

# Cylindrical Lévy processes

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Why do we need a model  
of random perturbations  
in infinite dimension?

# Stochastic heat equation

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Search solution  $X := (X(t, \cdot) : t \in [0, T])$  in  $L^2(\mathcal{O})$ :

$$\frac{dX}{dt}(t, \cdot) = AX(t, \cdot) + f(X(t, \cdot)) \underbrace{\frac{dN}{dt}(t, \cdot)}_{\text{noise in } L^2(\mathcal{O})} \quad \text{for all } t \in [0, T],$$

where  $A : \text{dom}(A) \subseteq L^2(\mathcal{O}) \rightarrow L^2(\mathcal{O})$  with  $Af = \frac{d^2 f}{dr^2}$ .

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where  $A : \text{dom}(A) \subseteq L^2(\mathcal{O}) \rightarrow L^2(\mathcal{O})$  with  $Af = \frac{d^2 f}{dr^2}$ . As integral equation:

$$X(t, \cdot) = X(0, \cdot) + \int_0^t AX(s, \cdot) ds + \underbrace{\int_0^t f(X(s, \cdot)) N(ds, \cdot)}_{\text{stochastic integral}} \quad \text{for all } t \in [0, T].$$

Why cannot we use  
a standard Brownian motion  
in infinite dimensions?

# Brownian motion

**Definition.** Let  $U$  be a Hilbert space. A stochastic process  $(W(t) : t \geq 0)$  with values in  $U$  is called a **Brownian motion**, if

- (1)  $W(0) = 0$ ;
- (2)  $W$  has independent, stationary increments;
- (3)  $W(t) - W(s) \stackrel{\mathcal{D}}{=} N(0, (t - s)Q)$  for all  $0 \leq s \leq t$ ,

where  $Q: U \rightarrow U$  is a linear operator with the following properties:

symmetric:  $\langle Qu, v \rangle_U = \langle u, Qv \rangle_U$  for all  $u, v \in U$ ;

non-negative:  $\langle Qu, u \rangle_U \geq 0$  for all  $u \in U$ ;

nuclear:  $\sum_{k=1}^{\infty} \langle Qe_k, e_k \rangle_U < \infty$  for an orthonormal basis  $\{e_k\}_{k \in \mathbb{N}}$ .

# Why are Brownian motions not sufficient?

There does not exist a Brownian motion with independent components in an infinite dimensional Hilbert space:

Let  $U$  be a Hilbert space with orthonormal basis  $\{e_k\}_{k \in \mathbb{N}}$  and  $W(t) \stackrel{\mathcal{D}}{=} N(0, tQ)$  with independent components. Then we have

$$t\langle Qe_k, e_l \rangle_U = E\left[\langle W(t), e_k \rangle_U \langle W(t), e_l \rangle_U\right] = \begin{cases} t, & \text{if } k = l, \\ 0, & \text{if } k \neq l, \end{cases}$$

which implies  $Q = \text{Id}$ . However,  $Q = \text{Id}$  is not a nuclear operator:

$$\sum_{k=1}^{\infty} \langle Qe_k, e_k \rangle_U = \sum_{k=1}^{\infty} \|e_k\|_U^2 = \infty.$$

What is a  
cylindrical Brownian motion?

## Common definition

**Working definition:** Let  $H$  be a Hilbert space with orthonormal basis  $\{e_k\}_{k \in \mathbb{N}}$  and let  $\{b_k\}_{k \in \mathbb{N}}$  be independent real valued Brownian motions.

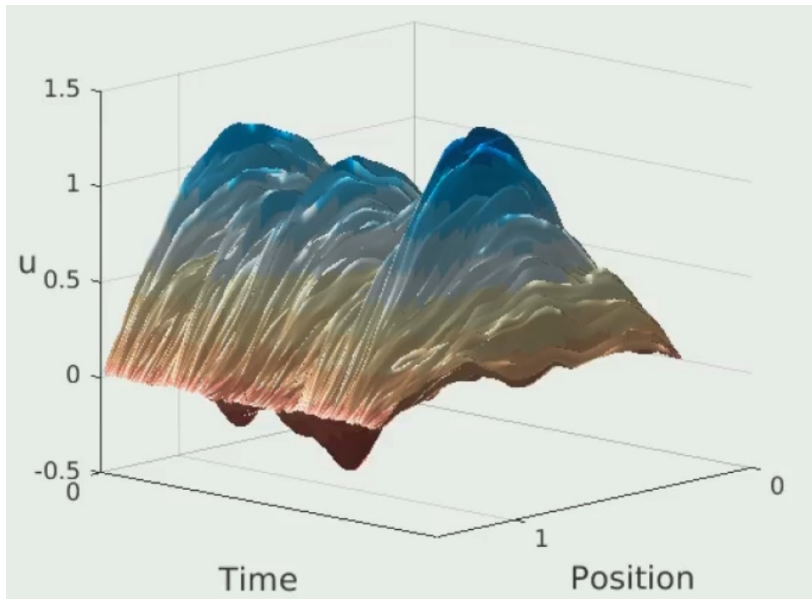
A cylindrical Brownian motion in  $H$  is a family  $(W(t) : t \geq 0)$  such that

$$W(t) = \sum_{k=1}^{\infty} e_k b_k(t), \quad t \geq 0,$$

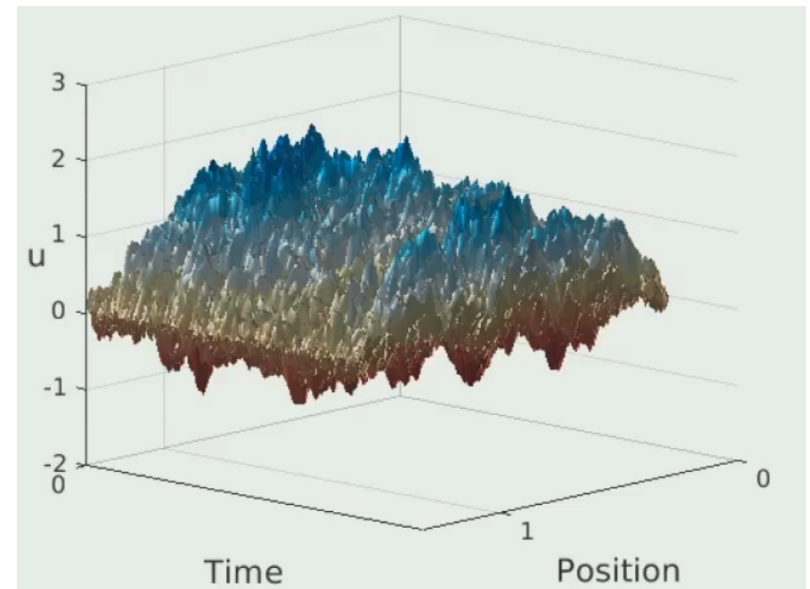
converges in square mean in a larger Hilbert space  $H_1$  containing  $H$ .

# Genuine vs. cylindrical

Solution of  $X(t, \cdot) = \int_0^t \Delta X(s, \cdot) ds + W(t)$ :



genuine Brownian motion  $W$



cylindrical Brownian motion  $W$

Code & graphics by David Cohen (University of Gothenburg)

Cylindrical random variables  
and  
cylindrical measures

# Cylindrical measures

Let  $U$  be a Banach space with dual space  $U^*$  and dual pairing  $\langle \cdot, \cdot \rangle$  and let  $(\Omega, \mathcal{A}, P)$  denote a probability space.

The cylindrical algebra  $\mathfrak{Z}(U, \Gamma)$  for some  $\Gamma \subseteq U^*$  is defined by

$$\mathfrak{Z}(U, \Gamma) := \left\{ \{u \in U : (\langle u, u_1^* \rangle, \dots, \langle u, u_n^* \rangle) \in B\} : u_i^* \in \Gamma, B \in \mathfrak{B}(\mathbb{R}^n), n \in \mathbb{N} \right\}$$

If  $\Gamma$  is finite, then  $\mathfrak{Z}(U, \Gamma)$  is a  $\sigma$ -algebra.

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If  $\Gamma$  is finite, then  $\mathfrak{Z}(U, \Gamma)$  is a  $\sigma$ -algebra.

**Definition:** A map  $\mu: \mathfrak{Z}(U, U^*) \rightarrow [0, 1]$  is called a cylindrical (probability) measure if for each finite  $\Gamma \subseteq U^*$ , the restriction  $\mu|_{\mathfrak{Z}(U, \Gamma)}$  is a probability measure.

# Operations for cylindrical measures

**Image of a cylindrical measure:**

If  $\mu: \mathfrak{Z}(U, U^*) \rightarrow [0, 1]$  and  $F: U \rightarrow V$  linear and continuous, then

$$F(\mu) := \mu \circ F^{-1}: \mathfrak{Z}(V, V^*) \rightarrow [0, 1]$$

is a cylindrical measure defined by

$$\begin{aligned} F(\mu) & \left( \{v \in V : (\langle v, v_1^* \rangle, \dots, \langle v, v_n^* \rangle) \in B\} \right) \\ & := \mu \left( \{u \in U : (\langle u, F^* v_1^* \rangle, \dots, \langle u, F^* v_n^* \rangle) \in B\} \right) \end{aligned}$$

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## Convolution of cylindrical measures:

If  $\mu, \nu: \mathfrak{Z}(U, U^*) \rightarrow [0, 1]$  then the convolution

$$(\mu * \nu): \mathfrak{Z}(U, U^*) \rightarrow [0, 1], \quad (\mu * \nu)(C) := \int_U \mu(C - u) \nu(du).$$

## Finite-dimensional projections

For  $\mu: \mathfrak{B}(U, U^*) \rightarrow [0, 1]$  and  $u_1^*, \dots, u_n^* \in U^*$  define

$$\pi_{u_1^*, \dots, u_n^*}: U \rightarrow \mathbb{R}^n, \quad \pi_{u_1^*, \dots, u_n^*}(u) = (\langle u, u_1^* \rangle, \dots, \langle u, u_n^* \rangle).$$

Then the image of  $\mu$  under  $\pi_{u_1^*, \dots, u_n^*}$  defines a probability measure

$$\mu \circ \pi_{u_1^*, \dots, u_n^*}^{-1}: \mathfrak{B}(\mathbb{R}^n) \rightarrow [0, 1].$$

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The family  $\{\mu \circ \pi_{u_1^*, \dots, u_n^*}^{-1} : u_1^*, \dots, u_n^* \in U^*, n \in \mathbb{N}\}$  satisfies the consistency condition:

$$\mu \circ \pi_{u_1^*, \dots, u_n^*}^{-1} \circ A^{-1} = \mu \circ \pi_{A(u_1^*, \dots, u_n^*)}^{-1} \quad \text{for all } A \in \mathbb{R}^{m, n}.$$

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**Lemma.** If a family  $\{\mu_{u_1^*, \dots, u_n^*} : u_1^*, \dots, u_n^* \in U^*, n \in \mathbb{N}\}$  of Borel measures  $\mu_{u_1^*, \dots, u_n^*}$  on  $\mathfrak{B}(\mathbb{R}^n)$  satisfies the consistency condition

$$\mu_{u_1^*, \dots, u_n^*} \circ A^{-1} = \mu_{A(u_1^*, \dots, u_n^*)} \quad \text{for all } A \in \mathbb{R}^{m, n}, u_i^* \in U^*, n \in \mathbb{N},$$

then it defines a cylindrical measure on  $\mathfrak{Z}(U, U^*)$ .

# Characteristic function

For a cylindrical measure  $\mu$  the mapping

$$\varphi_\mu : U^* \rightarrow \mathbb{C}, \quad \varphi_\mu(u^*) := \int_U e^{i\langle u, u^* \rangle} \mu(du)$$

is called **characteristic function of  $\mu$** .

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**Theorem.** (Kolmogorov 1935)

For cylindrical measures  $\mu$  and  $\nu$  the following are equivalent:

(1)  $\mu = \nu$ ;

(2)  $\varphi_\mu = \varphi_\nu$ .

# Bochner's theorem

**Bochner's Theorem.** (finite dimensions)

Let  $\varphi: \mathbb{R}^d \rightarrow \mathbb{C}$  be a function. Then  $\varphi$  is the characteristic function of a probability measure on  $\mathfrak{B}(\mathbb{R}^d)$  if and only if  $\varphi$  is continuous with  $\varphi(0) = 1$  and positive-definite.

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**Bochner's Theorem.** (infinite dimensions) (Bochner 1955)

Let  $\varphi: U^* \rightarrow \mathbb{C}$  be a function. Then  $\varphi$  is the characteristic function of a **cylindrical** probability measure if and only if  $\varphi$  is continuous with  $\varphi(0) = 1$  and positive-definite.

## Genuineness test

**Lemma.** Let  $\mu: \mathfrak{B}(U) \rightarrow [0, 1]$  be a genuine Radon measure with characteristic function

$$\varphi_\mu: U^* \rightarrow \mathbb{C}, \quad \varphi_\mu(u^*) := \int_U e^{i\langle u, u^* \rangle} \mu(du).$$

Then  $\varphi_\mu$  is weak\* sequentially continuous.

**Example.** The function  $h \mapsto e^{-\frac{1}{2}\|h\|^2}$  is not sequentially weak\* continuous.

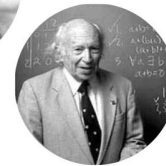
# Cylindrical random variables

**Definition:** A cylindrical random variable  $Z$  in  $U$  is a mapping

$$Z : U^* \rightarrow L_P^0(\Omega; \mathbb{R}) \quad \text{linear and continuous.}$$

A cylindrical process in  $U$  is a family  $(Z(t) : t \geq 0)$  of cylindrical random variables.

- I. E. Segal, 1954
- I. M. Gel'fand 1956: Generalized Functions
- L. Schwartz 1969: seminaire rouge,  
radonifying operators



# Cylindrical measures and cylindrical random variables

## Theorem.

(a) Let  $Z: U^* \rightarrow L_0^P(\Omega; \mathbb{R})$  be a cylindrical random variable. By defining

$$\mu(\{u \in U : (\langle u, u_1^* \rangle, \dots, \langle u, u_n^* \rangle) \in B\}) := P((Zu_1^*, \dots, Zu_n^*) \in B)$$

for all  $u_i^* \in U^*$ ,  $B \in \mathfrak{B}(\mathbb{R}^n)$  we obtain a cylindrical measure on  $\mathfrak{Z}(U, U^*)$ .

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for all  $u_i^* \in U^*$ ,  $B \in \mathfrak{B}(\mathbb{R}^n)$  we obtain a cylindrical measure on  $\mathfrak{Z}(U, U^*)$ .

(b) Let  $\mu$  be a cylindrical measure on  $\mathfrak{Z}(U, U^*)$ . Then there exists a probability space  $(\Omega', \mathcal{A}', P')$  and a cylindrical random variable  $Z: U^* \rightarrow L_P^0(\Omega; \mathbb{R})$  such that

$$\mu(\{u \in U : (\langle u, u_1^* \rangle, \dots, \langle u, u_n^* \rangle) \in B\}) = P'((Zu_1^*, \dots, Zu_n^*) \in B)$$

for all  $u_i^* \in U^*$ ,  $B \in \mathfrak{B}(\mathbb{R}^n)$ .

# Cylindrical measures and cylindrical random variables

**Definition.** Let  $Z: U^* \rightarrow L_0^P(\Omega; \mathbb{R})$  be a cylindrical random variable.

**(a)** The cylindrical measure  $\mu$  defined previously is called the (cylindrical) distribution of  $Z$ .

**(b)** The function

$$\varphi_Z: U^* \rightarrow \mathbb{C}, \quad \varphi_Z(u^*) := E \left[ e^{iZu^*} \right].$$

is called characteristic function of  $Z$ .

**It follows:**  $\varphi_\mu = \varphi_Z$ .

## Example: induced cylindrical random variable

**Example:** Let  $X : \Omega \rightarrow U$  be a (classical) random variable. Then

$$Z : U^* \rightarrow L_P^0(\Omega; \mathbb{R}), \quad Zu^* := \langle X, u^* \rangle$$

defines a cylindrical random variable.

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**But:** not for every cylindrical random variable  $Z : U^* \rightarrow L_P^0(\Omega; \mathbb{R})$  there exists a classical random variable  $X : \Omega \rightarrow U$  satisfying

$$Zu^* = \langle X, u^* \rangle \quad \text{for all } u^* \in U^*.$$

# Cylindrical Lévy processes

## Definition: cylindrical Lévy process

**Definition.** (Applebaum, Riedle (2010))

A cylindrical process  $(L(t) : t \geq 0)$  is called a *cylindrical Lévy process*, if for all  $u_1^*, \dots, u_n^* \in U^*$  and  $n \in \mathbb{N}$  the stochastic process :

$$\left( (L(t)u_1^*, \dots, L(t)u_n^*) : t \geq 0 \right)$$

is a Lévy process in  $\mathbb{R}^n$ .

A stochastic process  $(\ell(t) : t \geq 0)$  with values in  $\mathbb{R}^n$  is called Lévy process, if:

- (1)  $\ell(0) = 0$ ;
- (2)  $\ell$  has stationary, independent increments;
- (3)  $\ell$  has càdlàg paths and jumps only at random times.

## Verifying a cylindrical Lévy process

**Lemma.** A cylindrical process  $(L(t) : t \geq 0)$  in  $U$  is a cylindrical Lévy process if and only if the following two conditions are satisfied:

(i) for each  $u_1^*, \dots, u_n^* \in U^*$ ,  $t_0 \leq t_1 \leq \dots \leq t_n$  and  $n \in \mathbb{N}$  the random variables

$$(L(t_1) - L(t_0))u_1^*, \dots, (L(t_n) - L(t_{n-1}))u_n^*$$

are independent;

(ii)  $(L(t)u^* : t \geq 0)$  is a Lévy process for all  $u^* \in U^*$ .

## Example: genuine Lévy process

If  $(Y(t) : t \geq 0)$  is a genuine Lévy process in  $U$  then

$$L(t) : U^* \rightarrow L_P^0(\Omega; \mathbb{R}), \quad L(t)u^* := \langle Y(t), u^* \rangle$$

defines a cylindrical Lévy process  $(L(t) : t \geq 0)$ .

## Example: cylindrical compound Poisson process

Let  $(Y_k : k \in \mathbb{N})$  be a sequence of independent cylindrical random variables in  $U$  with identical cylindrical distribution. If  $(n(t) : t \geq 0)$  is a real valued Poisson process which is independent of  $\{Y_k u^* : k \in \mathbb{N}, u^* \in U^*\}$  then the *cylindrical compound Poisson process*  $(L(t) : t \geq 0)$  is defined for each  $u^* \in U^*$  by

$$L(t)u^* := \begin{cases} 0, & \text{if } n(t) = 0, \\ Y_1 u^* + \cdots + Y_{n(t)} u^*, & \text{else.} \end{cases}$$

Example:

Hedgehog cylindrical Lévy processes

## Example: hedgehog processes

**Theorem.** Let  $U$  be a Hilbert space with ONB  $(e_k)_{k \in \mathbb{N}}$  and  $(\sigma_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}$ ;

$(h_k)_{k \in \mathbb{N}}$  be a sequence of independent, real-valued Lévy processes.

**1) (weak convergence)** If for all  $u^* \in U$  and  $t \geq 0$  the sum

$$L(t)u^* := \sum_{k=1}^{\infty} \langle e_k, u^* \rangle \sigma_k h_k(t)$$

converges  $P$ -a.s. then it defines a cylindrical Lévy process  $(L(t) : t \geq 0)$ .

**2) (strong convergence)** If for all  $t \geq 0$  the sum

$$L(t) := \sum_{k=1}^{\infty} e_k \sigma_k h_k(t)$$

converges  $P$ -a.s. then it defines an genuine Lévy process  $(L(t) : t \geq 0)$ .

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**Example 0:** for  $h_k$  standard, real-valued Brownian motion:

$(\sigma_k)_{k \in \mathbb{N}} \in \ell^\infty \iff$  cylindrical (Brownian) Lévy process

$(\sigma_k)_{k \in \mathbb{N}} \in \ell^2 \iff$  genuine (Brownian) Lévy process

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**Example 1:** for  $h_k$  Poisson process with intensity 1:

$(\sigma_k)_{k \in \mathbb{N}} \in \ell^2 \iff$  cylindrical Lévy process

$(\sigma_k)_{k \in \mathbb{N}} \in \ell^1 \iff$  genuine Lévy process

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converges  $P$ -a.s. then it defines a cylindrical Lévy process  $(L(t) : t \geq 0)$ .

**Example 2:** for  $h_k$  symmetric, standardised,  $\alpha$ -stable:

$$(\sigma_k)_{k \in \mathbb{N}} \in \ell^{(2\alpha)/(2-\alpha)} \iff \text{cylindrical Lévy process}$$

$$(\sigma_k)_{k \in \mathbb{N}} \in \ell^\alpha \iff \text{genuine Lévy process}$$

Example:  
Subordination

## Example: subordination

### Theorem.

Let  $W$  be a cylindrical Brownian motion in a Banach space  $U$ ,  
 $\ell$  be a real-valued Lévy subordinator, independent of  $W$ .

Then, for each  $t \geq 0$ ,

$$L(t) : U^* \rightarrow L_P^0(\Omega; \mathbb{R}), \quad L(t)u^* = W(\ell(t))u^*$$

defines a cylindrical Lévy process  $(L(t) : t \geq 0)$  in  $U$ .

**Example.** If  $\ell$  is an  $\alpha/2$  stable process, then  $\varphi_{L(t)}(u^*) = \exp(-t \|u^*\|^\alpha)$   
for all  $u^* \in U^*$ .

Example:

Lévy basis

# Independently scattered random measures

For  $\mathcal{O} \subseteq \mathbb{R}^d$  define  $\mathfrak{B}_b(\mathcal{O}) := \{A \subseteq \mathcal{O} : A \text{ relatively compact}\}$ .

**Definition** (Rajput and Rosinski (1989)).

An **infinitely divisible random measure** is a map

$$M: \mathfrak{B}_b(\mathcal{O}) \rightarrow L^0(\Omega, P)$$

satisfying for each collection of disjoint sets  $A_1, A_2, \dots \in \mathfrak{B}_b(\mathcal{O})$ :

**(a)** the random variables  $M(A_1), M(A_2), \dots$  are independent;

**(b)** if  $\bigcup_{k \in \mathbb{N}} A_k \in \mathfrak{B}_b(\mathcal{O})$  then  $M\left(\bigcup_{k \in \mathbb{N}} A_k\right) = \sum_{k \in \mathbb{N}} M(A_k)$   $P$ -a.s.

**(c)** the random variable  $M(A)$  is infinitely divisible for each  $A \in \mathfrak{B}_b(\mathcal{O})$ .

# Independently scattered random measures

For  $\mathcal{O} \subseteq \mathbb{R}^d$  define  $\mathfrak{B}_b(\mathcal{O}) := \{A \subseteq \mathcal{O} : A \text{ relatively compact}\}$ .

**Definition** (Rajput and Rosinski (1989)).

An **infinitely divisible random measure** is a map

$$M: \mathfrak{B}_b(\mathcal{O}) \rightarrow L_P^0(\Omega; \mathbb{R})$$

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**(c)** the random variable  $M(A)$  is infinitely divisible for each  $A \in \mathfrak{B}_b(\mathcal{O})$ .

$M(A) \stackrel{\mathcal{D}}{=} (\gamma(A), \Sigma(A), \nu_A)$  **characteristics of  $M$**

$\lambda(A) = \|\gamma\|_{TV}(A) + \Sigma(A) + \int_{\mathbb{R}} (\beta^2 \wedge 1) \nu_A(d\beta)$  **control measure of  $M$**

# Lévy-valued random measure

**Definition.** A family  $(M(t) : t \geq 0)$  of infinitely divisible random measures  $M(t) : \mathfrak{B}_b(\mathcal{O}) \rightarrow L_R^0(\Omega, \mathbb{R})$  is called a **Lévy-valued random measure** if, for every  $A_1, \dots, A_n \in \mathfrak{B}_b(\mathcal{O})$ ,  $n \in \mathbb{N}$ , the stochastic process

$$((M(t)(A_1), \dots, M(t)(A_n)) : t \geq 0)$$

is a Lévy process in  $\mathbb{R}^n$ . We shall write  $M(t, A) := M(t)(A)$ .

## Example: stable noise (Balan (2014))

Define for  $B \in \mathfrak{B}_b([0, \infty) \times \mathbb{R}^d)$ :

$$\widetilde{M}(B) := \begin{cases} \int_{B \times \mathbb{R}} y N(ds, dx, dy), & \text{if } \alpha \in (0, 1], \\ \int_{B \times \mathbb{R}} y \widetilde{N}(ds, dx, dy), & \text{if } \alpha \in (1, 2), \end{cases}$$

where  $N$  is a Poisson random measure on  $[0, \infty) \times \mathbb{R}^d \times \mathbb{R}$  with intensity  $\text{leb} \otimes \text{leb} \otimes \nu_\alpha$  for  $\nu_\alpha(dy) = \frac{\alpha}{2} \frac{1}{|y|^{1+\alpha}} dy$ .

Then

$$M(t, A) := \widetilde{M}((0, t] \times A) \quad \text{for } A \in \mathfrak{B}_b(\mathbb{R}^d), t \geq 0,$$

defines a Lévy-valued random measure on  $\mathbb{R}^d$  with control measure

$$\lambda(A) = \frac{2}{2-\alpha} \text{leb}(A).$$

# Integration (Rajput and Rosinski (1989))

Let  $M$  be a Lévy-valued random measure. For a simple function

$$f: \mathcal{O} \rightarrow \mathbb{R}, \quad f(x) = \sum_{k=1}^n \alpha_k \mathbb{1}_{A_k}(x),$$

for  $\alpha_k \in \mathbb{R}$  and pairwise disjoint sets  $A_1, \dots, A_n \in \mathfrak{B}_b(\mathcal{O})$ , define

$$\int_{\mathcal{O}} f(x) M(t, dx) := \sum_{k=1}^n \alpha_k M(t, A_k) \quad \text{for all } t \geq 0.$$

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A measurable function  $f: \mathcal{O} \rightarrow \mathbb{R}$  is said to be  $M$ -integrable if

- (1) there exists a sequence of simple functions  $(f_n)_{n \in \mathbb{N}}$  such that  $f_n$  converges pointwise to  $f$   $\lambda$ -a.e., where  $\lambda$  is the control measure of  $M$ ;
- (2) for each  $t \geq 0$ , the sequence  $(\int_{\mathcal{O}} f_n(x) M(t, dx))_{n \in \mathbb{N}}$  converges in probability.

In this case:  $\int_A f(x) M(t, dx) := P\text{-}\lim_{n \rightarrow \infty} \int_A f_n(x) M(t, dx).$

# Integration (Rajput and Rosinski (1989))

Let  $M$  be a Lévy-valued random measure with control measure  $\lambda$ .

The space of  $M$ -integrable functions is given by the Musielak-Orlicz space

$$L_M(\mathcal{O}, \lambda) := \left\{ f \in L^0(\mathcal{O}, \lambda) : \int_{\mathcal{O}} \Phi_M(|f(x)|, x) \lambda(dx) < \infty \right\},$$

where  $\Phi_M: \mathbb{R} \times \mathcal{O} \rightarrow \mathbb{R}$  is a function depending on the distribution of  $M$ .

## From random measure to cylindrical

**Theorem.** Let  $M$  be a Lévy-valued random measure on  $\mathfrak{B}_b(\mathcal{O})$  with control measure  $\lambda$ . If  $U$  is a Banach space for which  $U^*$  is continuously embedded into  $L_M(\mathcal{O}, \lambda)$ , then

$$L(t)f := \int_{\mathcal{O}} f(x) M(t, dx) \quad \text{for all } f \in U^*,$$

defines a cylindrical Lévy processes  $L$  in  $U$ .

## Example: stable noise

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## Example: stable noise

For each  $\alpha \in (0, 2)$  we have

$$L_M(\mathcal{O}, \lambda) = L^\alpha(\mathcal{O}, \text{leb}).$$

Thus,  $M$  defines a cylindrical Levy process  $L$  on

$$\begin{cases} U = L^{\alpha'}(\mathcal{O}, \text{leb}), & \text{if } \alpha \in (1, 2), \\ U = L^p(\mathcal{O}, \text{leb}), & \text{if } \alpha \in (0, 1), \mathcal{O} \text{ bounded, any } p > 1. \end{cases}$$

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In this case, we have for  $f \in U^*$ :

$$\begin{aligned} \varphi_{L(t)}(f) &= E \left[ \exp \left( i \int_{\mathcal{O}} f(x) M(t, dx) \right) \right] \\ &= e^{-c_\alpha t \|f\|^\alpha}. \end{aligned}$$

for a constant  $c_\alpha > 0$ , i.e.  $L$  is the canonical  $\alpha$ -stable cylindrical process.

## From cylindrical to random measure

**Definition.** A cylindrical Lévy process  $(L(t) : t \geq 0)$  in  $L^p(\mathcal{O}, \zeta)$  for some  $p \geq 1$  is called **independently scattered** if

$$L(t)\mathbb{1}_{A_1}, \dots, L(t)\mathbb{1}_{A_n} \text{ are independent}$$

for any disjoint sets  $A_1, \dots, A_n \in \mathfrak{B}_b(\mathcal{O})$  and  $n \in \mathbb{N}$ .

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**Theorem** An independently scattered cylindrical Lévy process  $(L(t) : t \geq 0)$  in  $L^p(\mathcal{O}, \zeta)$  for some  $p \geq 1$  defines by

$$M(t, A) := L(t)\mathbb{1}_A \quad \text{for all } A \in \mathfrak{B}_b(\mathcal{O}),$$

a Lévy-valued random measure  $M$  on  $(\mathcal{O}, \mathfrak{B}_b(\mathcal{O}))$ .

# Counterexample

Let  $(h_k)$  be independent, identically distributed real-valued Lévy processes with characteristics  $(0, 0, \varrho)$ . Every cylindrical Lévy process  $L$  of the form

$$L(t)u^* := \sum_{k=1}^{\infty} \langle e_k, u^* \rangle h_k(t)$$

for an ONB  $(e_k)$  of  $U$  is **not independently scattered**.