n-1} \varphi_h(i) g(Y_{ih}) + \sum_{i=1}^{\infty} 1_{\{\tau_i \leq nh\}} \frac{1}{h} \varphi_h(\tau_i/h) c(X_{\tau_i}^i, X_{\tau_i}^{i+1}) \right], \quad (46)$$

$$J_h(x, V) := \limsup_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_x^V \left[\sum_{i=1}^n g(Y_{ih}) + \sum_{i=1}^{\infty} 1_{\{\tau_i \leq nh\}} \frac{1}{h} c(X_{\tau_i}^i, X_{\tau_i}^{i+1}) \right]; \quad (47)$$

Discrete time impulse control

$$\inf_{V \in \mathbb{V}_h} J_h^d(x, V), \quad \inf_{V \in \mathbb{V}_h} J_h(x, V), \quad x \in E.$$

two types of Bellman equations

$$w_h^d(k, x) = \min \left(\varphi_h(k) \left(g(x) - \lambda_h^d(k) \right) + \mathbb{E}_x \left[w_h^d(k+1, X_h) \right], M_h^d w_h^d(k, x) \right), \quad (48)$$

$$w_h(x) = \min \left(g(x) - \lambda_h + \mathbb{E}_x[w_h(X_h)], M_h w_h(x) \right), \quad (49)$$

where $k \in \mathbb{N}$, $x \in E$, and the operators M_h^d and M_h are given as

$$M_h^d w_h^d(k, x) := \inf_{\xi \in U} \left(\varphi_h(k) \frac{1}{h} c(x, \xi) + w_h^d(k, \xi) \right), \quad k \in \mathbb{N}, x \in E, \quad (50)$$

$$M_h w_h(x) := \inf_{\xi \in U} \left(\frac{1}{h} c(x, \xi) + w_h(\xi) \right), \quad x \in E. \quad (51)$$

Proposition 3 *There exists a unique up to a constant map $w_h^d \in \mathcal{C}_b(\mathbb{N} \times E)$ and a unique sequence of constants $(\lambda_h^d(k))_{k \in \mathbb{N}} \subset \mathbb{R}$ which are a solution to Equation (48).*

Main results in discrete time

Proposition 4 *Let (w_h^d, λ_h^d) be a solution to (48). Also, let the strategy $\hat{V}_h^d$ be given by Bellman this equation (48) , and let J_h^d be given by (46). Then, we get*

$$\inf_{V \in \mathbb{V}_h} J_h^d(x, V) = J_h^d(x, \hat{V}_h^d) = \limsup_{n \rightarrow \infty} \frac{\sum_{i=0}^{n-1} \varphi_h(i) \lambda_h^d(i)}{\sum_{i=0}^{n-1} \varphi_h(i)}, \quad x \in E.$$

Theorem 7 *Let (w_h, λ_h) be a solution to (49) and let the strategy $\hat{V}_h$ be given by the Bellman equation (49). Also, let J_h^d and J_h be given by (46) and (47), respectively. Then, we get*

$$\inf_{V \in \mathbb{V}_h} J_h^d(x, V) = J_h^d(x, \hat{V}_h) = \lambda_h = \inf_{V \in \mathbb{V}_h} J_h(x, v), \quad x \in E.$$

From discrete to continuous time

Theorem 8 *Let J^d and J_h^d be given by (42) and (46), respectively. Then, we get*

$$\lim_{h \rightarrow 0} \left| \inf_{V \in \mathbb{V}_h} J_h^d(x, V) - \inf_{V \in \mathbb{V}_h} J^d(x, V) \right| = 0.$$

Theorem 9 *Let J and J_h be given by (43) and (47), respectively. Then, we get*

$$\lim_{h \rightarrow 0} \left| \inf_{V \in \mathbb{V}_h} J_h(x, V) - \inf_{V \in \mathbb{V}_h} J(x, V) \right| = 0.$$

Theorem 10 *Let J^d be given by (42). Then, we get*

$$\lim_{h \rightarrow 0} \inf_{V \in \mathbb{V}_h} J^d(x, V) = \inf_{V \in \mathbb{V}} J^d(x, V).$$

Theorem 11 *Let J be given by (43). Then, we get*

$$\lim_{h \rightarrow 0} \inf_{V \in \mathbb{V}_h} J(x, V) = \inf_{V \in \mathbb{V}} J(x, V).$$

Main result for long run impulse control

Theorem 12 *Let J^d and J be given by (42) and (43), respectively. Then, we get*

$$\inf_{V \in \mathbb{V}} J^d(x, V) = \inf_{V \in \mathbb{V}} J(x, V).$$

Thank you for attention