

Backward Euler method for stochastic differential equations with non-Lipschitz coefficients

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the Workshop on Stochastics, Memory and Roughness 2024

01/17/2024 - 01/20/2024

Based on the paper

Zhou, Hao; Hu, Y.; Liu, Yanghui.

Backward Euler method for stochastic differential equations with non-Lipschitz coefficients driven by fractional Brownian motion.

BIT 63 (2023), no. 3, 37 pp.

Outline

1. Problems
2. Properties of the solution
3. Backward Euler scheme
4. Convergence and rate of the convergence
5. Limiting distribution
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1. Problems

Consider following m -dimensional stochastic differential equation:

$$dX_t = b(X_t)dt + dB_t, \quad t \in [0, T], \quad (1)$$

where

B_t is fBm (fractional Brownian motion) with Hurst parameter $H \in (1/2, 1)$

$X_0 = x_0$ is a constant in $\mathbb{R}^m$.

$b : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is one-sided Lipschitzian: namely, there is a constant κ such that

$$\langle x - y, b(x) - b(y) \rangle \leq \kappa |x - y|^2, \quad \forall x, y \in \mathbb{R}^m. \quad (2)$$

The coefficient b itself and its first, second and third derivatives are of polynomial growth: namely, there are constants κ and μ such that

$$|b(x)| + |\partial b(x)| + |\partial^2 b(x)| + |\partial^3 b(x)| \leq \kappa(1 + |x|^\mu), \quad \forall x \in \mathbb{R}^m. \quad (3)$$

Example: $m = 1$, $b(x) = -cx^{2m+1} + \text{lower order terms}$. $c > 0$.

Want to study

1. The existence and uniqueness of the solution, The properties of the solution (needed in the study of numerical solution)
2. Numerical scheme and its rate of convergence
3. Limiting distribution

2. Properties of the solution

Proposition

Let X_t be the solution of equation (1). Then for any integer $p \geq 1$, there exists a constant $C_{\kappa, \mu, p, T}$ depending only on κ , μ , p and T such that

$$\mathbb{E} \sup_{t \in [0, T]} |X_t|^p \leq C_{\kappa, \mu, p, T}. \quad (4)$$

Proposition

Let X_t be the solution of equation (1). Take α such that $0 < \alpha < H$. Then

$$\begin{aligned} \mathbb{E}|X_t - X_s|^p &\leq C|t - s|^{Hp} \quad p \geq 1 \quad \text{and} \\ \|X\|_\alpha &\leq \kappa \left(1 + \sup_{t \in [0, T]} |X_t|^\mu\right) + \|B\|_\alpha. \end{aligned}$$

Proposition

Let Assumptions be satisfied. Then for any $p \geq 1$, there is a constant $C_{\kappa,\mu,T,p}$ such that

$$\mathbb{E} \left[\sup_{0 \leq r, t \leq T} |D_r X_t|^p \right] \leq C_{\kappa,\mu,T,p} < \infty, \quad (5)$$

and

$$\mathbb{E} \left[\sup_{0 \leq r, r', t \leq T} |D_{rr'}^2 X_t|^p \right] \leq C_{\kappa,\mu,T,p} < \infty. \quad (6)$$

3. Backward Euler scheme

Let

$$\pi : 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T \quad (7)$$

be a partition of the time interval $[0, T]$. Denote

$$\Delta_k = t_{k+1} - t_k, \quad \Delta B_k = B_{t_{k+1}} - B_{t_k}, \quad |\pi| = \sup_{0 \leq k \leq n-1} \Delta_k. \quad (8)$$

The classical backward Euler scheme applied to (1) is

$$Y_{t_{k+1}}^\pi = Y_{t_k}^\pi + b(Y_{t_{k+1}}^\pi) \Delta_k + \Delta B_k, \quad k = 0, 1, \dots, n-1. \quad (9)$$

This is an implicit scheme. To find $Y_{t_{k+1}}^\pi$ from $Y_{t_k}^\pi$, we need to solve a function equation: $Y_{t_{k+1}}^\pi - b(Y_{t_{k+1}}^\pi) \Delta_k = Y_{t_k}^\pi + \Delta B_k$. This scheme gives all the values of the approximation at the partition points $t_0, t_1, \dots, t_n$.

To compare the approximation solution with the original solution, we need to know the value of the approximation solution at all time instant. We shall use the following interpolation: let Y_t^π satisfy

$$\begin{aligned} Y_t^\pi = Y_{t_k}^\pi + b(Y_t^\pi)(t - t_k) + (B_t - B_{t_k}), \\ t \in (t_k, t_{k+1}], \quad k = 0, 1, \dots, n-1. \end{aligned} \quad (10)$$

Proposition

Let Y_t^π be the solution of the backward Euler scheme defined by (10). Then for any $p \geq 1$, there is a constant $C_{p,T} < \infty$ independent of the partition π such that we have

$$\mathbb{E} \sup_{0 \leq t \leq T} |Y_t^\pi|^p \leq C_{p,T};$$

$$\mathbb{E} \sup_{s,t \in [0,T]} |D_s Y_t|^p \leq C_{p,T};$$

$$\mathbb{E} \sup_{r,s,t \in [0,T]} |D_{rs}^2 Y_t|^p \leq C_{p,T}.$$

4. Convergence and rate of convergence

Theorem

Let X_t satisfy (1) and let Y_t satisfy (10). There is a constant C independent of the partition π (but dependent on κ, μ, T) such that

$$\sup_{0 \leq t \leq T} \mathbb{E} \left[|Y_t - X_t|^2 \right] \leq C|\pi|^2, \quad (11)$$

and

$$\int_0^T \mathbb{E} |Y_t - X_t|^2 dt \leq C|\pi|^2. \quad (12)$$

5. Limiting distribution

Theorem

For any $p \in [1, 2)$, we have the following convergence in L_p . Let $Z_t = X_t - Y_t$. Then

$$nZ_t \rightarrow U_t := \frac{1}{2} \int_0^t \phi_t \phi_s^{-1} \partial b b(X_s) ds + \frac{1}{2} \int_0^t \phi_t \phi_s^{-1} \partial b(X_s) dB_s. \quad (13)$$

Moreover, U_t satisfies

$$dU_t = \partial b(X_t) U_t dt + \frac{1}{2} \partial b b(X_t) dt + \frac{1}{2} \partial b(X_t) dB_t. \quad (14)$$

Let $B_t = (B_t^1, B_t^2)$ be a two-dimensional fractional Brownian motion with Hurst parameter $H \in (0, 1)$ and assume that B_t^1 and B_t^2 are mutually independent. In this example we consider the two-dimensional stochastic differential equation:

$$\begin{cases} dX_t = [X_t - Y_t - X_t^3 - X_t Y_t^2] dt + dB_t^1, \\ dY_t = [X_t + Y_t - X_t^2 Y_t - Y_t^3] dt + dB_t^2, \end{cases} \quad (15)$$

with the initial value $X(0) = 1, Y(0) = 1$ for $t \in [0, 1]$. It is easy to verify that equation (15) satisfies the one-sided Lipschitz condition.

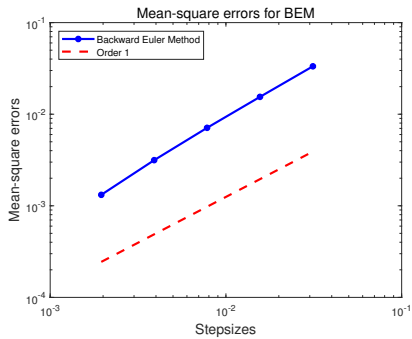


Figure: $H=0.6$

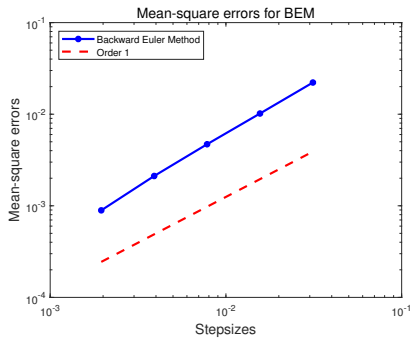


Figure: $H=0.7$

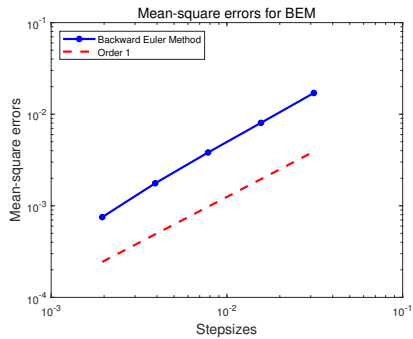


Figure: $H=0.8$

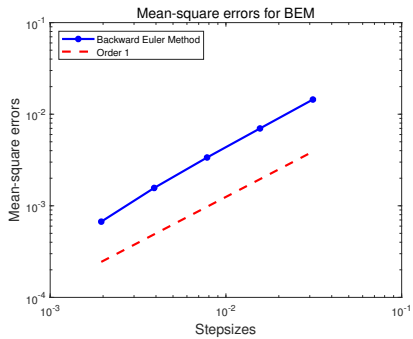


Figure: $H=0.9$

To find the convergence rate of the backward Euler method, in our numerical computation we take the time step sizes 2^{-k} , $k = 5, 6, \dots, 9$. The numerical solution with a much smaller step size (in this example we take the step size 2^{-11}) is used to replace the exact solution. We perform $M = 1000$ simulations, and we compute the mean square error by

$$\epsilon(T) = \sqrt{\frac{1}{M} \sum_{\ell=1}^M (|X_T^\ell - X_N^\ell|^2 + |Y_T^\ell - Y_N^\ell|^2)}.$$

For the four subfigures, the red dotted reference line has a slope of 1. When $H = 0.6$, the slope of the blue line is 1.1616 through least squares fitting the error, and when $H = 0.9$, the slope is 1.1016, the rest of the blue lines have slopes between these two values.

6. Idea of Proof

We can write the original equation as

$$\begin{aligned} X_t &= X_{t_k} + \int_{t_k}^t b(X_s) ds + (B_t - B_{t_k}) \\ &= X_{t_k} + b(X_t)(t - t_k) + (B_t - B_{t_k}) + R_k(t), \end{aligned} \quad (16)$$

where $t_k \leq t \leq t_{k+1}$, and

$$R_k(t) = \int_{t_k}^t b(X_s) - b(X_t) ds. \quad (17)$$

For ease of notation, we denote $R_k := R_k(t_{k+1})$ and $R_k^t := R_k(t_{k+1} \wedge t)$.

Denote the error process $Z_t = X_t - Y_t$. For convenience we will also denote $Z_k = Z_{t_k}$. Subtracting (10) from (16), we get for $t_k \leq t \leq t_{k+1}$

$$Z_t = Z_k + (t - t_k)\{b(X_t) - b(Y_t)\} + R_k(t).$$

We can rewrite the above formula as

$$Z_t - (t - t_k)\{b(X_t) - b(Y_t)\} = Z_k + R_k(t).$$

That is

$$Z_t - (t - t_k) \left\{ \int_0^1 J_b(uX_t + (1 - u)Y_t) du \right\} Z_t = Z_k + R_k(t). \quad (18)$$

We denote $A_k = A_{t_k}$. Therefore, multiplying A_t on both sides of (18) we get

$$Z_t = A_t Z_k + A_t R_k(t).$$

Taking $t = t_{k+1}$ particularly arrives at

$$Z_{k+1} = A_{k+1} Z_k + A_{k+1} R_k. \quad (19)$$

Denote

$$A_j^t = A_t \left[\prod_{\ell=j+1}^k A_\ell \right], \quad (20)$$

where k is such that $t_k < t \leq t_{k+1}$ and here we use the convention that if $\{\ell : j+1 \leq \ell \leq k\} = \emptyset$ then $\prod_{\ell=j+1}^k A_\ell = 1$. By iterating (20) and then using (19) and $R_k^t := R_k(t_{k+1} \wedge t)$ we obtain

$$Z_t = \sum_{j=0}^k A_j^t R_j^t. \quad (21)$$

Now we have

$$\begin{aligned}\|A_j^t\| &\leq (1 - \kappa(t - t_k))^{-1} \cdots (1 - \kappa\Delta_{j+1})^{-1}(1 - \kappa\Delta_j)^{-1} \\ &\leq e^{2\kappa(t-t_j)} \leq C.\end{aligned}\tag{22}$$

This can be used to obtain some rate of convergence in a previous work. However, this rate will not be optimal. To obtain the optimal rate estimate, we need to bound the Malliavin derivative of A_j^t .

A straightforward computation for $D_\xi A_j^t$ and $D_{\xi\zeta}^2 A_j^t$ yields

$$\begin{aligned}
 D_\xi A_j^t &= D_\xi A_t \left[\prod_{\ell=j+1}^k A_\ell \right] + \sum_{\ell=j+1}^k A_t \left[A_{j+1} \cdots A_{\ell-1} \cdot D_\xi A_\ell \cdot A_{\ell+1} \cdots A_k \right]; \\
 D_{\xi\zeta}^2 A_j^t &= D_{\xi\zeta}^2 A_t \left[\prod_{\ell=j+1}^k A_\ell \right] + \sum_{\ell=j+1}^k A_t \left[A_{j+1} \cdots A_{\ell-1} \cdot D_{\xi\zeta}^2 A_\ell \cdot A_{\ell+1} \cdots A_k \right] \\
 &\quad + \sum_{\ell=j+1}^k D_\xi A_t \left[A_{j+1} \cdots A_{\ell-1} \cdot D_\zeta A_\ell \cdot A_{\ell+1} \cdots A_k \right] \\
 &\quad + \sum_{\ell=j+1}^k D_\zeta A_t \left[A_{j+1} \cdots A_{\ell-1} \cdot D_\xi A_\ell \cdot A_{\ell+1} \cdots A_k \right] \\
 &\quad + \sum_{\ell \neq \ell', \ell, \ell' = j+1}^k A_t \left[A_{j+1} \cdots D_\xi A_\ell \cdots D_\zeta A_{\ell'} \cdots A_k \right].
 \end{aligned}$$

These two formulas yield

$$\mathbb{E} \left[\sup_{0 \leq \xi, t \leq T, j=0,1,\dots,n} |D_{\xi} A_j^t|^p \right] \leq C_p < \infty; \quad (23)$$

$$\mathbb{E} \left[\sup_{0 \leq \xi, \zeta, t \leq T, j=0,1,\dots,n} |D_{\xi\zeta}^2 A_j^t|^p \right] \leq C_p < \infty \quad (24)$$

for any $p \geq 1$.

Now we come back to the main proof. We write (21) as

$$Z_t = \sum_{j=0}^k A_j^t R_j^t.$$

Thus, we have

$$\begin{aligned} \mathbb{E}(|Z_t|^2) &= \mathbb{E} \left[\left| \sum_{j=0}^k A_j^t R_j^t \right|^2 \right] \\ &= \sum_{j,j'=0}^k \mathbb{E} \left\{ A_j^t R_j^t \cdot A_{j'}^t R_{j'}^t \right\} =: \sum_{j,j'=0}^k l_{j,j'}. \end{aligned} \quad (25)$$

It is worthy to notice from (17) that

$$\begin{aligned} R_j^t &= \int_{t_j}^{t_{j+1} \wedge t} \left\{ \int_0^1 J_b(uX_{t_{j+1} \wedge t} + (1-u)X_s) du \right\} \Delta X_{s, t_{j+1} \wedge t} ds \\ &= \int_{t_j}^{t_{j+1} \wedge t} b_1(s, j) \Delta X_{s, t_{j+1} \wedge t} ds, \end{aligned}$$

where $\Delta X_{s,r} = X_r - X_s$. So we can write

$$\begin{aligned} I_{j,j'} &= \int_{t_j}^{t_{j+1} \wedge t} \int_{t_{j'}}^{t_{j'+1} \wedge t} \mathbb{E} \left(A_j^t b_1(s, j) \Delta X_{s, t_{j+1} \wedge t} \right. \\ &\quad \left. \cdot A_{j'}^t b_1(s', j') \Delta X_{s', t_{j'+1} \wedge t} \right) ds' ds. \end{aligned} \quad (26)$$

Since

$$\Delta X_{s,t_{j+1} \wedge t} = X_{t_{j+1} \wedge t} - X_s = \int_s^{t_{j+1} \wedge t} b(X_u) du + \Delta B_{s,t_{j+1} \wedge t},$$

we can rewrite (26) as follows

$$\begin{aligned} I_{j,j'} &= \int_{t_j}^{t_{j+1} \wedge t} \int_{t_{j'}}^{t_{j'+1} \wedge t} \mathbb{E} \left[A_j^t b_1(s, j) \left[\int_s^{t_{j+1} \wedge t} b(X_u) du + \Delta B_{s,t_{j+1} \wedge t} \right] \right. \\ &\quad \left. \cdot A_{j'}^t b_1(s', j') \left[\int_{s'}^{t_{j'+1} \wedge t} b(X_v) dv + \Delta B_{s',t_{j'+1} \wedge t} \right] \right] ds' ds \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned} \tag{27}$$

where

$$I_1 := \int_{t_j}^{t_{j+1} \wedge t} \int_{t_{j'}}^{t_{j'+1} \wedge t} \int_s^{t_{j+1} \wedge t} \int_{s'}^{t_{j'+1} \wedge t} \mathbb{E} \left(A_j^t b_1(s, j) b(X_u) \right. \\ \left. \cdot A_{j'}^t b_1(s', j') b(X_v) \right) du dv ds' ds; \quad (28)$$

$$I_2 := \int_{t_j}^{t_{j+1} \wedge t} \int_{t_{j'}}^{t_{j'+1} \wedge t} \int_s^{t_{j+1} \wedge t} \mathbb{E} \left[A_j^t b_1(s, j) b(X_u) \right. \\ \left. \cdot A_{j'}^t b_1(s', j') \Delta B_{s', t_{j'+1} \wedge t} \right] du ds' ds; \quad (29)$$

$$I_3 := \int_{t_j}^{t_{j+1} \wedge t} \int_{t_{j'}}^{t_{j'+1} \wedge t} \int_{s'}^{t_{j'+1} \wedge t} \mathbb{E} \left[A_j^t b_1(s, j) \Delta B_{s, t_{j+1} \wedge t} \right. \\ \left. \cdot A_{j'}^t b_1(s', j') b(X_v) \right] dv ds' ds; \quad (30)$$

$$I_4 := \int_{t_j}^{t_{j+1} \wedge t} \int_{t_{j'}}^{t_{j'+1} \wedge t} \mathbb{E} \left[A_j^t b_1(s, j) \Delta B_{s, t_{j+1} \wedge t} \right. \\ \left. \cdot A_{j'}^t b_1(s', j') \Delta B_{s', t_{j'+1} \wedge t} \right] ds' ds.$$

In the above we have omitted the dependence of l_1, l_2, l_3, l_4 on j, j' for the sake of simplicity.

In the following we derive the upper-bound for $l_1, \dots, l_4$. First, let us consider l_1 . From the polynomial growth assumptions on b , Proposition 1, (22), we can obtain that

$$\mathbb{E} \left| A_j^t b_1(s, j) b(X_u) \cdot A_{j'}^t b_1(s', j') b(X_v) \right| \leq C < \infty.$$

Thus

$$|l_1| \leq C \int_{t_j}^{t_{j+1} \wedge t} \int_{t_{j'}}^{t_{j'+1} \wedge t} \int_s^{t_{j+1} \wedge t} \int_{s'}^{t_{j'+1} \wedge t} du dv ds' ds \leq C \Delta_j^2 \Delta_{j'}^2. \quad (31)$$

We turn to I_2 . From (22), (23)-(24), the polynomial growth assumption on b , Proposition 1 and Proposition 3, we can prove (by using the product rule of Malliavin derivative) that

$$\mathbb{E} \sup_{s,s',t,u,j,j',\xi} \left| D_\xi \left[A_j^t b_1(s,j) b(X_u) \cdot A_{j'}^t b_1(s',j') \right] \right| < \infty.$$

We can prove

$$\left| \mathbb{E} \left[A_j^t b_1(s,j) b(X_u) \cdot A_{j'}^t b_1(s',j') \Delta B_{s',t_{j'+1} \wedge t} \right] \right| \leq C \Delta_{j'}.$$

Consequently, we have

$$|I_2| \leq C\Delta_{j'} \int_{t_j}^{t_{j+1} \wedge t} \int_{t_{j'}}^{t_{j'+1} \wedge t} \int_s^{t_{j+1} \wedge t} du ds' ds \leq C\Delta_j^2 \Delta_{j'}^2. \quad (32)$$

Exactly in the same way we have

$$|I_3| \leq C\Delta_j^2 \Delta_{j'}^2. \quad (33)$$

Denote

$$F(s, s', t, j, j') := A_j^t b_1(s, j) A_{j'}^t b_1(s', j').$$

Similar to the above argument we can show that

$$\mathbb{E} \left[\sup_{s, s', t, j, j', \xi, \zeta} (|F(s, s', t, j, j')| + |D_\xi F(s, s', t, j, j')| + |D_{\xi\zeta}^2 F(s, s', t, j, j')|) \right] \leq C < \infty.$$

$$\begin{aligned}
|I_4| &\leq \int_{t_j}^{t_{j+1} \wedge t} \int_{t_{j'}}^{t_{j'+1} \wedge t} \left| \mathbb{E} \left[F(\mathbf{s}, \mathbf{s}', t, j, j') \Delta B_{\mathbf{s}, t_{j+1} \wedge t} \Delta B_{\mathbf{s}', t_{j'+1} \wedge t} \right] \right| d\mathbf{s}' d\mathbf{s} \\
&\leq C \int_{t_j}^{t_{j+1} \wedge t} \int_{t_{j'}}^{t_{j'+1} \wedge t} \left[\langle \mathbf{1}_{[\mathbf{s}, t_{j+1} \wedge t]}, \mathbf{1}_{[\mathbf{s}', t_{j'+1} \wedge t]} \rangle \right. \\
&\quad \left. + |t_{j+1} - \mathbf{s}| \cdot |t_{j+1} - \mathbf{s}'| \right] d\mathbf{s}' d\mathbf{s} \\
&\leq C \Delta_j \Delta_{j'} \langle \mathbf{1}_{[t_j, t_{j+1} \wedge t]}, \mathbf{1}_{[t_{j'}, t_{j'+1} \wedge t]} \rangle + C \Delta_j^2 \Delta_{j'}^2.
\end{aligned}
\tag{34}$$

Plugging (31)-(34) into (27), we conclude that

$$\begin{aligned}\mathbb{E}(|Z_t|^2) &\leq \sum_{j,j'=0}^k |l_{j,j'}| \leq C \sum_{j,j'=0}^k \Delta_j^2 \Delta_{j'}^2 \\ &\quad + C \Delta_j \Delta_{j'} \sum_{j,j'=0}^k \langle \mathbf{1}_{[t_j, t_{j+1} \wedge t]}, \mathbf{1}_{[t_{j'}, t_{j'+1} \wedge t]} \rangle \\ &\leq C |\pi|^2.\end{aligned}$$