

# INTEGRATION-BY-PARTS CHARACTERIZATIONS OF GAUSSIAN PROCESSES

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# REFERENCES

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Integration-by-Parts Characterizations of Gaussian Processes  
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# ABSTRACT

The Stein's lemma characterizes the Gaussian distribution via an integration-by-parts formula.

We show that a similar integration-by-parts formula characterizes a wide class of Gaussian processes, the so-called Gaussian Fredholm processes.

# OUTLINE

- 1 STEIN'S (MULTIVARIATE) LEMMA
- 2 FREDHOLM REPRESENTATION
- 3 PATHWISE MALLIAVIN DIFFERENTIATION
- 4 STRONG FORM INTEGRATION-BY-PARTS  
CHARACTERIZATION
- 5 WEAK FORM INTEGRATION-BY-PARTS  
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# STEIN'S (MULTIVARIATE) LEMMA

STEIN'S LEMMA, a.k.a. the INTEGRATION-BY-PARTS CHARACTERIZATION, states that a random variable  $X$  is standard normal if and only if

$$\mathbf{E} [f'(X)] = \mathbf{E} [Xf(X)]$$

for all smooth and bounded enough  $f: \mathbb{R} \rightarrow \mathbb{R}$ .

MULTIVARIATE STEIN'S LEMMA states that  $X = (X_1, \dots, X_d)$  is centered Gaussian with covariance  $R$  if and only if

$$\mathbf{E} \left[ \sum_{i=1}^d X_i \frac{\partial}{\partial x_i} f(X) \right] = \mathbf{E} \left[ \sum_{i=1}^d \sum_{j=1}^d R_{ij} \frac{\partial^2}{\partial x_i \partial x_j} f(X) \right]$$

for all smooth and bounded enough  $f: \mathbb{R}^d \rightarrow \mathbb{R}$

# STEIN'S (MULTIVARIATE) LEMMA

Let  $X = (X_t)_{t \in [0,1]}$  be a centered process with covariance  $R$ . The Multivariate Stein's Lemma suggests us to guess (**WRONGLY!**) that  $X$  is Gaussian if and only if

$$\mathbf{E} \left[ \int_0^1 X_t D_t f(X) dt \right] = \mathbf{E} \left[ \int_0^1 \int_0^1 R(t,s) D_{t,s}^2 f(X) ds dt \right],$$

where  $D$  is some kind of **MALLIAVIN DERIVATIVE**.

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# FREDHOLM REPRESENTATION

## THE THEOREM

### THEOREM (FREDHOLM REPRESENTATION)

*Let  $X = (X_t)_{t \in [0,1]}$  be a separable centered Gaussian process. Then there exists a kernel  $K \in L^2 \times L^2 = L^2([0,1]^2)$  and a Brownian motion  $W = (W_t)_{t \geq 0}$  such that*

$$X_t = \int_0^1 K(t, s) dW_s$$

*if and only if the covariance  $R$  of  $X$  satisfies the trace condition*

$$\int_0^1 R(t, t) dt < \infty.$$

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# PATHWISE MALLIAVIN DIFFERENTIATION

Let  $\mathcal{C}_p^\infty(\mathbb{R}^n)$  denote the space of all polynomially bounded functions with polynomially bounded partial derivatives of all orders. Consider functionals  $f: L^2 \rightarrow \mathbb{C}$  of the form

$$f(x) = g(z_1, \dots, z_n),$$

where  $n \in \mathbb{N}$  and  $g \in \mathcal{C}_p^\infty(\mathbb{R}^n)$ , and

$$z_k = \int_0^1 e_k(t) dx(t)$$

for some elementary functions  $e_k \in \mathcal{E}$ . For such  $f$  we write  $f \in \mathcal{S}$ . We call the elements of class  $\mathcal{S}$  the **SMOOTH** functionals. The **PATHWISE MALLIAVIN DERIVATIVE** of such  $f \in \mathcal{S}$  is

$$D_t f(x) = \sum_{k=1}^n \frac{\partial}{\partial z_k} g(z_1, \dots, z_n) e_k(t).$$

# PATHWISE MALLIAVIN DIFFERENTIATION

More generally, by iteration for every  $m \in \mathbb{N}$ , the pathwise Malliavin derivative of order  $m$  is defined as follows: for every  $t_1, \dots, t_m \in [0, 1]$ ,

$$\begin{aligned} D_{t_m, \dots, t_1}^m f(x) \\ = \sum_{1 \leq k_1, \dots, k_m \leq n} \frac{\partial^m}{\partial z_{k_1} \dots \partial z_{k_m}} g(z_{k_1}, \dots, z_{k_m}) (e_{k_1} \otimes \dots \otimes e_{k_m}) (t_1, \dots, t_m). \end{aligned}$$

## REMARK

Let  $f \in \mathcal{S}$  and  $y \in L^2$ . Then

$$\langle \nabla f(x), ly \rangle_{L^2} = \langle Df(x), y \rangle_{L^2}$$

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# STRONG FORM INTEGRATION-BY-PARTS CHARACTERIZATION

Let  $K^*$  extend linearly the relation  $K^* \mathbf{1}_t(s) = K(t, s)$ , where  $\mathbf{1}_t = \mathbf{1}_{[0,t)}$ .

## THEOREM

*Let  $K \in L^2 \times L^2$ . The co-ordinate process  $X: \Omega \rightarrow L^2$  is centered Gaussian with Fredholm kernel  $K$  if and only if*

$$\mathbf{E} [X_t D_t f(X)] = \mathbf{E} \left[ \int_0^1 K(t, s) K^* [D_{t, \cdot}^2 f(X)](s) ds \right]$$

*for all  $t \in [0, 1]$  and  $f \in \mathcal{S}$ .*

## STRONG IBP, PROOF OF IF PART

Suppose the co-ordinate process  $X: \Omega \rightarrow L^2$  satisfies the Strong IBP Formula. We begin by considering the covariance function of  $X$ , which will justify the use of the Fubini theorem later and make a tedious variance calculations unnecessary. For this, take  $f(X) = \frac{1}{2}X_u^2$  for some  $u \in [0, 1]$ . Then  $f \in \mathcal{S}$ . We have  $D_t f(X) = X_u \mathbf{1}_u(t)$  and  $D_{t,s}^2 f(X) = \mathbf{1}_u(s) \mathbf{1}_u(t)$ . Consequently,

$$\begin{aligned} \mathbf{E} [X_t X_u] \mathbf{1}_u(t) &= \mathbf{E} \left[ \int_0^1 K(t, s) K^* [\mathbf{1}_u](s) ds \right] \mathbf{1}_u(t) \\ &= \int_0^1 K(t, s) K(u, s) ds \mathbf{1}_u(t) \\ &= R(t, u) \mathbf{1}_u(t). \end{aligned}$$

This shows that  $X$  has the covariance function  $R$  given by the Fredholm kernel  $K$ .

# STRONG IBP, PROOF OF IF PART

In particular, we have

$$\mathbf{E} [X_t^2] = \int_0^1 K(t, s)^2 \, ds,$$

and since  $K \in L^2 \times L^2$ , we have  $\int_0^1 \mathbf{E} [X_t^2] \, dt < \infty$  which justifies the use of the Fubini theorem in the rest of the proof. Next we are going to show that any finite linear combination

$$Z = \sum_{k=1}^n a_k (X_{t_k} - X_{t_{k-1}}) = \int_0^1 e(t) \, dX_t$$

with  $e = \sum_{k=1}^n a_k \mathbf{1}_{(t_{k-1}, t_k]} \in \mathcal{E}$  is a Gaussian random variable. Now, note that for every  $\theta$  the complex-valued exponential functional  $e^{i\theta Z} = \cos(\theta Z) + i \sin(\theta Z)$  belongs to  $\mathcal{S}$ , meaning that the real and imaginary parts both belong to  $\mathcal{S}$ .

# STRONG IBP, PROOF OF IF PART

Let  $\varphi$  be the characteristic function of  $Z$ . Then

$$\begin{aligned}D_t e^{i\theta Z} &= i\theta e(t) e^{i\theta Z}, \\D_{t,s}^2 e^{i\theta Z} &= -\theta^2 e(t) e(s) e^{i\theta Z}.\end{aligned}$$

Hence  $\mathbf{E} [X_t D_t e^{i\theta Z}] = i\theta e(t) \mathbf{E} [X_t e^{i\theta Z}]$ . Also, by a direct application of Fubini theorem

$$\begin{aligned}&\mathbf{E} \left[ \int_0^1 K(t,s) K^* \left[ D_{t,\cdot}^2 e^{i\theta Z} \right] (s) ds \right] \\&= -\mathbf{E} \left[ \int_0^1 K(t,s) K^* \left[ \theta^2 e(t) e(\cdot) e^{i\theta Z} \right] (s) ds \right] \\&= -\theta^2 e(t) \mathbf{E} \left[ \int_0^1 K(t,s) K^* \left[ e(\cdot) e^{i\theta Z} \right] (s) ds \right] \\&= -\theta^2 e(t) \int_0^1 K(t,s) e^*(s) ds \mathbf{E} \left[ e^{i\theta Z} \right],\end{aligned}$$

## STRONG IBP, PROOF OF IF PART

where we have denoted  $e^* = K^* e$ . Consequently, the Strong IBP Formula yields

$$i \mathbf{E} \left[ X_t e^{i\theta Z} \right] = -\theta \int_0^1 K(t, s) e^*(s) ds \varphi(\theta).$$

By Fubini theorem justified by the covariance computation, we also have

$$\varphi'(\theta) = \mathbf{E} \left[ iZ e^{i\theta Z} \right].$$

Thus we obtain that  $\varphi'(\theta) = -c\theta \varphi(\theta)$ , where we have denoted

$$c = \int_0^1 \left( \sum_{k=1}^n a_k (K(t_k, s) - K(t_{k-1}, s)) e^*(s) \right) ds < \infty.$$

This implies that  $\varphi(\theta) = e^{-\frac{1}{2}c\theta^2}$ , and since  $\varphi$  is a characteristic function,  $c > 0$ . Consequently,  $Z$  is a centered Gaussian random variable with variance  $c$ .

# STRONG IBP, PROOF OF ONLY IF PART

Since the co-ordinate process  $X: \Omega \rightarrow L^2$  is Gaussian, we have the full power of Malliavin calculus at our disposal.

In particular, we can use Malliavin IBP Formula

$$\mathbf{E}[FG] = \mathbf{E}[F]\mathbf{E}[G] + \mathbf{E} \left[ \langle DF, -DL^{-1}G \rangle_{\mathcal{H}} \right].$$

with  $F = D_t f(X)$  and  $G = X_t$ . Since  $\mathbf{E}[X_t] = 0$ , we obtain

$$\mathbf{E} [X_t D_t f(X)] = \mathbf{E} \left[ \langle D_{t,\cdot}^2 f(X), -DL^{-1}X_t \rangle_{\mathcal{I}} \right]$$

But  $-DL^{-1}X_t = \mathbf{1}_t$  and  $K^*$  is an isometry between  $\mathcal{I}$  and  $L^2$ . Therefore, by noticing that  $K^*\mathbf{1}_t(s) = K(t, s)$ , we obtain

## STRONG IBP, PROOF OF ONLY IF PART

$$\begin{aligned}\mathbf{E} [X_t D_t f(X)] &= \mathbf{E} \left[ \langle D_{t,\cdot}^2 f(X), \mathbf{1}_t \rangle_{\mathcal{I}} \right] \\ &= \mathbf{E} \left[ \langle K^* D_{t,\cdot}^2 f(X), K^* \mathbf{1}_t \rangle_{L^2} \right] \\ &= \mathbf{E} \left[ \int_0^1 K^* [D_{t,\cdot}^2 f(X)](s) K^* \mathbf{1}_t(s) \, ds \right] \\ &= \mathbf{E} \left[ \int_0^1 K^* [D_{t,\cdot}^2 f(X)](s) K(t, s) \, ds \right]\end{aligned}$$

showing the claim.

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# WEAK FORM INTEGRATION-BY-PARTS CHARACTERIZATION

By using Fubini to the Strong IBP Formula, we obtain

## THEOREM (WEAK INTEGRATION-BY-PARTS CHARACTERIZATION)

*Let  $K \in L^2 \times L^2$ . Assume that the co-ordinate process  $X: \Omega \rightarrow L^2$  satisfies  $X \in L^2(dt \otimes \mathbf{P})$ , i.e.*

$$\int_0^1 \mathbf{E} [X_t^2] dt < \infty.$$

*Then  $X$  is centered Gaussian with the Fredholm kernel  $K$  if and only if*

$$\mathbf{E} \left[ \int_0^1 X_t D_t f(X) dt \right] = \mathbf{E} \left[ \int_0^1 \int_0^1 K(t, s) K^* [D_{t, \cdot}^2 f(X)](s) ds dt \right]$$

*for all  $f \in S$ .*

# BROWNIAN MOTION

## COROLLARY (BROWNIAN MOTION)

*Assume that the co-ordinate process  $X: \Omega \rightarrow L^2$  satisfies  $X \in L^2(dt \otimes \mathbf{P})$ , i.e.*

$$\int_0^1 \mathbf{E} [X_t^2] dt < \infty.$$

*Then  $X$  is the Brownian motion if and only if*

$$\mathbf{E} \left[ \int_0^1 X_t D_t f(X) dt \right] = \mathbf{E} \left[ \int_0^1 \int_0^t D_{t,s}^2 f(X) ds dt \right]$$

*for all  $f \in \mathcal{S}$ .*

Thank you for listening!  
Any questions?