

Limit distributions for the discretization error of stochastic Volterra equations with fractional kernel

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Introduction

Consider a standard d -dimensional SDE and its discretization:

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s,$$
$$\hat{X}_t = X_0 + \int_0^t b(\hat{X}_{\lfloor ns \rfloor / n})ds + \int_0^t \sigma(\hat{X}_{\lfloor ns \rfloor / n})dW_s,$$

where W is an m -dimensional BM.

Theorem (Kurtz and Protter (1991), Jacod and Protter (1998))

The scaled error $\mathbf{U}^n = \sqrt{n}(X - \hat{X})$ converges stably in law to the solution $\mathbf{U} = (U^1, \dots, U^d)^\top$ of the SDE (with Einstein summation convention)

$$U_t^i = \int_0^t U_s^k \left(\partial_k b^i(X_s)ds + \partial_k \sigma_j^i(X_s)dW_s^j \right) - \frac{1}{\sqrt{2}} \int_0^t \partial_k \sigma_j^i(X_s) \sigma_l^k(X_s) dB_s^{l,j}$$

as $n \rightarrow \infty$, where B is an m^2 -dimensional BM independent of W .

Recent extensions include

- stochastic time partitions: Fukasawa and Obloj (2020)
- SDEs driven by a fractional Brownian motion:
Hu et al. (2016), Liu and Tindel (2019), Aida and Naganuma (2020)
- Multi-Level Monte Carlo: Ben Alaya and Kebaier (2015)

Motivated by rough volatility modeling, we consider an extension to Stochastic Volterra Equations (SVE) with fractional kernel

$$X_t = X_0 + \int_0^t K(t-s)b(X_s)ds + \int_0^t K(t-s)\sigma(X_s)dW_s,$$

where

$$K(u) = \frac{u^{H-1/2}}{\Gamma(H+1/2)}$$

and $H \in (0, 1)$.

Discretization of SVE

First we consider

$$\hat{X}_t = X_0 + \int_0^t K(t-s)b(\hat{X}_{\frac{[ns]}{n}})ds + \int_0^t K(t-s)\sigma(\hat{X}_{\frac{[ns]}{n}})dW_s.$$

We can also deal with the Euler-type scheme:

$$\check{X}_t = X_0 + \int_0^t K\left(t - \frac{[ns]}{n}\right)b(\check{X}_{\frac{[ns]}{n}})ds + \int_0^t K\left(t - \frac{[ns]}{n}\right)\sigma(\check{X}_{\frac{[ns]}{n}})dW_s.$$

Zhang (2008), Li et al. (2020) and Richard et al. (2021) gave the rate of convergence for $\check{X}$.

Note: our 1st version (arXiv, 2021) only dealt with $\hat{X}$. Inspired by Nualart and Saikia (arXiv, 2022), our 2nd version treats $\check{X}$ as well. Our published version assumes $H \leq 1/2$, but the result remains true for $H \in (0, 1)$.

Note on the construction of $\hat{X}$

We can express $\hat{X}$ as

$$\hat{X}_t = X_0 + \sum_{j=0}^{[nt]} b(\hat{X}_{\frac{j}{n}}) \int_{\frac{j}{n}}^{\frac{j+1}{n} \wedge t} K(t-s) ds + \sum_{j=0}^{[nt]} \sigma(\hat{X}_{\frac{j}{n}}) \int_{\frac{j}{n}}^{\frac{j+1}{n} \wedge t} K(t-s) dW_s.$$

In particular, we can construct $(\hat{X}_{t_1}, \dots, \hat{X}_{t_n})$ on the regular grid $t_k = \frac{k}{n}$ by generating the independent sequence of Gaussian vectors (matrices when the dimension of W is larger than two)

$$\left(\int_{t_{k-1}}^{t_k} K(t_k - s) dW_s, \int_{t_{k-1}}^{t_{k+1}} K(t_{k+1} - s) dW_s, \dots, \int_{t_{k-1}}^{t_n} K(t_n - s) dW_s \right)$$

using the Cholesky decomposition of the identical covariance matrix

$$\Sigma = [\Sigma_{ij}], \quad \Sigma_{ij} = \int_0^{\frac{1}{n}} K\left(\frac{i}{n} - s\right) K\left(\frac{j}{n} - s\right) ds.$$

The main result (for $\hat{X}$)

$(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$: a filtered probability space with the usual conditions.

W : an m -dimensional BM on $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$.

$b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$ with bounded and unif. conti. derivatives.

C_0^λ : the set of the λ -Hölder continuous functions on $[0, T]$ vanishing at 0.

Theorem

Let $\epsilon \in (0, H)$. Then $\mathbf{U}^n := n^H(X - \hat{X})$ stably converges in law in $C_0^{H-\epsilon}$ to the unique continuous solution $\mathbf{U} = (U^1, \dots, U^d)^\top$ of the SVE

$$\begin{aligned} U_t^i &= \int_0^t K(t-s) U_s^k \left(\partial_k b^i(X_s) ds + \partial_k \sigma_j^i(X_s) dW_s^j \right) \\ &\quad - \frac{1}{\sqrt{\Gamma(2H+2) \sin \pi H}} \int_0^t K(t-s) \partial_k \sigma_j^i(X_s) \sigma_l^k(X_s) dB_s^{l,j}, \end{aligned}$$

where B is an m^2 -dimensional standard Brownian motion, independent of $\mathcal{F}$ and defined on some extension of $(\Omega, \mathcal{F}, \mathbb{P})$.

Key ingredients in the proof

- 1 Jacod's theory of stable convergence of semimartingales
 - a key in Jacod and Protter (1998).
 - a simplified version (continuous, 1-dim case) of Jacod (1994) is as follows: For 1-dim continuous local martingales M^n with

$$\langle M^n \rangle_t \rightarrow V_t, \quad \langle M^n, W \rangle_t \rightarrow 0$$

in prob. (for each t) for a continuous V as $n \rightarrow \infty$, we have $M^n \rightarrow B_V$ stably in law in C . Cf. an asymptotic version of Knight's theorem.

- 2 Continuity of fractional derivative map between Hölder spaces
 - Hardy-Littlewood
 - For $\alpha, \beta \in (0, 1)$,

$$C_0^\alpha \ni f \mapsto \left(t \mapsto \int_0^t (t-s)^{\beta-\alpha} df(s) \right) \in C_0^\beta$$

is continuous, where the integral is defined via integration-by-parts.

Properties of Hölder spaces

Let $0 < \lambda < 1$.

- (trivial) The inclusion $C_0^\lambda \rightarrow C$ is continuous.
- so the stable convergence in law in C_0^λ implies that in C .
- Not separable; for example $\{f_a\}_{a \in (0,1)}$, $f_a(t) := (t-a)_+^\lambda$ is a family of uncountably many functions with $\|f_a - f_b\|_\lambda \geq 1$ for any $a \neq b$.
Prohorov's theorem still ensures that a tight sequence of probability measures has a convergent subsequence (separability needed for its converse).
- The bounded set $\{f \in C_0^\lambda; \|f\|_\lambda \leq \delta\}$ is compact in $C_0^{\lambda-\epsilon}$ for any $\delta > 0$ and $\epsilon \in (0, \lambda)$.
- By Kolmogorov's continuity theorem (or Garsia-Rodemich-Rumsey inequality), λ -Hölder property in the L^p sense for all $p \geq 1$ implies $(\lambda - \epsilon)$ Hölder continuity in the almost sure pathwise sense.

Sketch of proof

Step 1: Taylor expansion

$$\begin{aligned}U_t^{n,i} &= n^H(X_t - \hat{X}_t) \\&= n^H \int_0^t K(t-s)((b^i(X_s) - b^i(\hat{X}_{\frac{[ns]}{n}}))ds + (\sigma_j^i(X_s) - \sigma_j^i(\hat{X}_{\frac{[ns]}{n}}))dW_s^j \\&\approx \int_0^t K(t-s)(\nabla b^i(\hat{X}_s)^\top \mathbf{U}_s^n ds + \nabla \sigma_j^i(\hat{X}_s)^\top \mathbf{U}_s^n dW_s^j) \\&+ \int_0^t K(t-s)n^H \nabla b^i(\hat{X}_s)^\top (\hat{X}_s - \hat{X}_{\frac{[ns]}{n}})ds + \int_0^t K(t-s)\partial_k \sigma_j^i(\hat{X}_s)dV_s^{n,k,j},\end{aligned}$$

where $\mathbf{V}^n = \{V^{n,k,j}\}$ is defined as

$$V^{n,k,j} = n^H \int_0^\cdot (\hat{X}_s^k - \hat{X}_{\frac{[ns]}{n}}^k) dW_s^j, \quad 1 \leq k \leq d, 1 \leq j \leq m.$$

Step 2: For all $t \in [0, T]$, $(k_1, k_2) \in \{1, \dots, d\}^2$, $1 \leq j \leq m$,

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$$\langle V^{n,k_1,j}, V^{n,k_2,j} \rangle_t \xrightarrow[\text{in } L_1]{n \rightarrow \infty} \frac{1}{\Gamma(2H+2) \sin \pi H} \sum_{l=1}^m \int_0^t \sigma_l^{k_1}(X_s) \sigma_l^{k_2}(X_s) ds,$$

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$$\langle V^{n,k,j}, W^j \rangle_t \xrightarrow[\text{in } L_1]{n \rightarrow \infty} 0.$$

This part is the most computationally heavy. Once we have this, Jacod's theorem implies the stable convergence $\mathbf{V}^n \rightarrow \mathbf{V} = \{V^{i,j}\}$. Further by Kurt and Protter (1994),

$$\int_0^\cdot \partial_k \sigma_j^i(\hat{X}_s) dV_s^{n,k,j} \rightarrow \int_0^\cdot \partial_k \sigma_j^i(X_s) dV_s^{k,j}$$

in C .

Step 3: The tightness of $\mathbf{U}^n$ in $C_0^{H-\epsilon}$, and of Φ^n in $C^{1/2-\epsilon}$, where

$$\begin{aligned}\Phi^n = & \int_0^t (\nabla b^i(\hat{X}_s)^\top \mathbf{U}_s^n ds + \nabla \sigma_j^i(\hat{X}_s)^\top \mathbf{U}_s^n dW_s^j) \\ & + \int_0^t n^H \nabla b^i(\hat{X}_s)^\top (\hat{X}_s - \hat{X}_{\frac{[ns]}{n}}) ds + \int_0^t \partial_k \sigma_j^i(\hat{X}_s) dV_s^{n,k,j},\end{aligned}$$

- This follows the L^p boundedness of their Hölder norms.

Step 4: We show any subsequence of $\mathbf{U}^n$ has a further subsequence with the unique limit. By Step 1,

$$\mathbf{U}^n - J\Phi^n \rightarrow 0,$$

where

$$Jf(t) = \int_0^t K(t-s) df(s).$$

Since $J : C_0^{1/2-\epsilon} \rightarrow C_0^{H-\epsilon}$ is continuous, $\mathbf{U}^n - J\Phi^n \rightarrow \mathbf{U} - J\Phi = 0$, with the limit law $\mathbf{U}$ uniquely specified by Step 2.

A lemma to compute the limit of the quadratic variation

Lemma

Assume $g \in L_2(0, 1)$. Let $H^{(n)}$ and H be stochastic processes on $[0, T]$ such that

$$\mathbb{E} \left[\int_0^T |H_s^{(n)} - H_s|^2 ds \right] \rightarrow 0$$

with H being almost surely continuous. Then, for all $t \in [0, T]$,

$$\int_0^t H_s^{(n)} g(ns - [ns]) ds \xrightarrow[\text{in } L_2]{n \rightarrow \infty} \int_0^1 g(r) dr \int_0^t H_s ds.$$

Roughly, the fractional part $X_n(s) := ns - [ns]$ converges in law to an independent uniform distribution on $[0, 1]$.

Cf. Tukey (1938), Delattre and Jacod (1997).

Euler-type schemes

Next, let us consider the solution $\check{X}$ of a discretized SVE

$$\check{X}_t = X_0 + \int_0^t K(t - \eta_n(t, s)) b(\check{X}_{\frac{[ns]}{n}}) ds + \int_0^t K(t - \eta_n(t, s)) \sigma(\check{X}_{\frac{[ns]}{n}}) dW_s,$$

where $\eta_n(t, s)$ is of the form

$$\eta_n(t, s) = \frac{[ns]}{n} + \frac{\zeta(\lceil nt - ns \rceil)}{n}$$

with a function ζ with $\zeta(i) \leq (i-1) \wedge \bar{\zeta}$ for $i \geq 1$, where $\bar{\zeta} > 0$ is a constant. The case $\zeta \equiv 0$ is known as the Euler scheme. On the regular grid, we can write

$$\check{X}_{\frac{k}{n}} = X_0 + \sum_{j=0}^{k-1} K\left(\frac{k-j-\zeta(k-j)}{n}\right) \left(b(\check{X}_{\frac{j}{n}}) \frac{1}{n} + \sigma(\check{X}_{\frac{j}{n}}) (W_{\frac{j+1}{n}} - W_{\frac{j}{n}}) \right).$$

Borrowing an idea from Nualart and Saikia (2022), define $\bar{X}$ by

$$\bar{X}_t = X_0 + \int_0^t K(t-s)b(\check{X}_{\lfloor ns \rfloor / n})ds + \int_0^t K(t-s)\sigma(\check{X}_{\lfloor ns \rfloor / n})dW_s,$$

and decompose the normalized error as

$$n^H(X - \check{X}) = n^H(X - \bar{X}) + n^H(\bar{X} - \check{X}) =: \mathbf{Y}^n + \mathbf{A}^n.$$

The process $\mathbf{A}^n$ does not converge in C_0 as observed in Nualart and Saikia (2022).

Note that $\bar{X}$ is also a feasible approximation to X . The 3R hybrid scheme (Fukasawa and Hirano (2021)) is applicable.

$\mathbf{Y}^n$ has a similar structure to $\mathbf{U}^n$: In fact...

$$\begin{aligned}
Y_t^{n,i} &= n^H(X_t - \bar{X}_t) \\
&= n^H \int_0^t K(t-s)((b^i(X_s) - b^i(\check{X}_{\frac{[ns]}{n}}))ds + (\sigma_j^i(X_s) - \sigma_j^i(\check{X}_{\frac{[ns]}{n}}))dW_s^j) \\
&\approx \int_0^t K(t-s)(\nabla b^i(\bar{X}_s)^\top \mathbf{Y}_s^n ds + \nabla \sigma_j^i(\bar{X}_s)^\top \mathbf{Y}_s^n dW_s^j) \\
&+ \int_0^t K(t-s)n^H \nabla b^i(\bar{X}_s)^\top (\bar{X}_s - \check{X}_{\frac{[ns]}{n}})ds + \int_0^t K(t-s)\partial_k \sigma_j^i(\bar{X}_s)d\check{V}_s^{n,k,j},
\end{aligned}$$

where $\check{\mathbf{V}}^n = \{\check{V}^{n,k,j}\}$ is defined as

$$\check{V}^{n,k,j} = n^H \int_0^\cdot (\bar{X}_s^k - \check{X}_{\frac{[ns]}{n}}^k) dW_s^j, \quad 1 \leq k \leq d, 1 \leq j \leq m.$$

The only difference is from the limit of $\langle \check{V}^{n,k_1,j}, \check{V}^{n,k_2,j} \rangle$.

Theorem

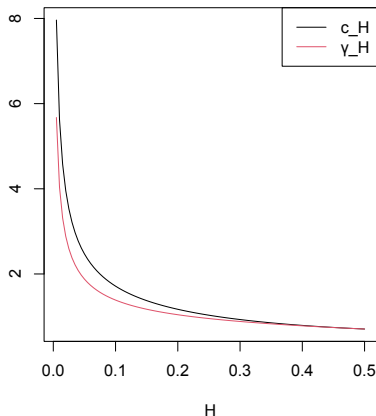
Let $\epsilon \in (0, H)$. Then the process $\mathbf{Y}^n = n^H(X - \bar{X})$ stably converges in law in $C_0^{H-\epsilon}$ to a process $\mathbf{Y} = (Y^1, \dots, Y^d)^\top$ which is the unique continuous solution of the SVE

$$Y_t^i = \int_0^t K(t-s) Y_s^k \left(\partial_k b^i(X_s) ds + \partial_k \sigma_j^i(X_s) dW_s^j \right) - \gamma_H \int_0^t K(t-s) \partial_k \sigma_j^i(X_s) \sigma_l^k(X_s) dB_s^{l,j},$$

where B is an m^2 -dimensional standard Brownian motion, independent of $\mathcal{F}$ and defined on some extension of $(\Omega, \mathcal{F}, P)$, and $\Gamma(H + 1/2)^2 \gamma_H^2 =$

$$\int_0^1 \int_0^\infty ((y+x)^{H-1/2} - (\lceil y \rceil - \zeta(\lceil y \rceil))^{H-1/2})^2 dy dx + \frac{1}{2H(2H+1)}.$$

Let $c_H^2 = \frac{1}{\Gamma(2H+2)\sin\pi H}$. When $\zeta \equiv 0$ (the Euler scheme),



We can also show that $\gamma_H^2/c_H^2 \rightarrow 1/2$ as $H \rightarrow 0$. $\bar{X}$ is better than $\hat{X}$!!

Interpretation

- The discretization of kernel indeed results in an additional error.
- On the other hand, since $K(t_j - t_i)(W_{t_{i+1}} - W_{t_i})$ has a smaller variance than $\int_{t_i}^{t_{i+1}} K(t_j - s) dW_s$, in the Euler-type iteration for $\check{X}$, propagation of error is mitigated.
- $\bar{X}$ picks the best of $\hat{X}$ and $\check{X}$

Minimization of γ_H with respect to ζ

Denoting by $c_k = (k - \zeta(k))^{H-1/2}$, we have $\Gamma(H + 1/2)^2 \gamma_H^2 =$

$$\sum_{k=1}^{\infty} \int_0^1 \int_0^1 ((k + x - y)^{H-1/2} - c_k)^2 dy dx + \frac{1}{2H(2H+1)}$$

which is minimized by

$$\begin{aligned} c_k &= c_k^* := \int_0^1 \int_0^1 (k + x - y)^{H-1/2} dx dy \\ &= \frac{(k+1)^{H+3/2} - 2k^{H+3/2} + (k-1)^{H+3/2}}{(H+1/2)(H+3/2)}. \end{aligned}$$

The optimal ζ is therefore given by $\zeta^*(k) = k - (c_k^*)^{1/(H-1/2)} < 0$.

Note however that the reduction of γ_H by replacing $\zeta \equiv 0$ with $\zeta = \zeta^*$ is small ($< 10^{-3}$) according to our numerical computations.

Summary

- We determine the limit distribution of the discretization error for SVE.
- We prove its stable convergence in Hölder spaces.
- The minimal discretization $\hat{X}$ is not always better than the Euler scheme $\check{X}$
- We propose to use $\bar{X}$ for $H < 1/2$.