

Standard and fractional Bessel and Cox-Ingersoll-Ross processes

Yuliya Mishura

joint work with Andrey Pilipenko, Anton Yurchenko-Tytarenko and Kostiantyn Ralchenko

Taras Shevchenko National University of Kyiv
Mälardalen University

17 January 2024

Workshop on Stochastics, Memory and Roughness 2024, University of Oslo, January 17-19th, 2024

Introduction

The talk is based on the following papers.



M., Y. and Ralchenko, K. (2024).

Fractional Bessel process with Hurst index $H \in (0, \frac{1}{2})$.

№ 110008. *Statistics and Probability Letters*, Vol.206, Iss.3 pp. 1 - 8, - 2024.



M., Y., Pilipenko, A., and Yurchenko-Tytarenko, A. (2024).

Low-dimensional Cox–Ingersoll–Ross process.

<https://doi.org/10.1080/17442508.2023.2300291>. *Stochastics*, pp. 1 - 21.



M., Y. and Yurchenko-Tytarenko, A. (2023).

Parameter Estimation in Rough Bessel Model.

Fractal and Fractional, Vol.7, Iss.508. pp. 1 - 17.



M., Y. and Yurchenko-Tytarenko, A. (2023).

Standard and fractional reflected Ornstein-Uhlenbeck processes as the limits of square roots of Cox-Ingersoll-Ross processes.

Stochastics, 95(1):99–117.



M., Y. and Yurchenko-Tytarenko, A. (2018).

Fractional Cox-Ingersoll-Ross process with non-zero “mean”.

Mod. Stoch. Theory Appl., 5(1):99–111.



M., Y. and Yurchenko-Tytarenko, A. (2019).

Fractional Cox-Ingersoll-Ross process with small Hurst indices.

Mod. Stoch. Theory Appl., 6(1):13–19.

Introduce the squared Bessel process

$$X(t) = x_0 + at + 2 \int_0^t \sqrt{X(s)} dW(s) \quad (1)$$

and its generalization, Cox-Ingersoll-Ross (CIR) process

$$X(t) = x_0 + \int_0^t (a - bX(s)) ds + \sigma \int_0^t \sqrt{X(s)} dW(s). \quad (2)$$

These processes and their respective square roots are widely used in various fields such as physics (see e.g. [Bray, 2000], [Horibe et al., 1983] and [Guarnieri et al., 2017]) or finance (see e.g. [Cox et al., 1981], [Cox et al., 1985a], [Cox et al., 1985b] and [Heston, 1993]).

One of the reasons for the popularity of these processes lies in the fact that $a > 0$ in (2) implies that $X(t) \geq 0$ for all $t \geq 0$ with probability 1.

If the *Feller condition* $2a \geq \sigma^2$ is satisfied, the paths of X in (2) are strictly positive a.s. which turns out to be very useful in multiple cases, for example, for the Heston model. Put $Y := \sqrt{X}$ and observe that under the Feller condition we have that

$$\int_0^t \frac{1}{Y(s)} ds < \infty \quad (3)$$

with probability 1 for all $t \geq 0$. This, in particular, implies that Y has the dynamics of the form

$$Y(t) = \sqrt{x_0} + \frac{1}{2} \int_0^t \left(\frac{a - \frac{\sigma^2}{4}}{Y(s)} - bY(s) \right) ds + \frac{\sigma}{2} W(t), \quad t \geq 0. \quad (4)$$

Problem

Empirical considerations indicate that the Feller condition can sometimes be too restrictive and models perform better when it is not satisfied.

Therefore, the process $Y = \sqrt{X}$ under relatively small values of a often turns out to be more relevant for reflecting real-life phenomena despite the associated analytical challenges.

But due to these challenges, the majority of sources in the literature pay more attention to the case when the Feller condition is satisfied.

Some notable exceptions:

- $\frac{\sigma^2}{4} < a < \frac{\sigma^2}{2}$: [Blei, 2012], [Cherny, 2000], [Cherny and Engelbert, 2005].
- $0 < a < \frac{\sigma^2}{4}$: [Bertoin, 1990a], [Bertoin, 1990b] consider some properties of Bessel processes (1) with rather involved methodology.

The number $d = \frac{4a}{\sigma^2}$ is called the dimension or the number of degrees of freedom (even being noninteger). So, Feller condition is fulfilled for the case where the dimension $d \geq 2$.

So, to recall: let $a, \sigma > 0$, $b \geq 0$,

$$X(t) = x_0 + \int_0^t (a - bX(s)) ds + \sigma \int_0^t \sqrt{X(s)} dW(s).$$

Since the paths of X are non-negative with probability 1 provided that $a > 0$, we can define the square-root process $Y = \{Y(t), t \geq 0\} := \{\sqrt{X(t)}, t \geq 0\}$.

In order to analyze the dynamics of Y , take $\varepsilon > 0$ and observe that, by Itô's formula,

$$\begin{aligned} \sqrt{X(t) + \varepsilon} &= \sqrt{x_0 + \varepsilon} + \frac{1}{2} \int_0^t \left(\frac{a}{\sqrt{X(s) + \varepsilon}} - \frac{\sigma^2}{4} \frac{X(s)}{(X(s) + \varepsilon)^{\frac{3}{2}}} \right) ds \\ &\quad - \frac{1}{2} \int_0^t \frac{bX(s)}{\sqrt{X(s) + \varepsilon}} ds + \frac{\sigma}{2} \int_0^t \frac{\sqrt{X(s)}}{\sqrt{X(s) + \varepsilon}} dW(s). \end{aligned}$$

- With probability 1, the LHS of the equation above converges to $Y(t)$ as $\varepsilon \downarrow 0$.
- By monotone convergence,

$$\int_0^t \frac{X(s)}{\sqrt{X(s)} + \varepsilon} ds \rightarrow \int_0^t Y(s) ds \quad a.s., \quad \varepsilon \downarrow 0.$$

- It is possible to prove that

$$\mathbb{E} \left(\sup_{t \in [0, T]} \left| \int_0^t \frac{\sqrt{X(s)}}{\sqrt{X(s)} + \varepsilon} dW(s) - W(t) \right| \right)^2 \rightarrow 0, \quad \varepsilon \downarrow 0.$$

Hence there exists a sequence $\{\varepsilon_n, n \geq 1\}$ such that

$$\sup_{t \in [0, T]} \left| \int_0^t \frac{\sqrt{X(s)}}{\sqrt{X(s)} + \varepsilon_n} dW(s) - W(t) \right| \rightarrow 0 \quad a.s., \quad n \rightarrow \infty.$$

Calculations

Fixing $T > 0$ and taking the limit as $\varepsilon_n \downarrow 0$, we obtain that $Y = \sqrt{X}$ satisfies the SDE of the form

$$Y(t) = Y(0) - \frac{b}{2} \int_0^t Y(s) ds + \frac{\sigma}{2} W(t) + L(t),$$

where $Y(0) = \sqrt{x_0}$ and

$$L(t) := \lim_{n \rightarrow \infty} \frac{1}{2} \int_0^t \left(\frac{a}{\sqrt{X(s) + \varepsilon_n}} - \frac{\sigma^2}{4} \frac{X(s)}{(X(s) + \varepsilon_n)^{\frac{3}{2}}} \right) ds. \quad (5)$$

Remark 1

Note that the limit in (5) is well-defined with probability 1 since all other terms converge as $\varepsilon_n \rightarrow 0$ a.s. Moreover, the process L is continuous a.s. since

$$L(t) = Y(t) - Y(0) + \frac{b}{2} \int_0^t Y(s) ds - \frac{\sigma}{2} W(t).$$

But if the Feller condition $2a \geq \sigma^2$ holds, this is neither reflection function nor local time because the process does not visit zero.

Question

What is the limit in

$$L(t) := \lim_{n \rightarrow \infty} \frac{1}{2} \int_0^t \left(\frac{a}{\sqrt{X(s) + \varepsilon_n}} - \frac{\sigma^2}{4} \frac{X(s)}{(X(s) + \varepsilon_n)^{\frac{3}{2}}} \right) ds$$

equal to?

Simple case: if the Feller condition $2a \geq \sigma^2$ holds, then trajectories of X are bounded away from zero a.s. and hence, with probability 1,

$$L(t) = \frac{a - \frac{\sigma^2}{4}}{2} \int_0^t \frac{1}{Y(s)} ds < \infty.$$

However, outside of the Feller condition, i.e. if $0 < a < \frac{\sigma^2}{2}$, the process Y hits zero and hence the behaviour of the integral $\int_0^t \frac{1}{Y(s)} ds$ is not clear and has to be discussed separately.

Case I: $a > \frac{\sigma^2}{4}$

Let $\frac{\sigma^2}{4} < a < \frac{\sigma^2}{2}$, i.e., $1 < d < 2$. This case turns out to be quite similar to $a \geq \frac{\sigma^2}{2}$. Namely, the following theorem holds.

Theorem 2

Let $a > \frac{\sigma^2}{4}$. Then, for any $t \geq 0$,

$$\int_0^t \frac{1}{Y(s)} ds < \infty \quad a.s.$$

and Y a.s. satisfies the SDE of the form

$$Y(t) = \sqrt{x_0} + \frac{1}{2} \left(a - \frac{\sigma^2}{4} \right) \int_0^t \frac{1}{Y(s)} ds - \frac{b}{2} \int_0^t Y(s) ds + \frac{\sigma}{2} W(t). \quad (6)$$

Remark 3

Using the same arguments as in [Cherny, 2000, Theorem 3.2], it is possible to prove that for $a > \frac{\sigma^2}{4}$ the process $Y = \sqrt{X}$ is the unique *non-negative* strong solution to the SDE (6). However, if $\frac{\sigma^2}{4} < a < \frac{\sigma^2}{2}$, (6) has other strong solutions; moreover, the uniqueness in law does not hold for (6). For a more detailed discussion of this phenomenon, we refer to [Blei, 2012] whereas a comprehensive overview of SDEs of the type (6) can be found in [Cherny and Engelbert, 2005].

Case II: $a = \frac{\sigma^2}{4}$

The case $a = \frac{\sigma^2}{4}$ turns out to be different from the one described above as the integral $\int_0^t \frac{1}{Y(s)} ds$ is infinite after zero hitting. However, the limit L from (5) has a simple interpretation in terms of Skorokhod reflections as summarized in the following theorem.

Theorem 4

Let $a = \frac{\sigma^2}{4}$ and denote $\tau := \inf\{s \geq 0 \mid X(s) = 0\}$.

1) For all $\gamma > 0$,

$$\int_0^{\tau+\gamma} \frac{1}{Y(s)} ds = \infty \quad \text{a.s.}$$

2) The processes $Y := \sqrt{X}$ and L defined by (5) is the (unique) solution to Skorokhod problem

$$Y(t) = \sqrt{x_0} - \frac{b}{2} \int_0^t Y(s) ds + \frac{\sigma}{2} W(t) + L(t), \quad (7)$$

with L being the corresponding Skorokhod reflection function, i.e. a continuous non-decreasing process starting at 0 with points of growth occurring only at zeros of Y and such that $Y(t) \geq 0$. In other words, Y is a reflected Ornstein-Uhlenbeck process.

Case II: $a = \frac{\sigma^2}{4}$

The connection between Cox-Ingersoll-Ross and reflected Ornstein processes described on the previous slide can be exploited to obtain an alternative representation of the Skorokhod reflection function. Namely, for an arbitrary sequence $\varepsilon_n \downarrow 0$, denote

$$X_{\varepsilon_n}(t) = x_0 + \int_0^t \left(\frac{\sigma^2}{4} + \varepsilon_n - bX_{\varepsilon_n}(s) \right) ds + \sigma \int_0^t \sqrt{X_{\varepsilon_n}(s)} dW(s).$$

Theorem 5

Let $Y_0(t) = \lim X_{\varepsilon_n}(t)$. Then

$$Y_0(t) = \sqrt{x_0} - \frac{b}{2} \int_0^t Y_0(s) ds + \frac{\sigma}{2} W(t) + L_0(t)$$

is a reflected Ornstein-Uhlenbeck process with Skorokhod reflection function L_0 , where, with probability 1,

$$\sup_{t \in [0, T]} \left| L_0(t) - \frac{1}{2} \int_0^t \frac{\varepsilon_n}{\sqrt{X_{\varepsilon_n}(s)}} ds \right| \rightarrow 0, \quad n \rightarrow \infty.$$

Case III: $0 < a < \frac{\sigma^2}{4}$

The case $0 < a < \frac{\sigma^2}{4}$ is arguably the most challenging one:

- $\int_0^t \frac{1}{Y(s)} ds$ is infinite after the first moment of zero hitting;
- Y is not a semimartingale [Mijatović and Urusov, 2015];
- L has infinite variation;
- L is decreasing on every interval (τ_1, τ_2) such that $X(t) > 0$ for all $t \in (\tau_1, \tau_2)$;
- in the same time, L is not strictly decreasing on the entire $[0, T]$: otherwise, $Y \leq U$, where U is the standard Ornstein-Uhlenbeck process defined by

$$U(t) = Y(0) - \frac{b}{2} \int_0^t U(s) ds + \frac{\sigma}{2} W(t).$$

However, it is not possible since Y cannot take negative values.

Case III: $0 < a < \frac{\sigma^2}{4}$

Our strategy in studying the limit in

$$L(t) := \lim_{n \rightarrow \infty} \frac{1}{2} \int_0^t \left(\frac{a}{\sqrt{X(s) + \varepsilon_n}} - \frac{\sigma^2}{4} \frac{X(s)}{(X(s) + \varepsilon_n)^{\frac{3}{2}}} \right) ds$$

is as follows.

- 1 Using the fact that the CIR process X is a non-negative diffusion, we represent it as

$$X(t) = S^{-1} \left(\widetilde{W}(\tau_t) \right),$$

where $\widetilde{W}(t)$ is a reflected Brownian motion, τ is a transformation of time and S^{-1} is a transformation of space.

- 2 Using the representation of X in terms of $\widetilde{W}$, we re-write the expression for L in terms of the local time $L^{\widetilde{W}}$ of $\widetilde{W}$.
- 3 Using the Hölder continuity of $L^{\widetilde{W}}$, we calculate the limit in terms of $L^{\widetilde{W}}$.

Case III: $0 < a < \frac{\sigma^2}{4}$

- Consider a reflected Brownian motion $\widetilde{W} = \{\widetilde{W}(t), t \geq 0\}$ with local time $L^{\widetilde{W}} = L^{\widetilde{W}}(t, x)$ i.e., for any bounded measurable f ,

$$\int_0^t f(\widetilde{W}(s)) ds = \int_0^\infty f(x) L^{\widetilde{W}}(t, x) dx.$$

- Define a function $S: [0, \infty) \rightarrow [0, \infty)$ by

$$S(x) := \int_0^x y^{-\frac{2a}{\sigma^2}} e^{\frac{2by}{\sigma^2}} dy \quad (8)$$

and observe that, since S is strictly increasing and $S(\infty) = \infty$, there exists its inverse S^{-1} .

- Put

$$\rho(x) := \frac{1}{\sigma^2} x^{\frac{4a}{\sigma^2}-1} e^{-\frac{4b}{\sigma^2}x} \mathbb{1}_{x>0}, \quad (9)$$

$$\varphi(t) := \int_0^t \rho\left(S^{-1}\left(\widetilde{W}(s)\right)\right) ds \quad (10)$$

and, since φ is strictly increasing and $\varphi(\infty) = \infty$ with probability 1, its inverse

$$\tau_t := \varphi^{-1}(t) \quad (11)$$

is well-defined a.s.

Case III: $0 < a < \frac{\sigma^2}{4}$

Theorem 6

There exists a Brownian motion $W = \{W(t), t \geq 0\}$ such that the process

$$X(t) := S^{-1}(\widetilde{W}(\tau_t)) \quad (12)$$

satisfies the equation

$$X(t) = x_0 + \int_0^t (a - bX(s))ds + \sigma \int_0^t \sqrt{X(s)}dW(s),$$

i.e. (12) is a CIR process.

Case III: $0 < a < \frac{\sigma^2}{4}$

Theorem 7

Let

$$Y(t) := \sqrt{X(t)} = \sqrt{S^{-1}(\widetilde{W}(\tau_t))}$$

be the square root of the Cox-Ingersoll-Ross process X given by representation (12). Then, for any bounded measurable f ,

$$\int_0^t f(Y(s))ds = \int_0^\infty f(y)L^Y(t, y)dy,$$

where, with probability 1,

$$L^Y(t, y) = y^{\frac{4a}{\sigma^2}-1} \ell(t, y)$$

with

$$\ell(t, y) := \frac{2}{\sigma^2} e^{-\frac{2b}{\sigma^2}y^2} L^{\widetilde{W}}(\varphi(t), S(y^2)) \quad (13)$$

Case III: $0 < a < \frac{\sigma^2}{4}$

Theorem 8

Let $X(t) = S^{-1}(\widetilde{W}(\tau_t))$ be a CIR process. Then, with probability 1,

$$\begin{aligned} L(t) &:= \lim_{\varepsilon \downarrow 0} \frac{1}{2} \int_0^t \left(\frac{a}{\sqrt{X(s) + \varepsilon}} - \frac{\sigma^2}{4} \frac{X(s)}{(X(s) + \varepsilon)^{\frac{3}{2}}} \right) ds \\ &= - \lim_{\varepsilon \downarrow 0} \frac{1}{2} \int_0^t \left(\frac{\frac{\sigma^2}{4} - a}{\sqrt{X(s) + \varepsilon}} - \frac{\sigma^2}{4} \frac{\varepsilon}{(X(s) + \varepsilon)^{\frac{3}{2}}} \right) ds \\ &= -\frac{1}{2} \left(\frac{\sigma^2}{4} - a \right) \int_0^\infty y^{\frac{4a}{\sigma^2} - 2} (\ell(t, y) - \ell(t, 0)) dy, \end{aligned} \tag{14}$$

where $\ell(t, y) := \frac{2}{\sigma^2} e^{-\frac{2b}{\sigma^2} y^2} L^{\widetilde{W}}(\varphi(t), S(y^2))$.

Case III: $0 < a < \frac{\sigma^2}{4}$

Consider again a reflected Ornstein-Uhlenbeck process

$$Y_0(t) = \sqrt{x_0} - \frac{b}{2} \int_0^t Y_0(s) ds + \frac{\sigma}{2} W(t) + L_0(t) \quad (15)$$

and, for $\delta \in (0, \frac{\sigma^2}{4})$, define a CIR process $\{X_\delta\}$ by

$$X_\delta(t) = x_0 + \int_0^t \left(\frac{\sigma^2}{4} - \delta - bX_\delta(s) \right) ds + \sigma \int_0^t \sqrt{X_\delta(s)} dW(s). \quad (16)$$

Denote also $Y_\delta := \sqrt{X_\delta}$ and recall that Y_δ satisfies the equation

$$Y_\delta(t) = \sqrt{x_0} - \frac{b}{2} \int_0^t Y_\delta(s) ds + \frac{\sigma}{2} W(t) + L_\delta(t),$$

where, with probability 1,

$$L_\delta(t) := - \lim_{\varepsilon_n \downarrow 0} \frac{1}{2} \int_0^t \left(\frac{\delta}{\sqrt{X_\delta(s) + \varepsilon_n}} - \frac{\sigma^2}{4} \frac{\varepsilon_n}{(X_\delta(s) + \varepsilon_n)^{\frac{3}{2}}} \right) ds,$$

and the limit does not depend on a particular sequence $\{\varepsilon_n, n \geq 1\}$, $\varepsilon_n \downarrow 0$.

Case III: $0 < a < \frac{\sigma^2}{4}$

Theorem 9

Let $\{\delta_n, n \geq 1\}$ be an arbitrary positive sequence such that $\delta_n \downarrow 0, n \rightarrow \infty$. Then, with probability 1, for any $T > 0$

$$\sup_{t \in [0, T]} |Y_{\delta_n}(t) - Y_0(t)| \rightarrow 0$$

and

$$\sup_{t \in [0, T]} |L_{\delta_n}(t) - L_0(t)| \rightarrow 0$$

as $n \rightarrow \infty$.

Comparing this result with Theorem 5, we see the "continuity" of the components of SDE from the top and from the bottom.

Fractional Cox–Ingersoll–Ross process (Case $H > \frac{1}{2}$)

Now, let $\{B^H(t), t \geq 0\}$ be a continuous modification of a fractional Brownian motion with Hurst index $H \in (0, 1)$. Let for the moment $H > 1/2$. Consider a stochastic differential equation of the form

$$Y^H(t) = Y(0) + \int_0^t \left(\frac{a}{Y^H(s)} - bY^H(s) \right) ds + \sigma B^H(t), t \geq 0, \quad (17)$$

where $Y(0) > 0$ is a given constant, $a, b, \sigma > 0$.

SDE (17) a.s. has a unique pathwise solution $\{Y^H(t), t \geq 0\}$ such that $Y^H(t) > 0$ for all $t \geq 0$, and the subset of Ω where this solution exists does not depend on $Y(0)$, a , b or σ (in fact, the solution exists for all $\omega \in \Omega$ such that $B^H(\omega, t)$ is locally Hölder continuous in t). Moreover, it was proved in [M. and Yurchenko-Tytarenko, 2018] that the process $X^H(t) = (Y^H(t))^2$, $t \geq 0$, satisfies the SDE of the form

$$X^H(t) = X(0) + 2 \int_0^t (a - bX^H(s))ds + 2\sigma \int_0^t \sqrt{X^H(s)}dB^H(s), \quad t \geq 0, \quad (18)$$

where $X(0) = Y^2(0)$ and the integral with respect to the fractional Brownian motion exists as the pathwise limit of the corresponding Riemann-Stieltjes integral sums. Taking into account the form of (18), the process $\{X^H(t), t \geq 0\}$ can be interpreted as a natural fractional generalisation of the Cox–Ingersoll–Ross process with $\{Y^H(t), t \geq 0\}$ being its square root.

Fractional Cox–Ingersoll–Ross process (Case $H < \frac{1}{2}$)

For $H < \frac{1}{2}$ the situation is much more involved. We can define the process Y^H for $H < 1/2$ as the pointwise limit as $\varepsilon \downarrow 0$ of the processes Y^ε that satisfy the SDE of the following type (also written in the integral form):

$$Y_\varepsilon^H(t) = Y(0) + \int_0^t \left(\frac{a}{Y_\varepsilon^H(s) \mathbb{1}_{Y_\varepsilon^H(s) > 0} + \varepsilon} - bY_\varepsilon^H(s) \right) ds + \sigma B^H(t), \quad t \geq 0. \quad (19)$$

This limiting process is well defined and has the following properties [M. and Yurchenko-Tytarenko, 2019].

- The limit

$$Y^H(t, \omega) = \lim_{\varepsilon \rightarrow 0} Y_\varepsilon^H(t, \omega)$$

exists, is nonnegative a.s. and is positive a.e. with respect to the Lebesgue measure a.s.

Fractional Cox–Ingersoll–Ross process (Case $H < \frac{1}{2}$)

- Y^H satisfies the equation of the type

$$Y^H(t) = Y(\alpha) + a \int_{\alpha}^t \frac{ds}{Y^H(s)} - b \int_{\alpha}^t Y^H(s) ds + \sigma \left(B^H(t) - B^H(\alpha) \right),$$

for all $t \in [\alpha, \beta]$ where (α, β) is any interval of Y^H 's strict positivity.

- For all $t \geq 0$

$$\int_0^t Y^H(s) ds < \infty, \quad \int_0^t \frac{1}{Y^H(s)} ds < \infty.$$

- For all $t \geq 0$

$$Y^H(t) \geq Y(0) + a \int_0^t \frac{ds}{Y^H(s)} - b \int_0^t Y^H(s) ds + \sigma B^H(t).$$

- The set $\{t > 0 \mid Y^H(t) > 0\}$ is open in the natural topology on $\mathbb{R}$.
- The trajectories of the process $Y^H = \{Y^H(t), t \geq 0\}$ are continuous on $\mathbb{R}^+$ a.s.

As before, the fractional Cox–Ingersoll–Ross process is the process $X = \{X(t), t \geq 0\}$, such that

$$X(t) = (Y^H(t))^2, \quad t \geq 0.$$

Now let us consider the process $X^H = \{X_t^H, t \geq 0\}$ that satisfies the following SDE:

$$X_t^H = X_0 + a \int_0^t \frac{ds}{X_s^H} + \sigma B_t^H, \quad t \geq 0, \quad (20)$$

where X_0 , a , σ are strictly positive constants.

By analogy with diffusion Bessel processes, whose study began intensively with the paper [McKean, 1960] and was continued in many works, see, for example, [Revuz and Yor, 1991, Ch. 11], we name this process *fractional Bessel process*.

Equation (20) is a particular case of equation (17).

According to comparison theorem for the equations involving fBm, the solution of equation (20) in the case $H > \frac{1}{2}$, exists and is strictly positive.

Fractional Bessel process (Case $H < \frac{1}{2}$)

Let us consider the case $H < \frac{1}{2}$. Similarly to (19), we can define the fractional Bessel process X^H for $H < 1/2$ as the pointwise limit as $\varepsilon \downarrow 0$ of the processes X^ε that satisfy the SDE of the following type [M. and Ralchenko, 2023]:

$$X_t^\varepsilon = X_0 + a \int_0^t \frac{1}{X_s^\varepsilon \mathbf{1}_{X_s^\varepsilon > 0} + \varepsilon} ds + \sigma B_t^H, \quad X_0 > 0. \quad (21)$$

Theorem 10

Process X is a unique nonnegative solution of the equation

$$X_t = X_0 + a \int_0^t \frac{1}{X_s} ds + \sigma B_t^H + L_t^H,$$

where L^H is the reflection function, i.e., it is nondecreasing, continuous and increasing only at zeros of X .

Fractional Bessel process (Case $H < \frac{1}{2}$)

Theorem 11

- ❶ For any $H \in (0, \frac{1}{2})$ and any $\alpha \in (0, \frac{1}{2})$ with probability 1,

$$\limsup_{t \rightarrow \infty} \frac{X_t^H}{t^\alpha} = +\infty.$$

- ❷ For any $H \in (0, \frac{1}{2})$ with probability 1,

$$\limsup_{t \rightarrow \infty} \frac{X_t^H}{\sqrt{t}} \geq \sqrt{2a}.$$

- ❸ For any $H \in (0, \frac{1}{2})$ with probability 1, $\lim_{t \rightarrow \infty} X_t^H = +\infty$.

- ❹ For any $H \in (0, \frac{1}{2})$ and for any $\alpha > \frac{1}{2}$ with probability 1, $\limsup_{t \rightarrow \infty} \frac{X_t^H}{t^\alpha} = 0$.

- ❺ Let $H \in (0, \frac{1}{2})$ be fixed. Then for any $\varepsilon > 0$ there exists $a_0 > 0$ such that for any $a \geq a_0$

$$\mathbb{P} \left\{ \inf_{t \geq 0} X_t^H = 0 \right\} < \varepsilon.$$

Simulation

The graphs below contain sample paths of X^ε , solution to SDE (left graph) and of the right-hand side of SDE (20), where X is replaced with X^ε , i.e.

$$F(X^\varepsilon) = X_0 + a \int_0^t \frac{ds}{X_s^\varepsilon} + \sigma B_t^H$$

(right graph) for the following values of parameters:

$$H = 0.25, \quad X_0 = 1, \quad a = 1, \quad \sigma = 1.$$

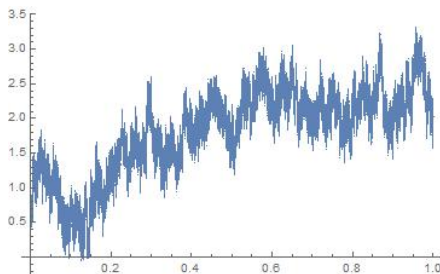


Figure: Case $a = 1$. A trajectory of X^H .

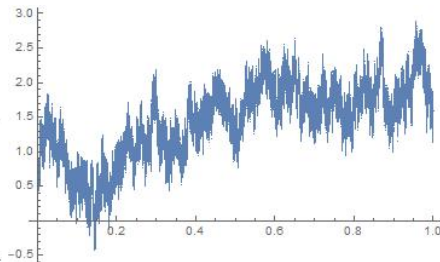


Figure: Case $a = 1$. The right-hand side of (20), i.e., a trajectory of $X^H - L^H$

Simulation

The difference between left-hand and right-hand sides of (20), i.e., the process L^H , is indeed positive as can be seen from its graph (Figure 3).

In order to illustrate the behavior of the fractional diffusion Bessel process for large values of a , let us take $a = 10$. A corresponding trajectory of X^H is represented in Figure 4. We see that in this case the process X^H never hits zero. Moreover, its asymptotic growth is comparable with a function $\sqrt{2at}$ (red curve) as stated in Theorem 11.

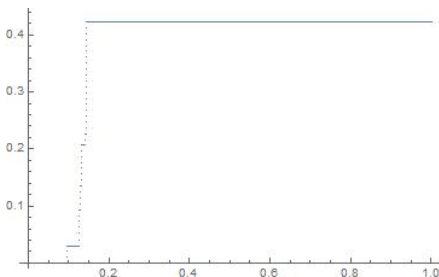


Figure: Case $a = 1$. A trajectory of L^H

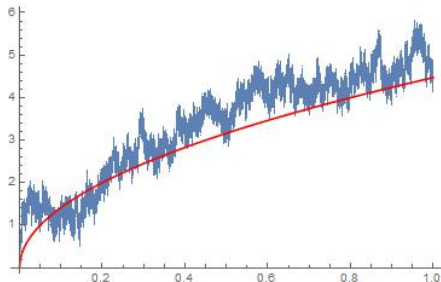


Figure: Case $a = 10$. A trajectory of X^H and a graph of $\sqrt{2at}$

Simulation

Finally, in order to illustrate the behaviour of $\tilde{L}_t^H := \lim_{a \downarrow 0} (a \int_0^t \frac{ds}{X_s^{H,a}} + L_t^{H,a})$ let us take a small value of a , namely $a = 0.01$. We see that it is still non-negative and non-decreasing, which confirms our theoretical results. Moreover, we see that $\tilde{L}^H$ has more points of growth, in comparison to L^H which testifies in favor of the fact that the reflecting function $\tilde{L}^H$ is the limit of two components: the prelimit integral and the prelimit reflecting function.

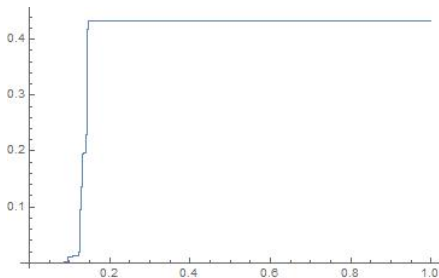


Figure: Case $a = 0.01$. A trajectory of $\tilde{L}^H$.

Fractional Bessel process (Case $H < \frac{1}{2}$): Parameter estimation

Consider a fractional Bessel process

$$X_t = X_0 + a \int_0^t \frac{1}{X_s} ds + \sigma B_t^H + L_t^H,$$

where $a, \sigma > 0$, $H \in (0, 1/2)$ and L^H is the reflection function, i.e., it is nondecreasing, continuous and increasing only at zeros of X .

Problem

We want to estimate parameters $H \in (0, 1/2)$, $\sigma > 0$ and $a > 0$.

Fractional Bessel process (Case $H < \frac{1}{2}$): Estimation of H

Notation

For a stochastic process $\xi = \{\xi_t, t \in [0, T]\}$ observed on a discrete uniform partition $0 = t_0 < t_1 < \dots < t_n = T$, $t_k := \frac{kT}{n}$, denote

$$V_{1,2}^n(\xi) := \sum_{k=0}^{n-1} (\xi_{t_{k+1}} - \xi_{t_k})^2, \quad V_{2,2}^n(\xi) := \sum_{k=0}^{n-2} (\xi_{t_{k+2}} - 2\xi_{t_{k+1}} + \xi_{t_k})^2.$$

Theorem 12

- ① Let X be a rough Bessel process observed in points $t_k = \frac{kT}{n}$, $k = 0, \dots, n$. Then, with probability 1, as $n \rightarrow \infty$,

$$\left(\frac{n}{T}\right)^{-1+2H} V_{1,2}^n(X) \rightarrow \sigma^2 T, \quad \left(\frac{n}{T}\right)^{-1+2H} V_{2,2}^n(X) \rightarrow (4 - 2^{2H})\sigma^2 T.$$

- ② The estimator

$$\hat{H} := \frac{\log\left(4 - \frac{V_{2,2}^n(X)}{V_{1,2}^n(X)}\right)}{2 \log 2}$$

is a (strongly) consistent estimator of the Hurst index H .

Fractional Bessel process (Case $H < \frac{1}{2}$): Estimation of σ

Theorem 13

Let the Hurst index H be known. Then the estimator

$$\hat{\sigma} = \frac{1}{T^H} \sqrt{n^{-1+2H} V_{1,2}^n(X)}$$

is a (strongly) consistent estimator of the volatility coefficient σ , i.e., with probability 1,

$$\hat{\sigma} \rightarrow \sigma, \quad n \rightarrow \infty.$$

Remark 14

$\hat{H}$ and $\hat{\sigma}$ are consistent estimators for any $H \in (0, 1)$, not only $H < 1/2$.

Fractional Bessel process (Case $H < \frac{1}{2}$): Estimation of a

Recall that

$$X_t = X_0 + a \int_0^t \frac{1}{X_s} ds + \sigma B_t^H + L_t^H.$$

Denote

$$\hat{a}(T) := \frac{X_T}{\int_0^T \frac{1}{X_t} dt}.$$

Theorem 15

With probability 1,

$$\hat{a}(T) \rightarrow a, \quad T \rightarrow \infty,$$

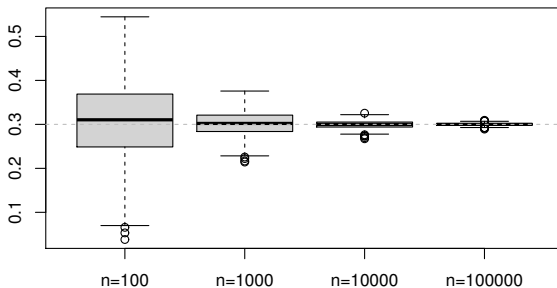
i.e. $\hat{a}(T)$ is a strongly consistent estimator of a .

- For each estimator, 1000 paths of X^H are generated and, for each path, the values of estimator for different n (for $\hat{H}$ and $\hat{\sigma}$) or T (for $\hat{a}(T)$) are computed.
- We simulate paths of X^H by using a standard Euler scheme for X_ε^H with $\varepsilon = 0.001$ and $\Delta t = 0.000001$.
- Real values: $H = 0.3$, $\sigma = 1$, $a = 2$.

Simulations I: $\hat{H}$ ($H = 0.3$)

	$n = 100$	$n = 1000$	$n = 10000$	$n = 100000$
Mean	0.3074612	0.3018724	0.299684	0.2999879
Variance	0.007608759	0.0006979747	0.00007438351	0.000007596678
CV	0.2837047	0.08751781	0.02877894	0.009187727

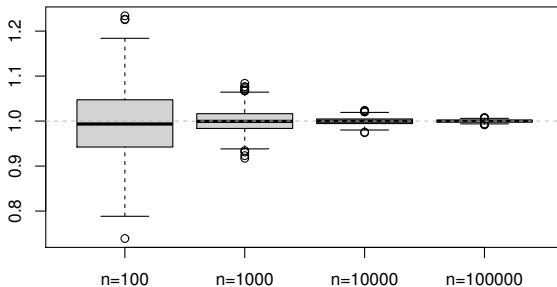
Estimation of the Hurst index H



Simulations II: $\hat{\sigma}$ (H is known)

	$n = 100$	$n = 1000$	$n = 10000$	$n = 100000$
Mean	0.9964649	0.9999765	0.9996666	0.9999794
Variance	0.005895953	0.0006022594	0.00005529735	0.000005777412
CV	0.07705752	0.02454155	0.007438699	0.002403674

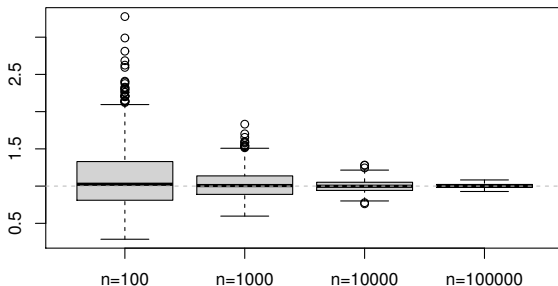
Estimation of the volatility coefficient, H is known



Simulations III: $\hat{\sigma}$ (H is unknown)

	$n = 100$	$n = 1000$	$n = 10000$	$n = 100000$
Mean	1.098972	1.021549	0.9991251	1.001703
Variance	0.1783677	0.03501052	0.006181774	0.000860674
CV	0.3843009	0.183164	0.0786931	0.02928738

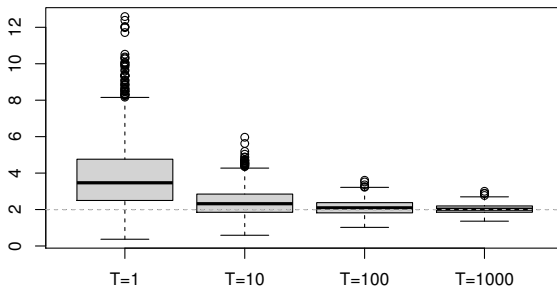
Estimation of the volatility coefficient, H is unknown



Simulations IV: $\hat{a}(T)$

	$T = 1$	$T = 10$	$T = 100$	$T = 1000$
Mean	3.872642	2.409692	2.109389	2.029289
Variance	4.09064	0.5914294	0.1749705	0.06310528
CV	0.5222618	0.3191463	0.1983014	0.1237909

Estimation of the drift coefficient



References I



Bertoin, J. (1990a).

Complements on the Hilbert transform and the fractional derivative of Brownian local times.
Kyoto journal of mathematics, 30(4):651–670.



Bertoin, J. (1990b).

Excursions of a $BES_0(d)$ and its drift term ($0 < d < 1$).
Probability theory and related fields, 84(2):231–250.



Blei, S. (2012).

On symmetric and skew Bessel processes.
Stochastic processes and their applications, 122(9):3262–3287.



Bray, A. J. (2000).

Random walks in logarithmic and power-law potentials, nonuniversal persistence, and vortex dynamics in the two-dimensional XY model.
Phys. Rev. E, 62:103–112.



Cherny, A. S. (2000).

On the strong and weak solutions of stochastic differential equations governing Bessel processes.
Stochastics and stochastics reports, 70(3–4):213–219.

References II



Cherny, A. S. and Engelbert, H.-J. (2005).
Singular Stochastic Differential Equations.
Springer Berlin Heidelberg, Berlin, Heidelberg.



Cox, J. C., Ingersoll, J. E., and Ross, S. A. (1981).
A re-examination of traditional hypotheses about the term structure of interest rates.
The journal of finance, 36(4):769–799.



Cox, J. C., Ingersoll, J. E., and Ross, S. A. (1985a).
An intertemporal general equilibrium model of asset prices.
Econometrica: journal of the Econometric Society, 53(2):363.



Cox, J. C., Ingersoll, J. E., and Ross, S. A. (1985b).
A theory of the term structure of interest rates.
Econometrica: journal of the Econometric Society, 53(2):385.



Guarnieri, F., Moon, W., and Wettlaufer, J. S. (2017).
Solution of the Fokker-Planck equation with a logarithmic potential and mixed eigenvalue spectrum.
Journal of mathematical physics, 58(9):093301.

References III



Heston, S. L. (1993).

A closed-form solution for options with stochastic volatility with applications to bond and currency options.

The review of financial studies, 6(2):327–343.



Horibe, M., Hosoya, A., and Sakamoto, J. (1983).

Stochastic quantization and the Gribov problem in non-abelian gauge theories.

Progress of theoretical physics, 70(6):1636–1647.



M., Y., Pilipenko, A., and Yurchenko-Tytarenko, A. (2023).

Low-dimensional Cox–Ingersoll–Ross process.

arXiv:2303.12911 [math.PR].

(submitted).



M., Y. and Ralchenko, K. (2023).

Fractional Bessel process with Hurst index $H \in (0, \frac{1}{2})$.

arXiv:2304.06328 [math.PR].

(submitted).



M., Y. and Yurchenko-Tytarenko, A. (2018).

Fractional Cox–Ingersoll–Ross process with non-zero “mean”.

Mod. Stoch. Theory Appl., 5(1):99–111.

References IV



M., Y. and Yurchenko-Tytarenko, A. (2019).
Fractional Cox-Ingersoll-Ross process with small Hurst indices.
Mod. Stoch. Theory Appl., 6(1):13–19.



M., Y. and Yurchenko-Tytarenko, A. (2023).
Standard and fractional reflected Ornstein-Uhlenbeck processes as the limits of square roots
of Cox-Ingersoll-Ross processes.
Stochastics, 95(1):99–117.



McKean, Jr., H. P. (1960).
The Bessel motion and a singular integral equation.
Mem. Coll. Sci. Univ. Kyoto Ser. A. Math., 33:317–322.



Mijatović, A. and Urusov, M. (2015).
On the loss of the semimartingale property at the hitting time of a level.
Journal of theoretical probability, 28(3):892–922.



Revuz, D. and Yor, M. (1991).
Continuous martingales and Brownian motion, volume 293 of *Grundlehren der
mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*.
Springer-Verlag, Berlin.

Thank you very much for your attention!