

Sufficient variability of paths and differential systems with BV -coefficients

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Dinner for today

- 1 Introduction
- 2 Variability of paths and existence of the integral
 - Examples
- 3 Differential systems with BV -coefficients
 - Differential systems, change of variables, and Doss transform
 - One-dimensional meets multidimensional
- 4 Tasty bites of details
- 5 Summary up to this far
- 6 Future consequences and applications

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Problem formulation in one dimension

Let $\varphi(x) = \mathbf{1}_{x>a}$ and consider the one dimensional case

- **problem:** usually $\mathbf{1}_{X_t>a}$ is not continuous or of bounded p -variation for any $p \geq 1$
- $\Rightarrow$ integral $\int_0^T \mathbf{1}_{X_t>a} dX_t$ cannot be defined as Young integral, as Föllmer integral, or through rough paths

One solution

generalised Lebesgue–Stieltjes (gLS) integrals ([studied, among others, by Zähle](#))

- if $X \in W_0^{\beta,1}(0, T)$, i.e. belongs to certain weighted fractional Sobolev space, and $Y \in W_T^{1-\beta,\infty}(0, T)$ for some $\beta \in (0, 1)$, then $\int_0^T X_s dY_s$ exists

Origin of the story

For $X = B^H$ the fractional Brownian motion with $H > \frac{1}{2}$, Azmoodeh-Mishura-Valkeila (2010) proved the existence of $\int_0^T \mathbf{1}_{B_s^H > a} dB_s^H$ that then generalises to the existence of any $\varphi \in BV(\mathbb{R})$ via representation

$$\varphi(x) \sim \int |x - a| \mu(da).$$

Key is to show $\int_0^T \int_0^T \frac{\mathbf{1}_{B_t^H > a > B_s^H}}{|t-s|^{\beta+1}} ds dt < \infty$ that can be obtained via last hitting times $T_a(t) = \sup\{s \leq t : B_s^H = a\}$, Hölder continuity, and by Gaussianity for any $\alpha \in (0, 1)$

$$\int_0^T \sup_{a \in \mathbb{R}} \mathbb{E} |B_t^H - a|^{-\alpha} dt < \infty.$$

One-dimensional probabilistic extensions

- Azmoodeh-V (2015): L^1 -rate of convergence $n^{1/2-H-\varepsilon}$ for Riemann-Stieltjes (RS) sums in the fBm-case
- Sottinen-V (2016): existence for α -Hölder continuous Gaussian processes X with $\alpha > 1/2$ (and certain technical assumptions on covariance) and for mixed processes $X + W$
- Chen-Leskelä-V (2019): existence of integral for non-Gaussian processes and L^1 -rate of convergence $n^{1-2H-\varepsilon}$ of RS-sums
- Azmoodeh-Ilmonen-Shafik-Sottinen-V (2023): Exact convergence rate n^{1-2H} in expectation for RS-sums in the case of fractional type processes
- Garino-V (2024+): exact L^2 -rate of convergence n^{1-2H} for RS-sums and connection to local time for fBm

What is known in the one dimensional case

Notion of sufficient variability

Let $X \in C^\alpha$, $Y \in C^\gamma$ with $\alpha + \gamma > 1$ and let $\varphi \in BV_{\text{loc}}(\mathbb{R})$. Suppose further that for $\theta \in (0, 1)$ (**suitably chosen**)

$$\sup_{a \in \mathbb{R}} \mathbb{E} \int_0^T |X_t - a|^{-\theta} dt < \infty. \quad (1)$$

Equation (1) guarantees that X is "**active**" enough and do not spend too much time on any point a , and in particular, on the bad points of φ . Roughly speaking, X may enter the bad region (**otherwise situation is boring**) but do not stay there too long

What is known in the one dimensional case

Selected results

Suppose assumptions of previous slide. Then

Chen, Leskelä, V (2019)

- $\int_0^T \varphi(X_s) dY_s$ exists in the sense of gLS-integral
- If $\alpha > \frac{1}{2}$ we have $F(X_T) = F(X_0) + \int_0^T \varphi(X_s) dX_s$ for continuous F with $F' = \varphi$

Torres, V (2023)

If $\gamma > \frac{1}{2}$, $\varphi(x) \geq \epsilon > 0$, and Y satisfies (1), then $dX_t = \varphi(X_t) dY_t$ has a solution $X \in C^\alpha$ satisfying (1) and $\alpha > \frac{1}{2}$

- the solution exists only if X_0 is suitably chosen
- there is at most one sufficiently variable (= satisfying (1)) solution

Target

Let $X : [0, T] \rightarrow \mathbb{R}^n$ be in C^α , $Y : [0, T] \rightarrow \mathbb{R}$ be in C^γ s.t. $\alpha + \gamma > 1$.

Consider

$$\int_0^t \varphi(X_s) dY_s$$

for some coefficient $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ belonging to $\varphi \in BV_{\text{loc}}(\mathbb{R}^n)$.

Note: in higher dimension we don't have $\varphi \sim \int |x - a| \mu(da)$ at our disposal

We aim to

- provide deeper understanding on the notion of (sufficient) variability
- extend the concept into higher dimensions
- give conditions for $\int_0^t \varphi(X_s) dY_s$ to exist
- study differential systems $dX_t = \varphi(X_t) dY_t$ in higher dimensions

Spoiler alert: discussion how to relax assumptions will be served as a dessert on top of other applications and consequences

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Variability of paths

Let $\varphi \in BV_{\text{loc}}(\mathbb{R}^n)$, that is, $\varphi \in L^1_{\text{loc}}(\mathbb{R}^n)$ and the distributional derivatives $D_i\varphi$ are signed Radon measures. Denote by $\|D\varphi\|$ the total variation measure of φ .

Definition

Let $p \in [1, \infty]$, $s \in (0, 1)$. We say that (continuous) X is (s, p) -variable with respect to φ if there exists a relatively compact open neighborhood $\mathcal{U}$ of $X([0, T])$ such that

$$\int_{\mathcal{U}} |X_{\cdot} - z|^{-n+1-s} \|D\varphi\|(dz) \in L^p(0, T).$$

Remarks

- under probabilistic condition (1) and $n = p = 1$, any φ goes
- for $p = 1$ and using μ as the occupation measure of X , variability reads as $\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x - z|^{-n+1-s} \|D\varphi\|(dz) \mu(dx) < \infty$.

Existence of the integral

Theorem (Hinz, Tölle, V (2023))

Suppose that $\varphi \in BV_{loc}(\mathbb{R}^n)$ and that $X \in C^\alpha([0, T], \mathbb{R}^n)$ is $(s, 1)$ -variable with respect to φ for some $s \in (0, 1)$.

- 1 For any $0 < \beta < \alpha s$ the composition $\varphi \circ X$ is an element of $W_0^{\beta, 1}(0, T)$
- 2 If, in addition, $Y \in C^\gamma([0, T])$ and $1 - \alpha s < \gamma$ then for any $t \in [0, T]$ the integral $\int_0^t \varphi(X_u) dY_u$ exists in the sense of gLS-integrals

Examples

Extreme behaviour of φ

If $\|D\varphi\|(dz)$ is *upper regular* of order $d > n - 1 + s$, then any path X is $(s, 1)$ -variable. This on the other hand forces φ to be Hölder continuous (and hence more "classical" approaches apply). Conversely, if φ locally Lipschitz, then for any $s \in (0, 1)$ any path is $(s, 1)$ -variable.

Extreme behaviour of the path X

If the occupation measure of X is *upper regular* of order $d > n - 1 + s$, then X is $(s, 1)$ -variable w.r.t. any φ . This should be compared to one-dimensional case where

$$\sup_{a \in \mathbb{R}} \mathbb{E} \int_0^T |X_t - a|^{-s} dt < \infty$$

implies that any φ is good enough.

Other examples can be constructed:

- Splitting the space into regions where either occupation measure of X or $\|D\varphi\|(dz)$ is upper regular
- "in between" cases where neither of the measures is (*sufficiently*) upper regular

Probabilistic examples

Let $X = B^H$ be the fBm. Then X is $(\frac{1}{H}, 1)$ -variable with respect to any φ satisfying

$$\int_{B(0,1)} |x|^{-n+1} \|D\varphi\|(dx) < \infty.$$

Remarks

- The moment condition for $\|D\varphi\|(dz)$ encodes the fact that $B_0^H = 0$ and at $t \approx 0$ fBm is not "active enough" which is compensated by assuming φ is "nice enough" around the origin
- This example can be easily generalised to cover many interesting probabilistic examples allowing to study stochastic differential systems and their variability solutions

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Differential systems

Recall the differential system

$$X_t = x_0 + \int_0^t \varphi(X_u) dY_u, \quad t \in [0, T].$$

Here we assume $T > 0$, $x_0 \in \mathbb{R}^n$, $\varphi : \mathbb{R}^n \mapsto \mathbb{R}^{n \times n}$ is a coefficient function $\varphi = (\varphi_{jk})_{1 \leq j, k \leq n}$ with $BV_{loc}(\mathbb{R}^n)$ components, and $Y = (Y^1, \dots, Y^n) : [0, T] \rightarrow \mathbb{R}^n$ is a given Hölder path.

We seek a solution $X : [0, T] \rightarrow \mathbb{R}^n$ that is $(s, 1)$ -variable w.r.t. *(components of)* φ

The change of variable formula

Theorem (Hinz, Tölle, V (2023))

Let $F \in W_{\text{loc}}^{1,1}(\mathbb{R}^n)$ be such that $\partial_k F \in BV_{\text{loc}}(\mathbb{R}^n)$ for $k = 1, \dots, n$. If $X \in C^\alpha([0, T], \mathbb{R}^n)$ with $\alpha > \frac{1}{2}$ is a path which is $(s, 1)$ -variable w.r.t. each $\partial_k F$ for some $s \in (\frac{1-\alpha}{\alpha}, 1)$, then we have

$$F(X_t) = F(x_0) + \sum_{k=1}^n \int_0^t \partial_k F(X_s) dX_s^k$$

for dt -a.e. $t \in [0, T]$, provided that x_0 is not on the bad points of F . If, in addition, F is continuous, then above holds for all $t \in [0, T]$ and no matter where $x_0 \in \mathbb{R}^n$ is located.

Change of variable allows to use Doss transform to solve differential systems

one-dimensional case..

In one-dimensional case one uses Doss transform and the change of variable formula **Torres, V**:

Details

- solve ODE $f' = \varphi(f)$ and use trial $X_t = f(C + Y_t)$
- Suitable initial X_0 gives correct C
- Assumptions on Y^a and φ (recall $\varphi \geq \epsilon > 0$) implies that X is $(s, 1)$ -variable (w.r.t. any φ)
- Assumptions on Y and φ shows that any variable solution has to be of form $f(C + Y_t)$ giving the uniqueness

$$^a Y \in C^\gamma, \gamma > 1/2, \text{ and } \sup_{a \in \mathbb{R}} \mathbb{E} \int_0^T |X_t - a|^{-\theta} dt < \infty$$

Remark: Uniqueness and existence actually up to $\tau = \inf\{t : \sigma(X_t) = 0\}$

..transformed into many

In multidimensional case we mimic the one-dimensional approach:

"Details"

- solve $\nabla f = \varphi(f)$ (technical assumptions on φ needed for solvability, including $\det(\varphi) \geq \epsilon > 0$)
- trial $X_t = f(C + Y_t)$
- uniqueness (more or less) as in one-dimensional case
- tricky part is to ensure existence, i.e. variability of $f(C + Y_t)$
 - in our case there is interplay between X and φ on the notion of variability

Remark: our results generalise (and give intuition) to the one-dimensional results as well

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Fractional Sobolev spaces and variability

- $W^{\theta,q}(0, T)$ denotes the fractional Sobolev space equipped with the norm

$$\|f\|_{\theta,q} = \|f\|_{L^q(0,T)} + \left(\int_0^T \int_0^T \frac{|f(t) - f(s)|^q}{|t - s|^{1+\theta q}} ds dt \right)^{\frac{1}{q}}.$$

- The latter part (Gagliardo seminorm) is denoted by $[f]_{\theta,q}$
- The Hölder space $C^\theta = W^{\theta,\infty}$.
- The class of paths X that are (s, p) -variable with respect to a given $\varphi \in BV(\mathbb{R}^n)$ is denoted by $V(\varphi, s, p)$.

Note that

If we denote by $U^\gamma \mu$ the Riesz-potential of measure μ defined by

$$(U^\gamma \mu)(x) = \int_{\mathbb{R}^n} |x - a|^{\gamma-n} \mu(da), \quad x \in \mathbb{R}^n$$

then $X \in V(\varphi, s, p)$ means $U^{1-s} \|D\varphi\| (X_\cdot) \in L^p(0, T)$.

gLS-integrals I

For f right-continuous at 0 and g left-continuous at T set

$$f_0(t) = f(t) - f(0) \quad \text{and} \quad g_T(t) = g(t) - g(T), \quad t \in [0, T].$$

For $\alpha \in (0, 1)$ set

$$D_{0+}^{\alpha} f_0(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(t) - f(0)}{t^{\alpha}} + \alpha \int_0^t \frac{f(t) - f(u)}{(t-u)^{\alpha+1}} du \right)$$

and

$$D_{T-}^{1-\alpha} g_T(t) = \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{g(t) - g(T)}{(T-t)^{1-\alpha}} + (1-\alpha) \int_t^T \frac{g(t) - g(u)}{(u-t)^{2-\alpha}} du \right).$$

gLS-integrals II

If $D_{0+}^{\alpha} f_0 \in L^p(0, T)$, $D_{T-}^{1-\alpha} g_T(t) \in L^q(0, T)$ for $\frac{1}{p} + \frac{1}{q} \leq 1$, then
 gLS-integral

$$\int_0^T f(u) dg(u) := (-1)^{\alpha} \int_0^T D_{0+}^{\alpha} f_0(t) D_{T-}^{1-\alpha} g_T(t) dt + f(0)(g(T) - g(0)), \quad (2)$$

exists, and its value does not depend on α .

gLS-integrals and Sobolev spaces

Theorem (Hinz, Tölle, V (2023))

Let $\gamma, \delta \in (0, 1)$, $\gamma + \delta > 1$, $1 \leq p, q \leq +\infty$ such that

$$\frac{1}{p} + \frac{1}{q} < \gamma + \delta. \quad (3)$$

If $f \in W^{\gamma,p}(0, T)$ and $g \in W^{\delta,q}(0, T)$ (+ one-sided continuities), then $\int_0^T f(u)dg(u)$ exists and

$$\left| \int_0^T f(u)dg(u) - f(0)(g(T) - g(0)) \right| \leq c \|f_0\|_{W^{\gamma,p}(0,T)} \|g_T\|_{W^{\delta,q}(0,T)}.$$

Remark: Weighted spaces mentioned earlier embedded into these.

Sobolev membership of compositions

Theorem (Hinz, Tölle, V (2023))

Let $\varphi \in BV(\mathbb{R}^n)$, X is a path, $s, \theta \in (0, 1)$, $1 \leq p < +\infty$ and $1 \leq q \leq +\infty$. Suppose that

$$X \in W^{\theta, q}(0, T; \mathbb{R}^n) \cap V(\varphi, s, p).$$

- (i) If $r \geq 1$ is such that $\frac{1}{p} + \frac{s}{q} \leq \frac{1}{r}$ and $\beta < s\theta$, then $\varphi \circ X \in W^{\beta, r}(0, T)$ and in particular,

$$[\varphi \circ X]_{\beta, r} \leq c [X]_{\theta, q}^s \|U^{1-s}\| \|D\varphi\| (X \cdot) \|_{LP(0, T)}$$

with a constant $c > 0$ depending only on $n, s, p, q, r, \theta, \beta$ and T .

- (ii) If $\frac{1}{p} + \frac{s}{q} < s\theta$, then for any $\theta' \leq \theta - \frac{1}{q}$ we have $X \in W^{\theta', \infty}(0, T; \mathbb{R}^n)$, and for any $\beta < s\theta - \frac{1}{p} - \frac{s}{q}$ we have $\varphi \circ X \in W^{\beta, \infty}(0, T)$.

Special cases and examples

Proof

Proof is based on maximal function inequality that in our setting can be stated as

$$|\varphi(X_t) - \varphi(X_\tau)| \leq c |X_t - X_\tau|^s [U^{1-s} \mu(X_t) + U^{1-s} \mu(X_\tau)]$$

for almost all $t, \tau \in [0, T]$ (+ Hölder embedding for the second statement).

Special case

The Hölder case can be recovered by plugging in $q = \infty$ in which case we can choose $r = 1$.

Remarks

- While the above is formulated for $\varphi \in BV(\mathbb{R}^n)$, the localised version can be obtained whenever $X([0, T]) \in \mathbb{R}^n$ is bounded.
- Result implies that one can define integrals $\int_0^T \varphi_1(X_s) d\varphi_2(Y_s)$. However, by Hölder embedding, conditions on parameters imply that one is Hölder continuous but only of very low order

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Summary

Notion of sufficient variability

Quantifies when a given path is active enough on bad points of a BV-function to ensure membership in fractional Sobolev spaces

Existence of (stochastic) integrals

Variability allows to define $\int_0^T \varphi(X_s) dY_s$ (as composition $\varphi(X)$ is not too "bad")

SDE-systems

Once the integrals are defined, one can use Doss transform to solve SDE-systems

$$dX = \varphi(X) dY$$

- existence and uniqueness results

Side dishes

geometric and measure theoretic views of variability, Fourier side approach, regularity properties of the integral, conditions for existence of Riemann-Stieltjes integrals, probabilistic examples, etc.

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Rough regime

We can generalise to the rough regime: let $X = B^H$ be fBm with $\frac{1}{3} < H < \frac{1}{2}$. Then one can give meaningful definition for $\int_0^T \varphi(B_s^H) dB_s^H$ provided that φ has BV -derivative

- fBm not needed, variability condition along lines connecting two points X_s and X_t sufficient $\Rightarrow$ Hölder Gaussians for example are okay
- Hölder continuity not needed, (fractional) Sobolev membership sufficient

Details: Preprint available in 2 months with probability ≈ 1

Variational problems

Recall the variability condition

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x - y|^{-n+1-s} \mu(dx) \|D\varphi\|(dy) < \infty.$$

For given $\varphi \in BV$, we can prove existence of variability-minimising path with occupation measure μ .

- related to continuous versions of discrete Coulomb potentials in mathematical physics
- works with more general measures than only gradient measures $\|D\varphi\|$ and occupation measure μ
- here p -variability is for $p = 1$ but general case straightforward

Details: Preprint available in 3 months with probability ≈ 1

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Thanks for your attention!