

Asymptotic analysis of integral equations with fractional covariance operators

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Introduction

The eigenvalue problem

Exact second order asymptotics for fBm

Application to small L^2 -ball probability problem

Fredholm equations

Innovation of mixed processes

Linear filtering

Statistical inference problems

About the method

Introduction

Fractional Brownian motion

Definition

The fBm $B^H = (B_t^H; t \geq 0)$ is the Gaussian process with zero mean and covariance function

$$\text{cov}(B_t^H, B_s^H) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H})$$

where $H \in (0, 1)$ is the Hurst parameter.

(Note: fBm with $H = \frac{1}{2}$ coincides with Brownian motion)

Fractional Brownian motion: properties for $H \neq \frac{1}{2}$

- H -self similar Gaussian process with stationary increments

$$B_t^H - B_s^H \sim N(0, |t - s|^{2H})$$

- paths are H -Hölder continuous
- neither a semimartingale, nor a Markov process
- the sequence of increments

$$\Delta B_n^H = B_n^H - B_{n-1}^H, \quad n = 1, 2, \dots$$

have long range dependence for $H > \frac{1}{2}$

$$\text{Cov}(\Delta B_1^H, \Delta B_n^H) \asymp C(H)n^{2H-2} \quad \text{as } n \rightarrow \infty$$

- Its formal derivative is a basic model for the "1/f noise" in engineering

Other “fractional” processes:

- fractional “noise”
- fractional Brownian bridge
- fractional OU process
- integrated fBm
- mixed fBm
- Riemann-Liouville process

Problems to study

- The eigenvalue problem
- Fredholm equations

They arise in

- optimal linear filtering/interpolation problems
- statistical inference of processes

The eigenvalue problem

The eigenvalue problem

- A random process $X = (X_t, t \in [0, T])$ with zero mean and covariance function ("kernel")

$$K(s, t) = \mathbb{E}X_s X_t.$$

- Covariance operator

$$(Kf)(t) = \int_0^T K(s, t)f(s)ds.$$

- Find all pairs (λ, φ) with $\lambda \in \mathbb{R}$ which solve

$$K\varphi = \lambda\varphi.$$

The eigenvalue problem

General theory

If $K \in L^2([0, T]^2)$ there are countably many solutions (λ_n, φ_n)

- the eigenvalues λ_n are nonnegative and $\lambda_n \searrow 0$
- the eigenfunctions φ_n form orthogonal basis in $L^2([0, T])$

Applications:

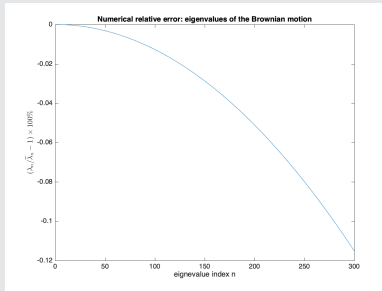
- Karhunen-Loeve expansion
- Statistical inference (nonparametric tests, etc.)
- Filtering and detection problems
- Small L^2 -ball probabilities
- Sampling from heavy tailed distributions
- etc.

Some comments: numerical methods

Numerical approximations...

- ...give reliable results for several *first* eigenelements
- ...**do not work** for **large** values of **n**

Numerical relative error for eigenvalues of Brownian motion



Why it is non trivial

Discontinuity of the second order term

Let $K(s, t) = s \wedge t - \varepsilon s t$ $\varepsilon \in [0, 1]$. Then

$$\lambda_n = \nu_n^{-2}$$

where

- $\nu_n = n\pi - \frac{1}{2}\pi + O(n^{-1})$ when $\varepsilon \in [0, 1)$
- $\nu_n = n\pi$ when $\varepsilon = 1$.

The eigenvalue problem

Example (Brownian motion)

For $K(s, t) = \min(s, t)$ with $s, t \in [0, 1]$

$$\lambda_n = \nu_n^{-2}, \quad \varphi_n(t) = \sqrt{2} \sin(\nu_n t), \quad n = 1, 2, \dots$$

where $\nu_n = (n - \frac{1}{2})\pi$.

Some other processes with exact solutions:

- Brownian bridge
- Demeaned Bm
- OU process
- integrated Bm and bridge
- etc.

The approach: reduction to BVP for ODEs (Sturm-Liouville)

Eigenvalue problem for the fBm

Find all solutions (λ, φ) to the integral equation

$$\int_0^1 K(s, t) \varphi(s) ds = \lambda \varphi(t), \quad t \in [0, 1]$$

with the kernel $K(s, t) = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H})$ with $H \in (0, 1)$.

Theorem (Bronski 03)

$$\lambda_n = \frac{\sin(\pi H) \Gamma(2H + 1)}{(n\pi)^{2H+1}} + o\left(n^{-\frac{(2H+2)(4H+3)}{4H+5} + \delta}\right) \quad \text{as } n \rightarrow \infty.$$

Other proofs: Nazarov and Nikitin 04, Luschgy and Pagès 04.

Theorem (Chigansky and Kleptsyna, 2018)

1. *The eigenvalues satisfy*

$$\lambda_n = \sin(\pi H) \Gamma(2H + 1) \nu_n^{-2H-1}, \quad n = 1, 2, \dots$$

where the sequence ν_n satisfies

$$\nu_n = \left(n - \frac{1}{2}\right)\pi - \frac{\left(H - \frac{1}{2}\right)^2}{H + \frac{1}{2}} \frac{\pi}{2} + O(n^{-1}), \quad n \rightarrow \infty.$$

- Exact **second order** asymptotics.

Eigenvalue problem for the fBm

Theorem (continued)

2. *The eigenfunctions satisfy*

$$\varphi_n(t) = \sqrt{2} \sin(\nu_n t + \eta_H) + \int_0^\infty \left(f_0(u) e^{-t\nu_n u} + f_1(u) e^{-(1-t)\nu_n u} \right) du + \nu_n^{-1} r_n(t), \quad t \in [0, 1]$$

where f_0 and f_1 are explicit (cumbersome) functions and

$$\eta_H = \frac{1}{4} \frac{(H - \frac{1}{2})(H - \frac{3}{2})}{H + \frac{1}{2}} \quad \text{and} \quad \sup_{t \in [0, 1]} |r_n(t)| \leq C(H).$$

- oscillatory term + boundary layer terms + residual

Theorem (continued)

3. *The eigenfunctions satisfy*

$$\varphi_n(1) = (-1)^n \sqrt{2H+1} (1 + O(n^{-1}))$$

and, with an explicit constant C_H ,

$$\int_0^1 \varphi_n(x) dx = C_H \nu_n^{-1}.$$

- additional “byproduct” of the method (useful in some applications)

An application: small ball probabilities

Small ball probabilities problem:

For a given process $X = (X_t, t \in [0, 1])$ and a norm $\|\cdot\|$ find the asymptotics of $P(\|X\| \leq \varepsilon)$ as $\varepsilon \rightarrow 0$.

- The most studied is the Gaussian case with L^2 -norm:

$$P(\|X\|_2 \leq \varepsilon) = P\left(\sum_{n=1}^{\infty} \lambda_n Z_n^2 \leq \varepsilon^2\right) \text{ with } Z_n \stackrel{\text{i.i.d.}}{\sim} N(0, 1)$$

- Some milestones: Sytaja 75 (Laplace transform), Li 92 (comparison theorem), Dunker, Lifshits and Linde 98 (checkable conditions), ...

Example (Bm, Cameron-Martin 44)

$$P(\|B\|_2 \leq \varepsilon) = \frac{4}{\sqrt{\pi}} \varepsilon \exp\left(-\frac{1}{8} \varepsilon^{-2}\right) (1 + o(1)), \quad \varepsilon \rightarrow 0.$$

An application: small ball probabilities

Nikitin and Nazarov 2004:

- The asymptotics of $-\log P(\|X\|_2 \leq \varepsilon)$ is determined by the **first order** asymptotics of λ_n
- The asymptotics of $P(\|X\|_2 \leq \varepsilon)$ is determined by the **second order** asymptotics of λ_n , up to a mult. “distortion” constant.

Corollary (Bronski 03)

For any $H \in (0, 1)$,

$$-\log P(\|B^H\|_2 \leq \varepsilon) = \beta(H)\varepsilon^{-1/H}(1 + o(\varepsilon)), \quad \varepsilon \rightarrow 0,$$

where

$$\beta(H) = H \frac{(\sin(\pi H)\Gamma(2H + 1))^{1/(2H)}}{((2H + 1) \sin \frac{\pi}{2H+1})^{1/(2H)+1}}$$

An application: small ball probabilities

Corollary (Chigansky and Kleptsyna 2018)

For all $H \in (0, 1)$,

$$P(\|B^H\|_2 \leq \varepsilon) = C_d(H) \varepsilon^{\gamma(H)} \exp\left(-\beta(H) \varepsilon^{-1/H}\right) (1 + o(1)), \quad \varepsilon \rightarrow 0$$

where $C_d(H)$ is (yet unknown) distortion constant and

$$\gamma(H) = \frac{1}{2H} \left(3/4 + H^2 + \frac{1}{2} \frac{1 - 2H}{(1 + 2H)^2} \right).$$

Fredholm equations

Find a function g which solves the equation

$$\varepsilon g(t) + \int_0^T K(s, t)g(s)ds = f(t), \quad t \in [0, T]$$

where

- $\varepsilon > 0$ is a parameter
- $f(t)$ is a given “forcing” function
- $K(s, t)$ is a covariance kernel of a random process
- $T < \infty$ or $T = \infty$

General theory

Existence and uniqueness of solutions in appropriate spaces depending on K and f . The solutions are rarely explicit.

A frequent objective in applications is to find the asymptotics of some functional of the solution with respect to a relevant parameter:

- Large time $T \rightarrow \infty$ (finite section limits)
- Small parameter $\varepsilon \rightarrow 0$ (singular perturbation)

Definition

Mixed fBm is the process

$$X_t = B_t^H + B_t, \quad t \in [0, T]$$

where B^H and B are independent fractional and standard Bm's.

Theorem (Cheredito 01)

X is a semimartingale if and only if $H \in (\frac{3}{4}, 1) \cup \{\frac{1}{2}\}$. For $H > \frac{3}{4}$, X and B induce equivalent probability measures.

Theorem (Cai, Chigansky and Kleptsyna, 16)

The process $M_t = E(B_t | \mathcal{F}_t^X)$ is the canonical innovating (fundamental) martingale for X :

$$\mathcal{F}_t^X = \mathcal{F}_t^M, \quad t \in [0, T].$$

The equation

$$g(s, t) + \frac{d}{ds} \int_0^t g(r, t) H |s - r|^{2H-1} \text{sign}(s - r) dr = 1,$$

has unique continuous solution and

$$M_t = \int_0^t g(s, t) dX_s \quad \text{and} \quad \langle M \rangle_t = \int_0^t g(s, t) ds.$$

Innovation of the mixed fBm

- For $H \in (\frac{1}{2}, 1)$ the derivative and integration interchange and

$$g(s, t) + \int_0^t g(r, t) c_H |s - r|^{2H-2} dr = 1, \quad 0 < s < t$$

with $c_H = H(2H - 1)$ and

$$\langle M \rangle_t = \int_0^t g(s, t) ds = \int_0^t g(s, s)^2 ds.$$

- Let $\varepsilon := t^{1-2H}$ then $\gamma_\varepsilon(x) := t^{2H-1} g(xt, t)$ solves

$$\varepsilon \gamma_\varepsilon(x) + \int_0^1 c_H |x - y|^{2-2H} \gamma_\varepsilon(y) dy = 1, \quad 0 < x < 1.$$

- $\langle M \rangle_t = \varepsilon^{\frac{2-2H}{2H-1}} \langle \gamma_\varepsilon, 1 \rangle$ and $\frac{d\langle M \rangle_t}{dt} = \varepsilon^2 \gamma_\varepsilon^2(1).$

- For $\varepsilon = 0$, the equation of the 2nd kind

$$\varepsilon \gamma_\varepsilon(x) + \int_0^1 c_H |x - y|^{2-2H} \gamma_\varepsilon(y) dy = 1, \quad 0 < x < 1.$$

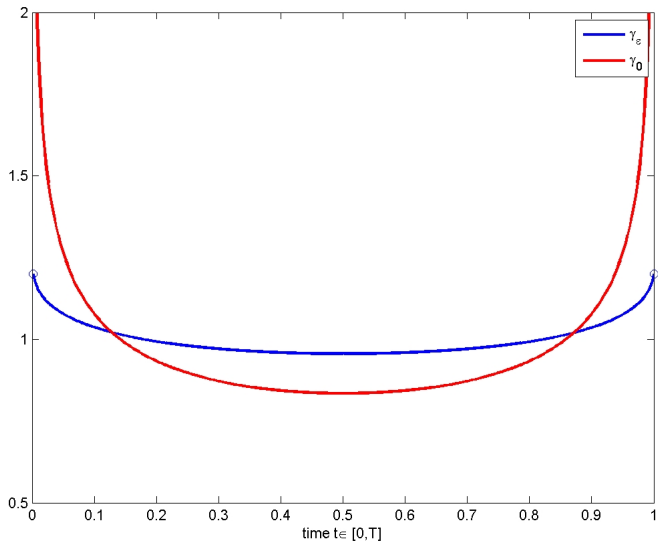
degenerates to the equation of the 1st kind

$$\int_0^1 c_H |x - y|^{2-2H} \gamma_0(y) dy = 1, \quad 0 < x < 1$$

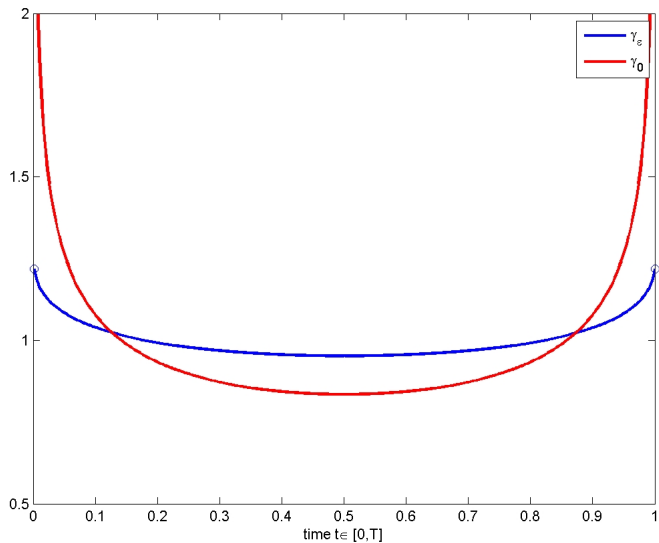
with the closed form solution (a_H is an explicit const.)

$$\gamma_0(x) = a_H x^{\frac{1}{2}-H} (1-x)^{\frac{1}{2}-H}, \quad x \in (0, 1).$$

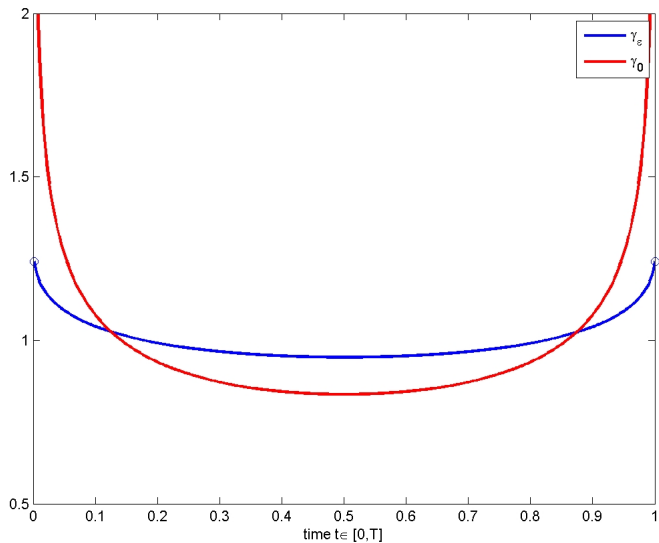
Singular perturbation



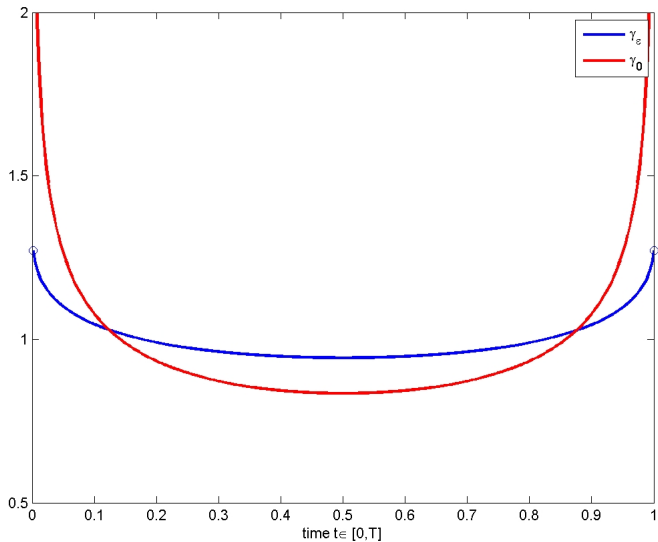
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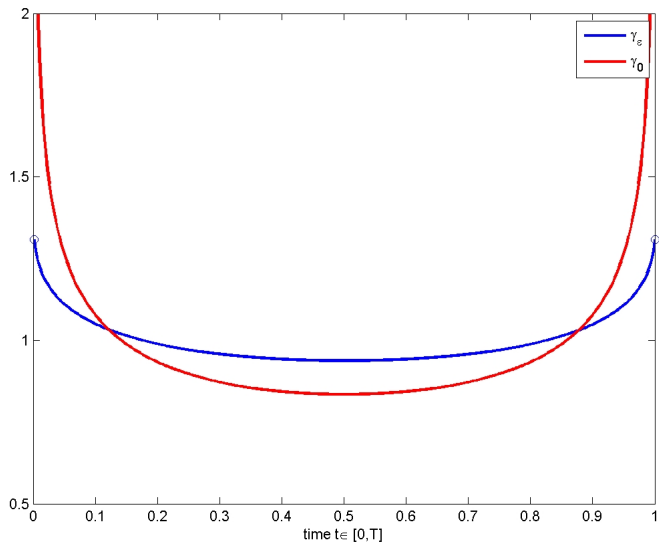
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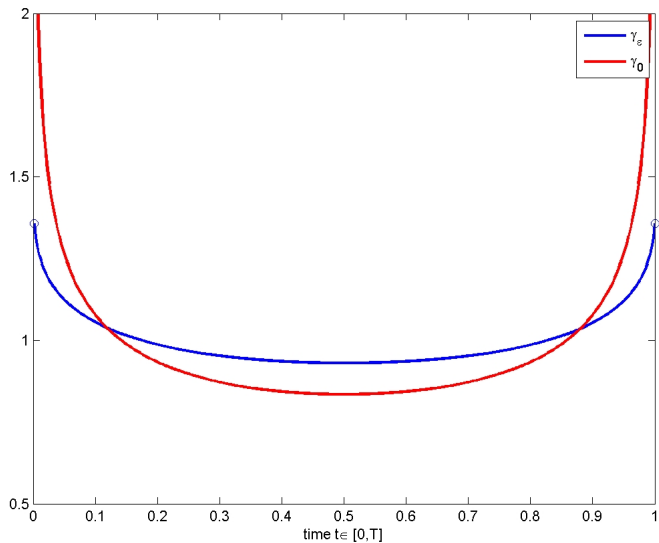
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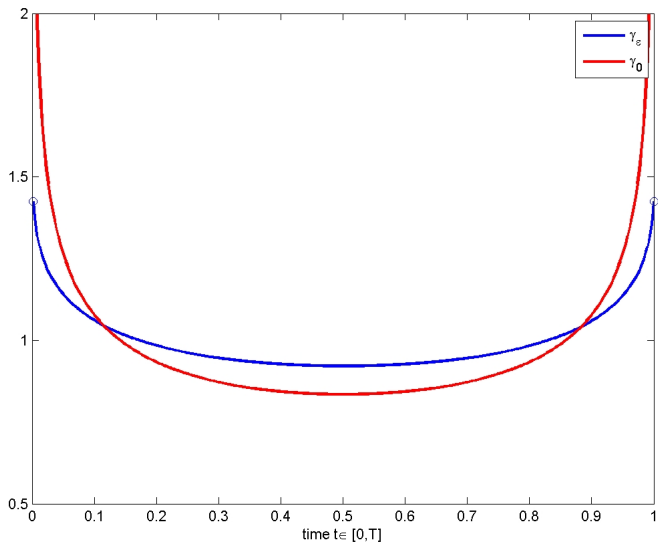
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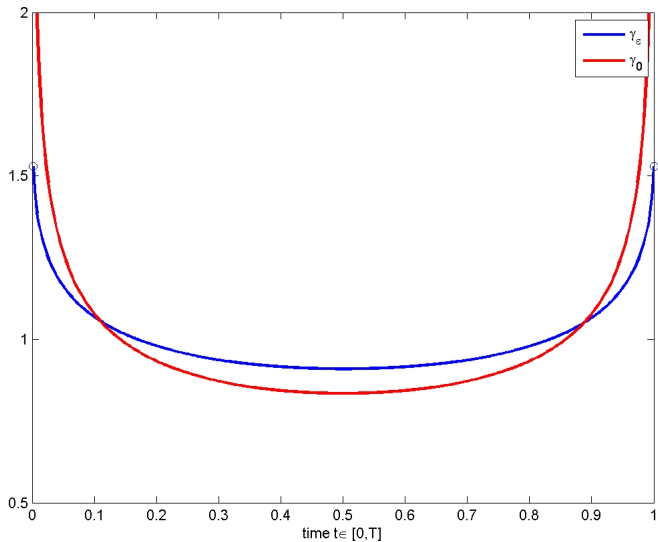
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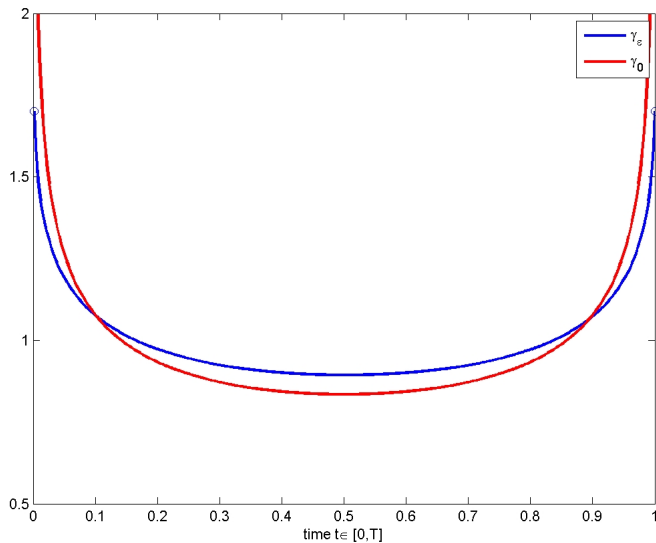
Singular perturbation



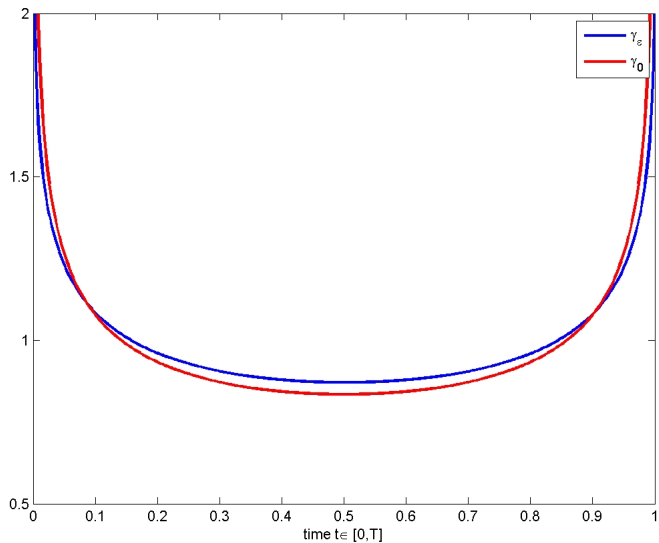
Singular perturbation



Singular perturbation



Singular perturbation



Corollary (Chigansky and Kleptsyna, 16)

(1) $\gamma_\varepsilon \rightarrow \gamma_0$ in $L^2(0, 1)$ and as $\varepsilon \rightarrow 0$,

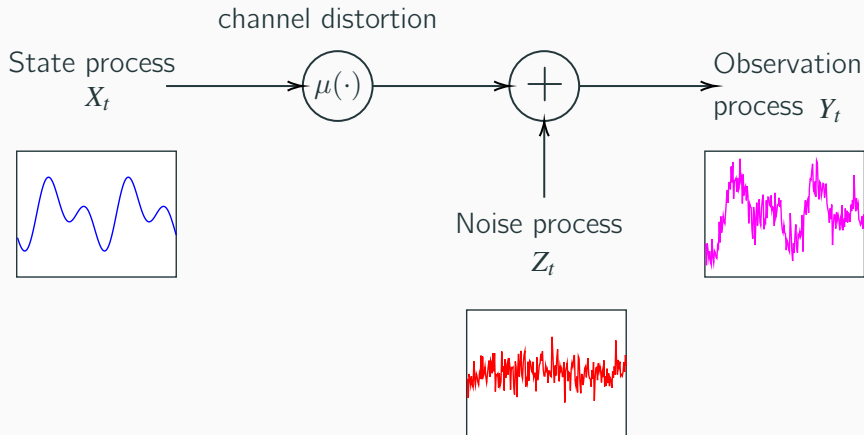
$$\|\gamma_\varepsilon - \gamma_0\|_2 \asymp \begin{cases} \varepsilon^{\frac{1-H}{2H-1}}, & H \in (\frac{2}{3}, 1), \\ \varepsilon \sqrt{\log \varepsilon^{-1}}, & H = \frac{2}{3}, \\ \varepsilon & H \in (\frac{1}{2}, \frac{2}{3}). \end{cases}$$

(2) The solution γ_ε diverges to $+\infty$ at the endpoints:

$$\gamma_\varepsilon(0) = \gamma_\varepsilon(1) \asymp \varepsilon^{-\frac{1}{2}}, \quad \varepsilon \rightarrow 0.$$

Proof: by means of expansion into series of eigenfunctions.

The filtering problem



The Kalman-Bucy filtering problem

A basic "Additive White Gaussian Noise" model in engineering:

- OU "state" process

$$X_t = \beta \int_0^t X_s ds + W_t, \quad t \in [0, T]$$

where $\beta \in \mathbb{R}$ is a constant and W is a Brownian motion.

- "Observation" process

$$Y_t = \mu \int_0^t X_s ds + \sqrt{\varepsilon} V_t, \quad t \in [0, T]$$

where $\mu \neq 0$ is the "channel gain" constant, $\varepsilon > 0$ is the "noise intensity" parameter and V is a Brownian motion independent of W .

- Find the optimal estimator of X_t and its MSE:

$$\hat{X}_t = E(X_t | \mathcal{F}_t^Y) \quad \text{and} \quad P_t = E(X_t - \hat{X}_t)^2.$$

The Kalman-Bucy filtering problem

Theorem (Kalman and Bucy, 61)

The optimal estimator solves the linear SDE

$$d\hat{X}_t = \beta\hat{X}_t dt + \frac{\mu P_t}{\varepsilon}(dY_t - \mu\hat{X}_t dt), \quad t \in [0, T]$$

subject to $\hat{X}_0 = 0$, where P_t solves the Riccati ODE

$$\dot{P}_t = 2\beta P_t + 1 - (\mu/\sqrt{\varepsilon})^2 P_t^2$$

started at $P_0 = 0$.

The Kalman-Bucy filtering problem

Elementary analysis reveals that

- the steady state error is

$$P_T\left(\beta, \frac{\mu}{\sqrt{\varepsilon}}\right) \xrightarrow{T \rightarrow \infty} \frac{\beta + \sqrt{\beta^2 + \mu^2/\varepsilon}}{\mu^2/\varepsilon}.$$

- and the small noise ("high SNR") asymptotics is

$$P_T\left(\beta, \frac{\mu}{\sqrt{\varepsilon}}\right) = \frac{\sqrt{\varepsilon}}{\mu}(1 + o(1)) \quad \text{as } \varepsilon \rightarrow 0, \quad \forall T > 0.$$

Kalman-Bucy model with general noises

- The optimal filter is

$$\hat{X}_T = \frac{1}{\sqrt{\varepsilon}} \int_0^T g(s) dY_s$$

- The weight function $g(s)$ solves **integro-differential equation**

$$\begin{aligned} \frac{\partial}{\partial s} \int_0^T g(r) \frac{\partial}{\partial r} K_V(r, s) dr + \\ \frac{\mu^2}{\varepsilon} \int_0^T K_X(r, s) g(r) dr = \frac{\mu}{\sqrt{\varepsilon}} K_X(s, T), \quad s \in [0, T], \end{aligned}$$

with the covariance kernels

$$K_X(s, t) = \text{Cov}(X_s, X_t) \quad \text{and} \quad K_V(s, t) = \text{Cov}(V_s, V_t).$$

- The minimal MSE is given by the **functional**

$$P_T\left(\beta, \frac{\mu}{\sqrt{\varepsilon}}\right) = \frac{\sqrt{\varepsilon}}{\mu} \left(\frac{\partial}{\partial s} \int_0^T g(r) \frac{\partial}{\partial r} K_V(r, s) dr \right) \Big|_{s := T}.$$

Kalman-Bucy model with fractional noises

Theorem (Afterman, Chigansky, Kleptsyna, Marushkevych 22)

For the Kalman-Bucy model with fBm's W^{H_1} and V^{H_2} , the steady state error exists:

$$P_{\infty}\left(\beta, \frac{\mu}{\sqrt{\varepsilon}}\right) = \lim_{T \rightarrow \infty} P_T\left(\beta, \frac{\mu}{\sqrt{\varepsilon}}\right),$$

and, for any $T > 0$,

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-\nu} P_T\left(\beta, \frac{\mu}{\sqrt{\varepsilon}}\right) = P_{\infty}(0, \mu) \quad \text{with } \nu = \frac{H_1}{1 + H_1 - H_2}.$$

- Asymptotic error increases with roughness of the noises

Kalman-Bucy model with fractional noises

In meaningful special cases more detailed picture is obtained. The main ingredients are:

- sectionally holomorphic “structural” function on $\mathbb{C} \setminus \mathbb{R}$:

$$\Lambda(z; H_1, H_2) = (z^2 - \beta^2) \kappa(H_2) \left(\pm \frac{z}{i} \right)^{1-2H_2} - \frac{\mu^2}{\varepsilon} \kappa(H_1) \left(\pm \frac{z}{i} \right)^{1-2H_1}$$

where $\kappa(H) = \Gamma(2H + 1) \sin(\pi H)$.

- $\Lambda(z; H_1, H_2)$ has zero z_0 in the first quadrant iff $H_1 > H_2$.
- the limit across the cut $\mathbb{R}$ in the upper half plane:

$$\Lambda^+(t; H_1, H_2) := \lim_{z \rightarrow t, \operatorname{Im}(z) > 0} \Lambda(z; H_1, H_2)$$

and its angle (explicit function)

$$\theta(t; H_1, H_2) = \arg \{ \Lambda^+(t; H_1, H_2) \}.$$

Kalman-Bucy model with fractional noises

Filtering of a fractional OU process in white noise:

Theorem (AChKM 2022)

Let $H := H_1 \in (0, 1) \setminus \{\frac{1}{2}\}$ and $H_2 = \frac{1}{2}$, then

$$P_\infty\left(\beta, \frac{\mu}{\sqrt{\varepsilon}}\right) = \frac{\varepsilon}{\mu^2} \left(\frac{1}{\pi} \int_0^\infty \theta(t; H, \tfrac{1}{2}) dt + \beta + 2\operatorname{Re}(z_0) \mathbf{1}_{\{H > \frac{1}{2}\}} \right)$$

Consequently, for any $T > 0$,

$$P_T\left(\beta, \frac{\mu}{\sqrt{\varepsilon}}\right) \asymp \frac{(\Gamma(2H+1) \sin(\pi H))^{\frac{1}{2H+1}}}{\sin \frac{\pi}{2H+1}} (\varepsilon/\mu^2)^{\frac{2H}{2H+1}}, \quad \text{as } \varepsilon \rightarrow 0.$$

- Reduces to the classical spectral formula in the stable case $\beta < 0$, but is valid also in the **unstable** case $\beta \geq 0$!

Estimation of the Hurst parameter from noisy continuous data

Problem

Estimate the parameters $H \in (0, 1)$ and $\sigma > 0$ from the sample

$$X_t = \sigma B_t^H + \sqrt{\varepsilon} B_t, \quad t \in [0, T]$$

where B^H and B are independent fractional and standard Bm's and $\varepsilon > 0$ is a known parameter.

- Both σ and H can be recovered exactly from the above sample if and only if $H \in (0, \frac{3}{4}]$ (by means of discrete power variations Dozzi, Mishura and Shevchenko, 15)

Estimation of the Hurst parameter from noisy continuous data

Theorem (Chigansky and Kleptsyna, 2023)

The parameters $H \in (\frac{3}{4}, 1)$ and $\sigma \in (0, \infty)$ can be estimated at the optimal local minimax rates

$$\varepsilon^{1/(4H-2)} \quad \text{and} \quad \varepsilon^{1/(4H-2)} \log \varepsilon^{-1}.$$

respectively, as $\varepsilon \rightarrow 0$.

Proof: verification of the LAN property, hinges on asymptotic analysis of the equation

$$\varepsilon g(t) + \int_0^T K(t-s)g(s)ds = K(t), \quad 0 < s < t,$$

with the weakly singular kernel

$$K(t) = \sigma H(2H-1)|t|^{2H-2}.$$

About the method

About the method

Inspired by math physics: Gibbs 66, **Ukai 71**, Pal'cev 74

Step 1. Reduce to an equation with differential and weakly singular integral operators.

Example

The eigenvalue problem for the fBm

$$\int_0^1 \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}) \varphi(s) ds = \lambda \varphi(t), \quad 0 < t < 1,$$

for $H \in (\frac{1}{2}, 1)$ reduces to the generalized eigenvalue problem

$$\int_0^1 c_\alpha |s - t|^{-\alpha} \psi(s) ds = -\lambda \psi''(t), \quad 0 < t < 1.$$

with $\alpha = 2 - 2H \in (0, 1)$ for $\psi(s) = \int_s^1 \varphi(s) ds$.

About the method

Step 2. Use the identity

$$|s - t|^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty \tau^{\alpha-1} e^{-\tau|s-t|} d\tau, \quad \alpha \in (0, 1)$$

to show that the Laplace transform

$$\hat{\psi}(z) = \int_0^1 \psi(t) e^{-tz} dt, \quad z \in \mathbb{C},$$

admits of the representation

$$\hat{\psi}(z) = \text{analytic function} + \frac{\Phi_0(z) + e^{-z}\Phi_1(-z)}{\Lambda(z)}$$

where

- Λ is an explicit function, sectionally holomorphic on $\mathbb{C} \setminus \mathbb{R}$ and, possibly, has isolated zeros.
- Φ_0 and Φ_1 are *unknown* functions, sectionally holomorphic on $\mathbb{C} \setminus \mathbb{R}_+$ with a priori growth estimates at 0 and at ∞ .

About the method

$$\hat{\psi}(z) = \text{analytic function} + \frac{\Phi_0(z) + e^{-z}\Phi_1(-z)}{\Lambda(z)}$$

Step 3. Since $\hat{\psi}$ is an entire function singularities in the ratio must be removable. Removal of the jump across the cut $\mathbb{R}$ gives a Hilbert BVP for $\Phi_0(z)$ and $\Phi_1(z)$ with boundary condition on $\mathbb{R}_+$, such as e.g.,

$$\begin{aligned}\Phi_0^+(t) - e^{2i\theta(t)}\Phi_0^-(t) &= 2ie^{-t}e^{i\theta(t)}\sin\theta(t)\Phi_1(-t) \\ \Phi_1^+(t) - e^{2i\theta(t)}\Phi_1^-(t) &= 2ie^{-t}e^{i\theta(t)}\sin\theta(t)\Phi_0(-t)\end{aligned}\quad t \in \mathbb{R}_+,$$

where $\theta(t) = \arg\{\Lambda^+(t)\}$ is an explicit function. If $\Lambda(z)$ has zeros, additional algebraic constraints apply.

About the method

Step 4. By means of the standard technique (the Sokhotski - Plemelj formula, etc.) for the Hilbert problems, the functions $\Phi_0(z)$ and $\Phi_1(z)$, and thus the Laplace transform $\widehat{\psi}(z)$, are expressed in terms of solutions to certain auxiliary integral equations on the semi-axis $\mathbb{R}_+$, such as e.g.,

$$p(t) = \frac{1}{\pi} \int_0^\infty \frac{h_0(s)e^{-\nu s}}{s+t} p(s) ds + f(t), \quad t \in \mathbb{R}_+$$

where

- ν is related to λ by a simple closed form formula
- h_0 is reasonably explicit function
- f is either a polynomial or an explicit function

Remarkably, these auxiliary equations are amenable to asymptotic analysis as $\nu \rightarrow \infty$ (corr. $\lambda \rightarrow 0$) and $p(t) = f(t) + o(1)$

Thank you!