

Inverse and implicit and function theorems

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1 Theorems

1.1 Inverse Function Theorem

The Inverse Function Theorem gives conditions under which a function has a differentiable local inverse. If $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuously differentiable and its Jacobian at a point is invertible, then f is locally invertible, and the inverse is also differentiable with derivative given by the inverse of the Jacobian.

Theorem 1 (Inverse Function Theorem). *Let $U \subset \mathbb{R}^n$ be an open set and*

$$f : U \rightarrow \mathbb{R}^n$$

a continuously differentiable function. Suppose that

$$a \in U \quad \text{and} \quad \det(Df(a)) \neq 0.$$

Then there exist open neighborhoods $V \subset U$ of a and $W \subset \mathbb{R}^n$ of $f(a)$ such that the restricted function

$$f : V \rightarrow W$$

is bijective (one-to-one and onto), and its inverse

$$f^{-1} : W \rightarrow V$$

is continuously differentiable. Moreover, for every $y \in W$, the derivative of f^{-1} is given by

$$(f^{-1})'(y) = [Df(f^{-1}(y))]^{-1}.$$

Here:

- **Domain of f :** the open set $U \subset \mathbb{R}^n$;

- **Range of f :** a subset of $\mathbb{R}^n$;
- Locally, f is invertible with a continuously differentiable inverse.

Remark 1. Theorem 6.7.1 in the book presents the Inverse Function Theorem in a general setting, where X and Y are complete normed vector spaces (Banach spaces), and $\mathbf{F} : U \rightarrow Y$ is a differentiable map defined on an open subset $U \subset X$. The theorem states that if the derivative $\mathbf{F}'(a)$ is invertible at a point $a \in U$ and $\mathbf{F}'$ is continuous at a , then $\mathbf{F}$ has a local inverse $\mathbf{G}$ near a , and this inverse is differentiable at $\mathbf{F}(a)$ with derivative

$$\mathbf{G}'(b) = \mathbf{F}'(a)^{-1}, \quad \text{where } b = \mathbf{F}(a).$$

This simplifies nicely when $X = Y = \mathbb{R}^n$ (as in the theorem above). In that case, $\mathbf{F}$ becomes a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, and $\mathbf{F}'(a)$ is just the Jacobian matrix $Df(a)$. The condition that $\mathbf{F}'(a)$ is invertible means this Jacobian has non-zero determinant. The conclusion is familiar: f is locally invertible around a , and its inverse f^{-1} is differentiable with

$$(f^{-1})'(f(a)) = (Df(a))^{-1}.$$

1.2 Implicit Function Theorem

The Implicit Function Theorem gives conditions under which an equation

$$F(x, y) = 0$$

can be solved for y as a function of x . If F is continuously differentiable and the derivative with respect to y is invertible at a point where $F(x, y) = 0$, then locally we can write $y = g(x)$, where g is differentiable.

Theorem 2 (Implicit Function Theorem). *Let $U \subset \mathbb{R}^n \times \mathbb{R}^m$ be an open set and*

$$F : U \rightarrow \mathbb{R}^m, \quad F = F(x, y), \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R}^m,$$

a continuously differentiable function. Suppose that

$$F(a, b) = 0 \quad \text{for some } (a, b) \in U,$$

and that the partial derivative with respect to the last m variables,

$$D_y F(a, b) = \frac{\partial F}{\partial y}(a, b),$$

is invertible. Then there exist open neighborhoods $A \subset \mathbb{R}^n$ of a and $B \subset \mathbb{R}^m$ of b , and a unique continuously differentiable function

$$g : A \rightarrow B$$

such that

$$g(a) = b \quad \text{and} \quad F(x, g(x)) = 0 \quad \text{for all } x \in A.$$

In this context:

- **Domain of F :** an open subset $U \subset \mathbb{R}^{n+m}$;
- **Range of F :** a subset of $\mathbb{R}^m$;
- The function g defines locally the solution in the y -coordinates as a function of the x -coordinates.

Remark 2. In Theorem 6.8.1 from the book, the Implicit Function Theorem is stated in the more general setting of normed vector spaces. Here, X , Y , and Z are complete normed spaces, and $\mathbf{F} : U \rightarrow Z$ is a continuously differentiable function defined on an open subset $U \subset X \times Y$. The notation $\frac{\partial \mathbf{F}}{\partial y}(x, y)$ represents the partial derivative of $\mathbf{F}$ with respect to the second variable y , which is a bounded linear operator from Y to Z . The condition that this derivative is a bijection ensures we can locally solve $\mathbf{F}(x, y) = 0$ for y in terms of x .

Working on $\mathbb{R}^n$ (as in the theorem above), this abstract formulation simplifies. We take

$$X = \mathbb{R}^n, \quad Y = \mathbb{R}^m, \quad Z = \mathbb{R}^m,$$

so that $\mathbf{F}$ becomes a function $F(x, y)$ from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^m$. The condition that $\frac{\partial F}{\partial y}$ is a bijection just means that the Jacobian matrix $\frac{\partial F}{\partial y}$ with respect to y is invertible. The conclusion is the same: there exists a differentiable function

$$y = g(x), \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R}^m$$

such that

$$F(x, g(x)) = 0,$$

and its derivative is given by the familiar formula:

$$\underbrace{g'(x)}_{m \times n \text{ matrix}} = - \underbrace{\left(\frac{\partial F}{\partial y}(x, g(x)) \right)^{-1}}_{m \times m \text{ matrix}} \underbrace{\frac{\partial F}{\partial x}(x, g(x))}_{m \times n \text{ matrix}}.$$

2 Examples

2.1 Example using the Inverse Function Theorem

Example 1. Let

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = x^3 + x,$$

be defined on the domain $U = \mathbb{R}$ with range contained in $\mathbb{R}$. We want to show that there exists a local inverse function

$$f^{-1},$$

in a neighborhood of $x = 0$. In other words, the function f is locally invertible around $x = 0$.

Solution.

First, compute the derivative of $f(x)$:

$$f'(x) = 3x^2 + 1.$$

At $x = 0$, we obtain

$$f'(0) = 3 \cdot 0^2 + 1 = 1 \neq 0.$$

Since f is continuously differentiable on $\mathbb{R}$ and $f'(0) \neq 0$, Theorem 1 implies that there exist open intervals $V \subset \mathbb{R}$ (around $x = 0$) and $W \subset \mathbb{R}$ (around $f(0) = 0$) such that $f : V \rightarrow W$ is a bijection and its inverse

$$f^{-1} : W \rightarrow V$$

is continuously differentiable. Moreover,

$$(f^{-1})'(f(0)) = \frac{1}{f'(0)} = \frac{1}{1} = 1.$$

□

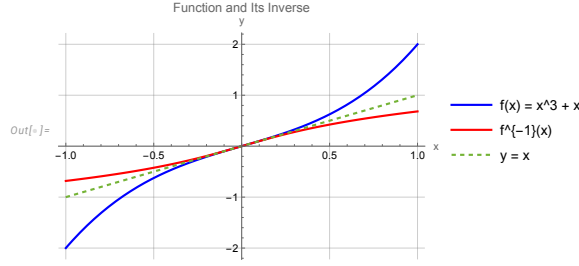


Figure 1: Example 1. Plot of $f(x) = x^3 + x$ and its local inverse near $x = 0$.

2.2 Example using the Implicit Function Theorem

Example 2. Consider the function

$$F : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad F(x, y) = x^2 + y^2 - 1,$$

defined on the domain $U = \mathbb{R}^2$. The range of F is $\mathbb{R}$. Let us examine the point

$$(x_0, y_0) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right),$$

which satisfies

$$F\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 - 1 = \frac{1}{4} + \frac{3}{4} - 1 = 0.$$

We wish to use the Implicit Function Theorem (Theorem 2) to show that near (x_0, y_0) the equation $F(x, y) = 0$ defines y as a function of x , say

$$y = g(x), \quad F(x, g(x)) = 0,$$

with $g\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2}$.

Solution.

First, compute the partial derivative of $F(x, y)$ with respect to y :

$$\frac{\partial F}{\partial y}(x, y) = 2y.$$

At (x_0, y_0) we have

$$\frac{\partial F}{\partial y}\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3} \neq 0.$$

Since F is continuously differentiable on $\mathbb{R}^2$, by Theorem 2 there exist open intervals $A \subset \mathbb{R}$ (around $x = \frac{1}{2}$) and $B \subset \mathbb{R}$ (around $y = \frac{\sqrt{3}}{2}$) together with a unique continuously differentiable function

$$g : A \rightarrow B$$

such that

$$F(x, g(x)) = 0 \quad \text{for all } x \in A,$$

and

$$g\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2}.$$

Additional: Differentiating (via the chain rule) the identity $F(x, g(x)) = 0$ with respect to x yields

$$\frac{\partial F}{\partial x}(x, g(x)) + \frac{\partial F}{\partial y}(x, g(x)) g'(x) = 0,$$

so that the derivative of g is given by

$$g'(x) = -\frac{\frac{\partial F}{\partial x}(x, g(x))}{\frac{\partial F}{\partial y}(x, g(x))} = -\frac{2x}{2g(x)} = -\frac{x}{g(x)} = -\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}.$$

(One can check all this against the formula $g(x) = \sqrt{1-x^2}$.)

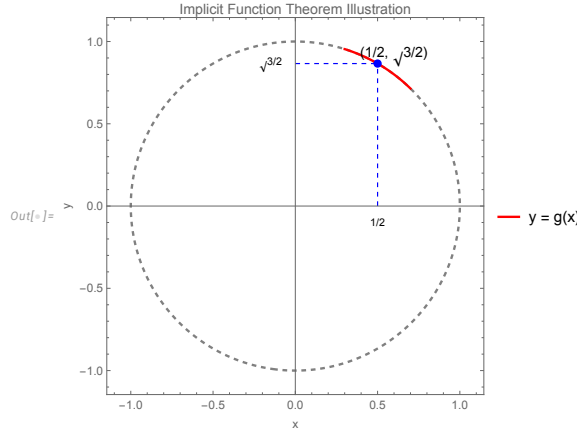


Figure 2: Example 2. Plot of the circle $x^2 + y^2 = 1$ and the branch $y = g(x)$ near $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

□

3 Two exercises

3.1 Based on the resit-exam in August 2016, problem 3

Exercise 1. Let the function $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ be defined by

$$F(x, y, z) = x \sin(xyz^2) + z.$$

- (a) Show that there exists an open neighborhood $U \subset \mathbb{R}^2$ of $(1, \frac{\pi}{2})$ and a function $Z : U \rightarrow \mathbb{R}$ such that

$$F(x, y, Z(x, y)) = 2 \quad \text{and} \quad Z(1, \frac{\pi}{2}) = 1.$$

- (b) Show that Z is differentiable at $(1, \frac{\pi}{2})$, and compute $\frac{\partial Z}{\partial x}(1, \frac{\pi}{2})$.

Solution.

(a) We want to solve $\tilde{F}(x, y, z) := F(x, y, z) - 2 = 0$ near the point $(1, \frac{\pi}{2}, 1)$. First, verify that $\tilde{F}(1, \frac{\pi}{2}, 1) = 0$:

$$\begin{aligned} F(1, \frac{\pi}{2}, 1) &= 1 \cdot \sin(1 \cdot \frac{\pi}{2} \cdot 1^2) + 1 \\ &= \sin(\frac{\pi}{2}) + 1 = 1 + 1 = 2. \end{aligned}$$

Next, compute $\frac{\partial \tilde{F}}{\partial z} = \frac{\partial F}{\partial z}$:

$$\begin{aligned} \frac{\partial F}{\partial z}(x, y, z) &= x \cos(xyz^2) \cdot xy \cdot 2z + 1 \\ &= 2x^2yz \cos(xyz^2) + 1. \end{aligned}$$

At the point $(1, \frac{\pi}{2}, 1)$:

$$\begin{aligned}\frac{\partial F}{\partial z}(1, \frac{\pi}{2}, 1) &= 2 \cdot 1^2 \cdot \frac{\pi}{2} \cdot 1 \cdot \cos(\frac{\pi}{2}) + 1 \\ &= \pi \cdot 0 + 1 = 1 \neq 0.\end{aligned}$$

Since $\tilde{F}$ is continuously differentiable and $\frac{\partial \tilde{F}}{\partial z} \neq 0$ at the point, the *Implicit Function Theorem* applies to $\tilde{F}(x, y, z) = 0$. Therefore, there exists an open neighborhood U of $(1, \frac{\pi}{2})$ and a differentiable function $Z : U \rightarrow \mathbb{R}$ such that

$$F(x, y, Z(x, y)) = 2, \quad \text{with} \quad Z(1, \frac{\pi}{2}) = 1.$$

(b) Differentiate the identity $F(x, y, Z(x, y)) = 2$ with respect to x :

$$\frac{d}{dx}F(x, y, Z(x, y)) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial Z}{\partial x} = 0.$$

Solving for $\frac{\partial Z}{\partial x}$:

$$\frac{\partial Z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}.$$

Now compute each partial derivative at $(1, \frac{\pi}{2}, 1)$. First,

$$\frac{\partial F}{\partial x}(x, y, z) = \sin(xyz^2) + x \cos(xyz^2)yz^2.$$

At $(1, \frac{\pi}{2}, 1)$:

$$\frac{\partial F}{\partial x}(1, \frac{\pi}{2}, 1) = \sin(\frac{\pi}{2}) + 1 \cdot \cos(\frac{\pi}{2}) \cdot \frac{\pi}{2} \cdot 1^2 = 1 + 0 = 1.$$

Previously, we found $\frac{\partial F}{\partial z}(1, \frac{\pi}{2}, 1) = 1$. Therefore

$$\frac{\partial Z}{\partial x}(1, \frac{\pi}{2}) = -\frac{1}{1} = -1.$$

□

3.2 Based on the resit-exam in August 2021, problem 6

Exercise 2. Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by

$$F(x) = \begin{pmatrix} x_1^3(x_2 + 1) - x_2^2 \\ x_1 - x_2^5 \end{pmatrix}, \quad \text{where } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

(a) Compute the Jacobian matrix $DF(x)$.

(b) Let

$$b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Note that $F(b) = c$. Show that there exists a neighborhood U of b and a neighborhood V of c such that the restriction $F : U \rightarrow V$ is bijective. Compute the derivative $(F^{-1})'(c)$.

Solution.

(a) Let

$$F_1(x) = x_1^3(x_2 + 1) - x_2^2, \quad F_2(x) = x_1 - x_2^5.$$

Then the Jacobian matrix is

$$DF(x) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{pmatrix}.$$

Compute the entries:

$$\begin{aligned} \frac{\partial F_1}{\partial x_1} &= 3x_1^2(x_2 + 1), & \frac{\partial F_1}{\partial x_2} &= x_1^3 - 2x_2, \\ \frac{\partial F_2}{\partial x_1} &= 1, & \frac{\partial F_2}{\partial x_2} &= -5x_2^4. \end{aligned}$$

Thus,

$$DF(x) = \begin{pmatrix} 3x_1^2(x_2 + 1) & x_1^3 - 2x_2 \\ 1 & -5x_2^4 \end{pmatrix}.$$

(b) We are given

$$b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and } F(b) = c.$$

We apply the Inverse Function Theorem. Compute the Jacobian matrix at b :

$$DF(b) = DF(1, 1) = \begin{pmatrix} 3(1)^2(1+1) & (1)^3 - 2(1) \\ 1 & -5(1)^4 \end{pmatrix} = \begin{pmatrix} 6 & -1 \\ 1 & -5 \end{pmatrix}.$$

Since

$$\det(DF(b)) = 6 \cdot (-5) - (-1) \cdot 1 = -30 + 1 = -29 \neq 0,$$

the matrix is invertible.

By the *Inverse Function Theorem*, there exist neighborhoods U of b and V of c such that $F : U \rightarrow V$ is bijective and differentiable, and F^{-1} is differentiable.

We compute:

$$(F^{-1})'(c) = (DF(b))^{-1}.$$

Using the formula for the inverse of a 2×2 matrix¹:

$$(DF(b))^{-1} = \frac{1}{-29} \begin{pmatrix} -5 & 1 \\ -1 & 6 \end{pmatrix} = \frac{1}{29} \begin{pmatrix} 5 & -1 \\ 1 & -6 \end{pmatrix}.$$

$$\text{Therefore, } (F^{-1})'(c) = \frac{1}{29} \begin{pmatrix} 5 & -1 \\ 1 & -6 \end{pmatrix}.$$

□

¹ $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$