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Cohomology of Monoid Algebras

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INTRODUCTION

The fundamental problem of deformation theory is finding a classification space for all deformations of some object. If the object is a commutative k-algebra A, we consider diagrams

$$\begin{array}{ccc} S & \longrightarrow & \tilde{A} \\ \downarrow & & \downarrow \\ k & \longrightarrow & A \end{array},$$

where S is a local artinian k-algebra, $\tilde{A}$ is flat over S, and $\tilde{A} \otimes_S k \simeq A$. The set of such diagrams, modulo natural isomorphisms, is called the set of deformations of A to the algebra S, denoted $\mathrm{Def}_A(S)$. $\mathrm{Def}_A(-)$ may be regarded as a functor from the category of local artinian k-algebras into the category of sets, and a natural question is whether the functor is representable. This was studied by Schlessinger in [Sch1] giving in general the answer to be "no," but under some conditions there is a surjection $\mathrm{Mor}(H, -) \to \mathrm{Def}_A(-)$, where the k-algebra H is the "hull" of the deformation functor, parametrizing all deformations.

In [L1] Laudal gives an explicit description of H in terms of algebra cohomology

$$A^p(k, A; A) = \varprojlim_{(k\text{-free}/A)^\circ}{}^{(p)} \mathrm{Der}_k(-, A) \qquad p \geq 0.$$

The vector space $A^1(k, A; A)$ is the dual of the tangent space of H and the products $A^1(k, A; A) \otimes A^1(k, A; A) \to A^2(k, A; A)$ give the local equations for H. Giving a formula for H in terms of A is of course, in general, impossible. If we restrict to monoid algebras, i.e., algebras where the multiplicative structure is determined by the additive structure of some monoid, the problem is much closer to its solution. In Section 2, we develop a

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theory for calculating algebra cohomology of monoid algebras, using only the combinatorial properties of the monoid. Thus the problem of finding the hull of the deformation functor is at least reducd to a combinatorial problem.

Several authors (among others [L–S; R; Sch2; Ch]) have tried to calculate the algebra cohomology groups for k-algebras of various kind. For two-dimensional torus embeddings over a field of characteristic zero, formulas for the lowest degree groups are known. But they have been computed by quite different methods, using also the additive structure of the monoid algebras, which are shown in Section 1 to be unneccessary.

The sections of the paper are organized as follows: In Section 1, we define various cohomology theories: algebra cohomology in the category of k-algebras as well as in the category of monoids, and Harrison cohomology, introduced by Harrison in [H], which is defined on the complex level. For monoid algebras $k[\Lambda]$ all theories coincide.

In Section 2, we show that the cohomology of Section 1 can be determined by using only the combinatorial properties of the monoid. We introduce the notion of monoid-like sets and Harrison cohomology of such. That leads to the isomorphism of Theorem 2.10,

$$\mathrm{Harr}^{p,\lambda}(\Lambda, k[\Lambda]) \simeq HA^p(\Lambda_+ - \Lambda(-\lambda), \Lambda_+; k), \qquad p \geqslant 0,$$

relating graded algebra cohomology with the purly combinatorial cohomology of monoid-like sets, and as a corollary the vanishing of algebra cohomology in positive degrees (Corollary 2.12).

In Section 3, we apply the theory to the case of two-dimensional torus embeddings. For those algebras we have a complete description of the monoid and we are able, because of certain vanishing properties, to give the algebra cohomology in terms of Harrison cohomology of finite subsets of the monoid.

In Section 4, we define products in algebra cohomology of monoid-like sets and relate them to the products of algebra cohomology. In the special case $A^1(k, A; A) \otimes A^1(k, A; A) \to A^2(k, A; A)$, giving the local equations for the formal moduli space H, we show (Theorem 4.8) that the product can be computed by an ordinary cup-product in stead of the much more complex composition product.

Throughout the paper we work over a ground field k. The monoids are always assumed to be finitely generated, i.e., we put a noetherian hypothesis on the monoid algebras. When the module of values in different cohomology groups are the ground field k, it may be omitted, without causing any confusion.

1. Cohomology of Monoids in Different Categories

Let k-**alg** be the category of commutative k-algebras, and let k-**free** be the full subcategory of free commutative k-algebras, i.e., polynomial rings over k. Let A be an object of k-**alg** and denote by (k-**free**/A) the category where the objects are the morphisms $\Gamma \xrightarrow{\delta} A$ of the polynomial ring Γ into A, and the morphisms are commutative diagrams

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\Phi} & \Gamma_2 \\ & {\scriptstyle\delta_1}\searrow \quad \swarrow{\scriptstyle\delta_2} & \\ & A & \end{array}.$$

If M is any A-module we define the functor

$$\mathrm{Der}_k(-, M) : (k-\mathbf{free}/A) \to \mathbf{Ab}$$

by the equality $\mathrm{Der}_k(\delta, M) = \mathrm{Der}_k(\Gamma, M)$, where M is considered as a Γ-module via $\delta : \Gamma \to A$.

Definition 1.1. With the notation as above we define the algebra cohomology groups of A with values in the A-module M by

$$A^p(k, A; M) = \varprojlim_{(k\text{-free}/A)^o}{}^{(p)} \mathrm{Der}_k(-, M) \qquad p \geqslant 0.$$

Let F_0 be a free k-algebra and $F_0 \xrightarrow{\delta} A$ a surjection. Consider the semi-simplicial k-algebra

$$F_\bullet : A \xleftarrow{\delta} F_0 \leftleftarrows F_1 \overset{\longleftarrow}{\leftleftarrows} F_2 \overset{\leftleftarrows}{\leftleftarrows} \cdots,$$

where $F_p = F_0 \times_A \cdots \times_A F_0$, the fibered product of $p+1$ copies of F_0.

Proposition 1.2. *There exists a Leray spectral sequence with*

$$E_2^{p,q} = H^p A^q(k, F_\bullet; M)$$

converging to the cohomology $A^\bullet(k, A; M)$, where $F_\bullet$ is as above and M is considered as an $F_\bullet$-module via δ.

Proof. Follows immediately from the Leray spectral sequence of [L1, (2.1.3)]. ∎

Let **mon** be the category of commutative monoids, and let **free mon** be the full subcategory of free monoids, i.e., monoids isomorphic to $\mathbb{Z}_+^n$. Let

Λ be an object of **mon** and consider the category (**free mon**/Λ) where the objects are morphisms $\Gamma \xrightarrow{\delta} \Lambda$, with Γ free. Define

$$\mathrm{Der}(-, M) : (\textbf{free mon}/\Lambda) \to \textbf{Ab},$$

where M is a $k[\Lambda]$-module (and therefore a Λ-module) and

$$\begin{aligned}\mathrm{Der}(\delta, M) &= \mathrm{Der}(\Gamma, M)\\ &= \{D : \Gamma \to M \mid D(\gamma_1 + \gamma_2) = \gamma_1 D(\gamma_2) + \gamma_2 D(\gamma_1)\}.\end{aligned}$$

M is considered as a Γ-module via δ.

DEFINITION 1.3. With the notation as above we define the algebra cohomology goups of Λ with values in the Λ-module M by

$$A^p(\Lambda; M) = \varprojlim_{(\text{free mon}/\Lambda)^o}{}^{(p)} \mathrm{Der}(-, M) \qquad p \geq 0.$$

Let $\Gamma \xrightarrow{\delta} \Lambda$ be surjective, with Γ a free abelian monoid. Consider the semi-simplicial monoid

$$\Gamma_\bullet : \Lambda \xleftarrow{\delta} \Gamma_0 = \Gamma \leftleftarrows \Gamma_1 = \Gamma \times_\Lambda \Gamma \overset{\leftarrow}{\leftleftarrows} \Gamma_2 = \times_\Lambda \Gamma \times_\Lambda \Gamma \overset{\leftleftarrows}{\leftleftarrows} \cdots.$$

PROPOSITION 1.4. *There exists a Leray spectral sequence with*

$$E_2^{p,q} = H^p A^q(\Gamma_\bullet; M)$$

converging to the cohomology $A^\bullet(\Lambda; M)$.

Proof. Follows from [L1, (2.1.3)]. ∎

Now observe that for an abelian monoid Λ and the associated monoid k-algebra $k[\Lambda]$ we have the equality

$$\mathrm{Der}(\Lambda; M) = \mathrm{Der}_k(k[\Lambda]; M).$$

Consider the semi-simplicial monoid $\Gamma_\bullet$ defined as above. It is easily seen that the associated k-algebra $k[\Gamma_\bullet]$ is a semi-simplicial object of the category k-**alg**. Moreover, there is a natural morphism of semi-simplicial k-algebras

$$\Phi : k[\Gamma_\bullet] \to k[\Gamma]_\bullet,$$

where for each $n \geq 1$

$$\Phi_n : k[\Gamma \times_\Lambda \Gamma \times_\Lambda \cdots \times_\Lambda \Gamma] \to k[\Gamma] \times_{k[\Lambda]} k[\Gamma] \times_{k[\Lambda]} \cdots \times_{k[\Lambda]} k[\Gamma]$$

is defined by $t^{(\gamma_1, \ldots, \gamma_n)} \mapsto (t^{\gamma_1}, \ldots, t^{\gamma_n})$, and Φ_0 is the identity.

Suppose Φ_n is surjective. Then it follows from the proof of (2.1.3) in [L1] that we may replace $k[\Gamma]_{\bullet}$ by $k[\Gamma_{\bullet}]$ in Proposition 1.2. The reason is that if we replace the complex $C_{\bullet} = \{(\mathbb{Z}, k[\Gamma]_p), \delta_p\}_{p \geqslant 0}$ of Lemma 2.1.1 in [L1] by $C'_{\bullet} = \{C(\mathbb{Z}, k[\Gamma_p]), \delta_p\}_{p \geqslant 0}$ we still have a resolution of $C(\mathbb{Z}, k[\Lambda])$ in the category $\mathbf{Ab}^{(k\text{-free})^{0}}$. So we must show that k_n is surjective.

$k[\Gamma]$ has a natural Λ-grading so we may work on the Λ-homogenous parts. Pick $\lambda \in \Lambda$. Since Λ is finitely generated there is a finite number of monomials $t^{\gamma_1}, \ldots, t^{\gamma_m}$ such that $\delta(t^{\gamma_i}) = t^{\lambda}$.

Let ω be a homogenous element of $k[\Gamma]_p$ of degree λ. We can write

$$\omega = \left(\sum_{i=1}^{m} a_i t^{\gamma_i}, \sum_{i=1}^{m} b_i t^{\gamma_i}, \ldots, \sum_{i=1}^{m} c_i t^{\gamma_i}, \sum_{i=1}^{m} d_i t^{\gamma_i} \right),$$

where a_i, b_i, c_i, and d_i are elements of the ground field k. We shall prove the surjectivity of Φ by constructing an element $W \in k[\Gamma_p]$ such that $\Phi(W) = \omega$.

It is easily seen that the element

$$\begin{aligned} W = {} & \sum_{i=1}^{m-1} a_i t^{(\gamma_i, \gamma_1, \ldots, \gamma_1)} + \left(b_1 - \sum_{i=1}^{m-1} a_i \right) t^{(\gamma_m, \gamma_1, \ldots, \gamma_1)} \\ & + \sum_{i=2}^{m-1} b_i t^{(\gamma_m, \gamma_i, \gamma_1, \ldots, \gamma_1)} + \cdots + \left(d_1 - \sum_{i=1}^{m-1} c_i \right) t^{(\gamma_m, \ldots, \gamma_m, \gamma_1)} \\ & + \sum_{i=2}^{m-1} d_i t^{(\gamma_m, \ldots, \gamma_m, \gamma_i)} + d_m t^{(\gamma_m, \ldots, \gamma_m)} \end{aligned}$$

has this property.

We have thus proved the following:

THEOREM 1.5. *With the notation as above we have an isomorphism of cohomology groups*

$$A^p(\Lambda; M) = A^p(k, k[\Lambda]; M) \qquad p \geqslant 0.$$

The given definitions of algebra cohomology are rather nice-looking, but no good when it comes to computations. For this purpose we rather introduce the Harrison complex and prove that Harrison cohomology, i.e., cohomology of the Harrison complex, and algebra cohomology coincide.

Let Γ be a commutative monoid and M a $k[\Gamma]$-module. We use the notation

$$\begin{aligned} & \mathrm{Mor}(\Gamma^n, M) \\ & \quad = \{\phi : \Gamma^n \to M \mid \phi(\gamma_1, \ldots, \gamma_n) = 0 \text{ if } \exists i \text{ such that } \gamma_i = 0\}. \end{aligned}$$

Note that for technical reasons we use the normalized complex, i.e., the complex consisting of morphisms vanishing on all tuples containing a zero.

The differential

$$d: \mathrm{Mor}(\Gamma^n, M) \to \mathrm{Mor}(\Gamma^{n+1}, M) \qquad n \geqslant 1$$

is defined by

$$\begin{aligned} d\phi(\gamma_1, ..., \gamma_{n+1}) = \gamma_1 \phi(\gamma_2, ..., \gamma_{n+1}) \\ + \sum_{i=1}^{n} (-1)^i \phi(\gamma_1, ..., \gamma_i + \gamma_{i+1}, ..., \gamma_{n+1}) \\ + (-1)^{n+1} \phi(\gamma_1, ..., \gamma_n)\, \gamma_{n+1}. \end{aligned}$$

LEMMA 1.6. $d^2 = 0$.

Proof. See, e.g., [C–E]. ∎

The definition of Harrison cohomology, firstly introduced in [H], is based on the notion of shufflings of the set $\{1, ..., n\}$. A permutation $\pi \in \Sigma_n$ is a shuffling if it satisfies the shuffle property; These exists $1 \leqslant i \leqslant n-1$ such that $\pi(j) < \pi(k)$ whenever $1 \leqslant j < k \leqslant i$ or $i+1 \leqslant j < k \leqslant n$. Such a permutation is called a $(i, n-i)$-shuffling. Let $s_{i,n-i} = \sum (\mathrm{sgn}\, \pi)\pi$, where π ranges over all $(i, n-i)$-shufflings of $\{1, ..., n\}$, and $\mathrm{sgn}\, \pi = \pm 1$ is the sign of the permutation π. We may view $s_{i,n-i}$ as an element of the group-ring $\mathbb{Q}[\Sigma_n]$. Now $\pi \in \Sigma_n$ acts on $\phi \in \mathrm{Mor}(\Lambda^n, M)$ from the right by

$$(\phi \cdot \pi)(\lambda_1, ..., \lambda_n) = \phi(\lambda_{\pi^{-1}(1)}, ..., \lambda_{\pi^{-1}(n)})$$

and we extend this action to all $\mathbb{Q}[\Sigma_n]$ by linearity. We define, as a generalization of the definition of Harrison in [H],

$$\begin{aligned} &\mathrm{Mor}_S(\Lambda^n, M) \\ &\quad = \{\phi \in \mathrm{Mor}(\Lambda^n, M) \mid \phi \circ s_{i,n-i} = 0\ \forall 1 \leqslant i \leqslant n-1\}. \end{aligned}$$

LEMMA 1.7. *The two-sided ideal of* $\mathbb{Q}[\Sigma_n]$ *generated by all* $(i, n-i)$-*shufflings equals the principal ideal generated by* s_n, *where* $s_n = \sum_{i=1}^{n-1} s_{i,n-i}$ *is the sum of all* $(i, n-i)$-*shufflings.*

Proof. Barr shows in [B] that there exists an element $e_n \in \mathbb{Q}[\Sigma_n]$ with the following properties:

(i) e_n is a polynomial in s_n without a constant term;

(ii) $e_n \cdot s_{i,n-i} = s_{i,n-i}$ for all $1 \leqslant i \leqslant n-1$.

Thus the ideal generated by all $(i, n-i)$-shufflings equals the principal ideal generated by s_n, equals the principal ideal generated by e_n. ▮

In [B], Barr also shows that the family $\{s_n\}_{n\geq 1}$ commutes with the differential d in the sense that $s_{n-1}d = ds_n$. Combining this information we may define the Harrison complex as

$$\mathrm{Mor}_S(\Lambda^n, M) = \{\phi \in \mathrm{Mor}(\Lambda^n, M) \mid \phi \circ s_n = 0\}$$

with differential d, respecting the shuffle submodules.

DEFINITION 1.8. Harrison cohomology of Λ with values in the $k[\Lambda]$-module M is defined by

$$\mathrm{Harr}^n(\Lambda, M) = H^n(\mathrm{Mor}_S(\Lambda^\bullet, M), d) \qquad n \geq 1.$$

The important property of the Harrison cohomology is its vanishing on free monoids. This result is obtained in two steps. The first proposition shows that Harrison cohomology commutes with (co-)products, and the second gives Harrison cohomology of the additive monoid $\mathbb{Z}_+$.

PROPOSITION 1.9. *Suppose* $\mathrm{char}\, k = 0$ *and let* M *be a* $k[\Gamma_1]$*- and a* $k[\Gamma_2]$*-module, and therefore a* $k[\Gamma_1 \times \Gamma_2]$*-module. There is an isomorphism of cohomology groups*

$$\mathrm{Harr}^n(\Gamma_1 \times \Gamma_2, M) \simeq \mathrm{Harr}^n(\Gamma_1, M) \oplus \mathrm{Harr}^n(\Gamma_2, M)$$

for all $n \geq 1$.

Proof. See Proposition 3.3 of [S]. ▮

PROPOSITION 1.10. *Let* M *be any* $k[\mathbb{Z}_+]$*-module. Then*

$$\mathrm{Harr}^n(\mathbb{Z}_+, M) = 0 \qquad n \geq 2.$$

Proof. See, e.g., Proposition 3.1 of [B] or Proposition 3.4 of [S]. ▮

The two propositions make up the following important result.

THEOREM 1.11. *If* $\Gamma \simeq \mathbb{Z}_+^r$ *is a free abelian monoid and* M *is any* $k[\Gamma]$*-module, then*

$$\mathrm{Harr}^n(\Gamma, M) = 0 \qquad n \geq 2.$$

We end this section by proving the coincidence of Harrison cohomology and algebra cohomology of a monoid.

We can consider, for M a $k[\Lambda]$-module,

$$\mathrm{Mor}_S(-^{\bullet}, M) : (\textbf{free mon}/\Lambda) \to \textbf{complex of ab.gr.}$$

as a contravariant functor from the category of free monoids over Λ into the category of complexes of abelian groups. If we let $C^{\bullet}((\textbf{free mon}/\Lambda)^{o}, -)$ be the resolving complex for the functor

$$\varprojlim_{(\textbf{free mon}/\Lambda)^{o}} -$$

we get the double complex

$$K^{\bullet\bullet} = C^{\bullet}((\textbf{free mon}/\Lambda)^{o}, \mathrm{Mor}_S(-^{\bullet}, M))$$

and the two spectral sequences

$$'E_2^{p,q} = \varprojlim_{(\textbf{free mon}/\Lambda)^{o}}{}^{(p)} H^q(\mathrm{Mor}_S(-^{\bullet}, M))$$

$$''E_2^{p,q} = H^q \varprojlim_{(\textbf{free mon}/\Lambda)^{o}}{}^{(p)} \mathrm{Mor}_S(-^{\bullet}, M)$$

both converging to the cohomology of the double complex $K^{\bullet\bullet}$. We know that for a free monoid Γ

$$H^q(\mathrm{Mor}_S(\Gamma^{\bullet}, M)) = \mathrm{Harr}^q(\Gamma, M) = 0 \qquad q \geqslant 2.$$

For $q = 1$, we get

$$\begin{aligned} \mathrm{Harr}^1(\Gamma, M) &= \ker\{\mathrm{Mor}_S(\Gamma, M) \to \mathrm{Mor}_S(\Gamma^2, M)\} \\ &= \{\phi \in \mathrm{Mor}(\Gamma, M) \mid \phi(\gamma_1 + \gamma_2) = \gamma_1\phi(\gamma_2) + \phi(\gamma_1)\gamma_2\} \\ &= \mathrm{Der}(\Gamma, M), \end{aligned}$$

and for the first spectral sequence, using the definition of algebra cohomology,

$$'E_2^{p,q} = \begin{cases} 0 & \text{if } q \neq 1 \\ A^p(\Lambda, M) & \text{if } q = 1. \end{cases}$$

To calculate the other sequence we need a lemma.

LEMMA 1.12. *With the above assumptions we have*

$$\varprojlim_{(\textbf{free mon}/\Lambda)^{o}} \mathrm{Mor}_S(-^{\bullet}, M) = \mathrm{Mor}_S(\Lambda^{\bullet}, M)$$

and

$$\varprojlim_{(\text{free nom}/\Lambda)^p}{}^{(p)} \ \mathrm{Mor}_S(-^q, M) = 0 \qquad \textit{for} \quad q \geqslant 1 \textit{ and } p \geqslant 1.$$

Proof. For each $q \geqslant 1$ and every $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_p) \in \Lambda^p$, we define a morphism

$$E(\boldsymbol{\lambda}) : \Gamma(p) \to \Lambda$$

given by $E(\boldsymbol{\lambda})(i) = \lambda_i$, where $\Gamma(p)$ is the free monoid on the set $\{1, ..., p\}$. We consider $E(\boldsymbol{\lambda})$ as an object of the category (**free mon**/Λ). If $\alpha : \Gamma \to \Lambda$ is another object in the same category, we have

$$\mathrm{Mor}(E(\boldsymbol{\lambda}), \alpha) = \{(\gamma_1, ..., \gamma_p) \in \Gamma^p \mid \alpha(\gamma_i) = \lambda_i, i = 1, ..., p\}$$

and, in fact,

$$\bigcup_{\boldsymbol{\lambda} \in \Lambda^p} \mathrm{Mor}(E(\boldsymbol{\lambda}), \alpha) = \Gamma^p.$$

We may construct a $\prod$-flabby object (see [L1]) in the category $(k[\Lambda]\text{-}\mathbf{mod})^{(\mathbf{free\ mon}/\Lambda)^p}$, defined by the objects $E(\boldsymbol{\lambda})$ and the $k[\Lambda]$-module M;

$$\alpha \mapsto \prod_{\bigcup_{\boldsymbol{\lambda} \in \Lambda^p} \mathrm{Mor}(E(\boldsymbol{\lambda}), \alpha)} M = \prod_{\Gamma^p} M = \mathrm{Mor}(\Gamma^p, M).$$

Thus by [L1] we have

$$\varprojlim_{(\text{free mon}/\Lambda)^p}{}^{(q)} \ \mathrm{Mor}(-^{\bullet}, M) = 0, \qquad q \geqslant 1,$$

and for $q = 0$,

$$\varprojlim_{(\text{free mon}/\Lambda)^p} \ \mathrm{Mor}(-^p, M) = \mathrm{Mor}(\Lambda^p, M).$$

The resolving complex for

$$\varprojlim_{(\text{fre mon}/\Lambda)^p}{}^{(p)}$$

is exact on functors. But $\mathrm{Mor}_S(-^{\bullet}, M) \subsetneq \mathrm{Mor}(-^{\bullet}, M)$ is a split injection and the lemma follows. ∎

Thus we have proved the following:

THEOREM 1.13. *With the notation as above, we have an isomorphism*

$$A^p(\Lambda, M) \simeq H^{p+1}(\mathrm{Mor}_S(\Lambda^{\bullet}, M), d) \qquad p \geqslant 0.$$

Combining this with Definition 1.7, we obtain

COROLLARY 1.14.

$$A^p(\Lambda, M) \simeq \mathrm{Harr}^{p+1}(\Lambda, M) \qquad p \geqslant 0.$$

2. COHOMOLOGY OF MONOID-LIKE SETS

Let Λ be a cancellative monoid with no non-trivial subgroups. Λ has the structure of an ordered set given by

$$\lambda_1 \leqslant \lambda_2 \qquad \text{if} \quad \exists \mu \in \Lambda \text{ such that } \lambda_1 + \mu = \lambda_2$$

whenever $\lambda_1, \lambda_2 \in \Lambda$. Let $L \subset \Lambda$ be some sub-ordered set.

DEFINITION 2.1. $L \subset \Lambda$ is said to be a monoid-like ordered set if for all relations $\lambda_1 \leqslant \lambda_2$ in L there exists $\mu \in L$ such that $\lambda_1 + \mu = \lambda_2$ as elements of the monoid.

Define

$$S_n(L) = \{(\lambda_1, ..., \lambda_n) \in L^n \mid w(\boldsymbol{\lambda}) \in L\},$$

where the weight $w(\boldsymbol{\lambda})$ of $(\boldsymbol{\lambda})$ is given by $w(\boldsymbol{\lambda}) = \sum_{i=1}^{n} \lambda_i \in \Lambda$. The permutation group Σ_n acts on $S_n(L)$ by

$$\sigma(\boldsymbol{\lambda}) = \sigma(\lambda_1, ..., \lambda_n) = (\lambda_{\sigma^{-1}(1)}, ..., \lambda_{\sigma^{-1}(n)}).$$

Let $C_n(L)$ be the vector space on $S_n(L)$. The action of Σ_n on $S_n(L)$ induces an action on $C_n(L)$ by permuting the basis elements.

We also define the dual groups; $C^n(L) = \mathrm{Hom}_k(C_n(L), k)$, with the action of Σ_n given by $(\phi \cdot \sigma)(\boldsymbol{\lambda}) = \phi(\sigma(\boldsymbol{\lambda}))$ for $\phi \in C^n(L)$, $\sigma \in \Sigma_n$ and $(\boldsymbol{\lambda}) \in S_n(L)$.

Denote by $Sh_n(L)$ the subspace of $C_n(L)$ generated by all shuffle-products, i.e., the submodule $s_n \cdot C_n(L)$, (see Lemma 1.7) and the dual, $C_S^n(L) = \mathrm{Hom}_k(C_n(L)/Sh_n(L), k)$. The differentials

$$\delta_n : C_n(L) \to C_{n-1}(L)$$

and

$$\delta^n : C^{n-1}(L) \to C^n(L)$$

are defined by their actions on the basis elements of $C_n(L)$, resp. $C^n(L)$;

$$\delta_n(\lambda_1, ..., \lambda_n) = (\lambda_2, ..., \lambda_n) + \sum_{i=1}^{n-1} (-1)^i (\lambda_1, ..., \lambda_i + \lambda_{i+1}, ..., \lambda_n) + (-1)^n (\lambda_1, ..., \lambda_{n-1})$$

resp.

$$\delta^n \xi(\lambda_1, ..., \lambda_n) = \xi(\lambda_2, ..., \lambda_n) + \sum_{i=1}^{n-1} (-1)^i \xi(\lambda_1, ..., \lambda_i + \lambda_{i+1}, ..., \lambda_n) + (-1)^n \xi(\lambda_1, ..., \lambda_{n-1}).$$

In the case $n = 1$ we put $\delta_1(\lambda) = \delta^1 \xi = 0$. It is easily seen that the differentials are dual, i.e., for $\xi \in C^n(L)$, we have

$$\delta^n \xi(\boldsymbol{\lambda}) = \xi(\delta_n(\boldsymbol{\lambda})).$$

LEMMA 2.2. (i) $\delta_{n-1} \delta_n = 0$.

(ii) *For* $x \in C_p(L)$, $y \in C_q(L)$, *we have*

$$\delta_{p+q}(x \bullet y) = \delta_p(x) \bullet y + (-1)^p x \bullet \delta_q(y),$$

where the shuffle-products are extended by linearity.

Proof. A simple computation. ▮

DEFINITION 2.3. The inhomogenous Harrison (co-)homology $HA_n(L)$ (resp. $HA^n(L, k)$) of the ordered set L is the (co-)homology of the complex $C^S_\bullet(L)$ (resp. $C^\bullet_S(L)$) with the inhomogenous differential δ_n (resp. δ^n).

Remark 2.4. There is also a relative version of Harrison (co-)homology. Let $L_0 \subset L \subset \Lambda$ and suppose L_0 is full in L, i.e., if $\gamma \in L$, $\gamma_0 \in L_0$ and $\gamma \geqslant \gamma_0$, then $\gamma \in L_0$. The relative Harison complex is given by

$$C^S_n(L - L_0, L) = C_n(L - L_0, L)/Sh_n(L)$$

$$\text{(resp. } C^n_S(L - L_0, L) = \{\phi \in C^n(L - L_0, L) \mid \phi(Sh_n(L)) = 0\},$$

where

$$C_n(L - L_0, L) \quad = C_n(L)/\{(\lambda_1, ..., \lambda_n) \mid \omega(\boldsymbol{\lambda}) \in L - L_0\}$$

$$\text{(resp. } C^n(L - L_0, L) = \{\phi \in C^n(L) \mid \phi(\boldsymbol{\lambda}) = 0 \text{ for } \omega(\boldsymbol{\lambda}) \in L - L_0\}$$

and relative Harrison (co-)homology is (co-)homology of the relative complex.

PROPOSITION 2.5. *With the notation as above there is a long-exact sequence*

$$0 \to HA^1(L - L_0, L; k) \to HA^1(L; k) \to HA^1(L - L_0; k) \to HA^2(L - L_0, L; k) \to \cdots$$

relating Harrison cohomology of the ordered sets L and $L - L_0$ with relative cohomology.

Proof. The relative complex gives rise to a short-exact sequence of complexes

$$0 \to C_S^{\bullet}(L - L_0, L; k) \to C_S^{\bullet}(L; k) \to C_S^{\bullet}(L - L_0; k) \to 0,$$

where $L - L_0$ is a monoid-like ordered set because L_0 is full in L. ∎

The next theorem is the first main result of this section. It relates the "local" cohomology $HA^p(\hat{\lambda}; k)$ for elements $\lambda \in \Lambda$ to the "global" cohomology $HA^{\bullet}(L; k)$. First, we explain the hat-set notation.

DEFINITION 2.6. Let $L \subset \Lambda$ be a monoid-like set and let $\lambda \in \Lambda$. We define

$$\hat{\lambda} = \{\lambda' \in L \mid \lambda' \leqslant \lambda\}$$

to be the set of all elements less than λ, including λ as the maximal element.

THEOREM 2.7. *There exists a spectral sequence given by*

$$E_2^{p,q} = \varprojlim_{\lambda \in L}{}^{(p)} HA^q(\hat{\lambda}; k)$$

converging to $HA^{\bullet}(L; k)$.

Proof. Using the definition of the complex it is easily seen that for some ordered set L

$$C_S^{\bullet}(L; k) = \varprojlim_{\lambda \in L} C_S^{\bullet}(\hat{-}; k).$$

The shuffle-products are homogeneous and the inhomogeneous differential is well defined on the sets $\hat{\lambda}$.

Now denote by $D^{\bullet}(L, -)$ the resolving complex for the invers limit over L. Let $K^{\bullet\bullet}$ be the double complex

$$K^{\bullet\bullet} = D^{\bullet}(L, C_S^{\bullet}(\hat{-}; k)).$$

We have two associated spectral sequences:

$$
\begin{aligned}
{}'E_2^{p,q} &= H^p H^q \overline{(D^{\cdot}(L, C_S^{\cdot}(\hat{-}; k)))} \\
&= H^p(D^{\cdot}(L, HA^q(\hat{-}; k))) \\
&= \varprojlim_L{}^{(p)} HA^q(\hat{-}; k),
\end{aligned}
$$

and the other one

$$
{}''E_2^{p,q} = H^p H^q \overline{(D^{\cdot}}(L, C_S^{\cdot}(\hat{-}; k))).
$$

If $\lambda_1 \leqslant \lambda_2$, the map

$$
C_S^n(\hat{\lambda}_2; k) \to C_S^n(\hat{\lambda}_1; k)
$$

is surjective and by [L2] we have

$$
\varprojlim_L{}^{(p)} C_S^n(\hat{-}; k) = 0 \qquad p > 0.
$$

For $p = 0$,

$$
H^q \varprojlim_L C_S^{\cdot}(\hat{-}; k) = H^q C_S^{\cdot}(L; k) = HA^q(L; k).
$$

The double complex is situated in the first quadrant and the two spectral sequences have the same abutment. The second sequence degenerates to $HA^q(L; k)$ and the theorem follows. ∎

There is a close relation between Harrison cohomology and the graded parts of algebra cohomology. From now on we consider the case $\Lambda + (-\Lambda) = \mathbb{Z}^r$ and we equip the complex $\mathrm{Mor}_S(\Lambda^q, k[\Lambda])$ with a $\mathbb{Z}^r$-grading. For $\lambda_0 \in \mathbb{Z}^r$, define

$$
\begin{aligned}
&\mathrm{Mor}_S^{\lambda_0}(\Lambda^q, k[\Lambda]) \\
&\quad = \{\phi \in \mathrm{Mor}_S(\Lambda^q, q[\Lambda]) \mid \phi\text{: homogenous of degree } \lambda_0\}.
\end{aligned}
$$

ϕ homogenous means that $\phi(\boldsymbol{\lambda})$ is homogenous and that the element

$$
\deg \phi = \omega(\phi(\boldsymbol{\lambda})) - \omega(\boldsymbol{\lambda}) \in \mathbb{Z}^r
$$

is independent of choice of $(\boldsymbol{\lambda})$. It is the $\mathbb{Z}^r$-degree of ϕ. Remembering the definitions of ch. 1, it is easy to see that the differential respects the grading, and that its degree is 0.

DEFINITION 2.8. The graded Harrison cohomology of Λ with values in $k[\Lambda]$ is defined by

$$\mathrm{Harr}^{n,\lambda}(\Lambda, k[\Lambda]) = H^n \, \mathrm{Mor}_S^{\lambda}(\Lambda^{\bullet}, k[\Lambda])$$

for $n \geqslant 0$, $\lambda \in \mathbb{Z}^r$.

Put as an abbreviation $M_S^n = \mathrm{Mor}_S(\Lambda^n, k[\Lambda])$ and $M_S^{n,\lambda} = \mathrm{Mor}_S^{\lambda}(\Lambda^n, k[\Lambda])$.

PROPOSITION 2.9. (a) *The inclusion* $\coprod_{\lambda \in \mathbb{Z}^r} M_S^{\bullet,\lambda} \to M_S^{\bullet}$ *of complexes induces an inclusion at the cohomology level*;

$$\coprod_{\lambda \in \mathbb{Z}^r} \mathrm{Harr}^{n,\lambda}(\Lambda, k[\Lambda]) \to \mathrm{Harr}^n(\Lambda, k[\Lambda]) \qquad n \geqslant 0.$$

(b) *The inclusion is an isomorphism whenever* $\mathrm{Harr}^n(\Lambda, k[\Lambda])$ *is a* $\mathbb{Z}^r$*-graded group.*

Proof. (a) Let $\phi \in M_S^{n,\lambda}$ be homogenous and suppose $\psi \in M_S^{n-1}$ satisfies $d\psi = \phi$. Let ψ_λ be the λ-graded homogenous part of ψ. Since $\deg d = 0$, we must have $\phi = d\psi_\lambda$.

(b) Suppose $\mathrm{Harr}^n(\Lambda, k[\Lambda])$ is $\mathbb{Z}^r$-graded and let $\psi \in M_S$, $d\psi = 0$. Then we may replace $(\mathrm{mod}\, \mathrm{im}(d))$ ψ by some ψ_0 which is sum of homogenous components. ∎

We are now in position to state the important relation between graded Harrison cohomology groups and Harrison cohomology of ordered sets, as defined in the previous section. This relation shows that we can compute algebra cohomology of a monoid algebra only using purely combinatorial properties of the monoid.

THEOREM 2.10. *With the notation as above there is an isomorphism in cohomology*

$$\mathrm{Harr}^{p,\lambda}(\Lambda, k[\Lambda]) \xrightarrow{\sim} HA^p(\Lambda_+ - \Lambda(-\lambda), \Lambda_+; k), \qquad p \geqslant 0,$$

where $\Lambda(\lambda) = (\lambda + \Lambda) \cap \Lambda_+$, *and* $\Lambda_+ = \Lambda - \{0\}$.

Proof. Put $\phi(\boldsymbol{\lambda}) = \phi_0(\boldsymbol{\lambda})(w(\boldsymbol{\lambda}) + \lambda)$ where $\phi_0(\boldsymbol{\lambda}) \in k$. The map $\phi \mapsto \phi_0$ is easily seen to induce an isomorphism of vector spaces

$$\mathrm{Mor}_S^{\lambda}(\Lambda^{\bullet}, k[\Lambda]) \xrightarrow{\sim} C_S^{\bullet}(\Lambda_+ - \Lambda(-\lambda), \Lambda_+; k).$$

It also takes the graded version of the differential d into the inhomogenous differential δ. Recall that in the definition of $\mathrm{Mor}_S^{\lambda}(\Lambda^{\bullet}, k[\Lambda])$ we agreed that $\phi(\lambda)=0$ if $\exists i$ such that $\lambda_i=0$. That is the reason why we use the positive part Λ_+ instead of Λ. ∎

We end this chapter with a couple of results about graded Harrison cohomology. A close study of the complexes $C^{\bullet}_S(\Lambda_+ - \Lambda(-\lambda), \Lambda_+; k)$ for various $\lambda \in \mathbb{Z}^r$ gives the next proposition.

PROPOSITION 2.11. *Fix some $n \geq 1$. The cohomology groups* $\mathrm{Harr}^{n,\lambda}(\Lambda, k[\Lambda])$ *are equal for all $\lambda \in \Lambda$.*

Proof. If $\lambda \in \Lambda$ we have $\Lambda(-\lambda) = (-\lambda + \Lambda_+) \cap \Lambda_+ = \Lambda_+$ and $\Lambda_+ - \Lambda(-\lambda) = \{0\}$. This means that for all $\lambda \in \Lambda$ the complexes are the same. ∎

COROLLARY 2.12. *Suppose the cohomology group $A^p(\Lambda, k[\Lambda])$ is of finite dimension over k. Then*

$$\mathrm{Harr}^{p+1,\lambda}(\Lambda, k[\Lambda]) = 0$$

for all $\lambda \in \Lambda$.

Proof. $k[\Lambda]$ has infinite dimension over k. ∎

3. Harrison Cohomology of 2-dim Torus Embeddings

The simplest, but still very important family of monoid-algebras are the two-dimensional torus embeddings $k[\Lambda]$ over a field of characteristic zero.

Let $\Lambda \subset \mathbb{Z}^2_+$ be a commutative saturated monoid such that $\Lambda + (-\Lambda) = \mathbb{Z}^2$ and let Λ_+ be the positive part, i.e., $\Lambda_+ = \Lambda - \{0\}$. As earlier we put $\Lambda(\lambda) = (\lambda + \Lambda) \cup \Lambda_+$.

We want to study the "local" and "global" Harrison cohomology of monoids, that is, the cohomology of monoid-like subsets of the monoid as well as the monoid itself. Also the submonoids $\Lambda_+ - \Lambda(\lambda)$ for various $\lambda \in \Lambda + (-\Lambda)$ are of great interest, and we start with a closer look at these objects.

Fix some $\lambda \in \Lambda + (-\Lambda)$ and let γ_1 and γ_2 be the generators for the one dimensional faces of Λ (see, for instance, [K–K–M–S] for details), and define

$$\Gamma_i = \Gamma_i(\Lambda, \lambda) = \{\lambda' \in \Lambda \mid \forall t \in \mathbb{Z}_+, \lambda' + t\gamma_i \notin \Lambda(\lambda)\} \qquad i = 1, 2.$$

Γ_i is an ordered set with the same ordering as Λ. Furthermore, it is easily seen that

$$\Gamma_1 \cup \Gamma_2 = \Lambda_+ - \Lambda(\lambda)$$

and

$$\Gamma_1 \cap \Gamma_2 = L'\lambda,$$

where L' is the "strong" link defined as follows: There is a unique description of λ given by

$$\lambda = a_1\gamma_1 + a_2\gamma_2,$$

where the a_i's are rational numbers. We put

$$L'\lambda = \{\lambda' = b_1\gamma_1 + b_2\gamma_2 \in \Lambda_+ \mid \lambda' \neq 0, 0 \leqslant b_i < a_i, i = 1, 2\}.$$

(Note: For the normal link the definition is $0 \leqslant b_i \leqslant a_i$, but $\lambda' \neq \lambda, 0$.) In the case where $\lambda \in -\Lambda$ we see that $\Gamma_1 = \Gamma_2 = L'\lambda = \Lambda_+ - \Lambda(\lambda) = \varnothing$.

Proposition 3.1. *With the notation as above there is a Mayer–Vietoris sequence*

$$\cdots \varprojlim_{\Lambda_+ - \Lambda(\lambda)}{}^{(p)} HA^q(\hat{-}) \to \varprojlim_{\Gamma_1}{}^{(p)} HA^q(\hat{-}) \times \varprojlim_{\Gamma_2}{}^{(p)} HA^q(\hat{-})$$
$$\to \varprojlim_{L'\lambda}{}^{(p)} HA^q(\hat{-}) \to \varprojlim_{\Lambda_+ - \Lambda(\lambda)}{}^{(p+1)} HA^q(\hat{-}) \to \cdots$$

for all $q \geqslant 1$. (*We have without any possible confusion skipped the ground field in* $HA^q(\hat{-}; k)$.)

Proof. Using the functor $HA^q(\hat{-})$ on the system of inclusions

$$I: \quad \Gamma_1 \leftarrow \Gamma_1 \cap \Gamma_2 \rightarrow \Gamma_2$$

of ordered sets, we get

$$\varprojlim_{\Gamma_1 \cap \Gamma_2} HA^q(\hat{-}) \simeq \varprojlim_{I} \varprojlim_{\Gamma_i} HA^q(\hat{-})$$

and for the higher derivatives, a spectral sequence

$$E_2^{p,q} = \varprojlim_{I}{}^{(p)} \varprojlim_{\Gamma_i}{}^{(r)} HA^q(\hat{-})$$

converging to

$$\varprojlim_{\Gamma_1 \cap \Gamma_2}{}^{(\cdot)} HA^q(\hat{-}).$$

$E_2^{p,q}=0$ for $p\neq 0, 1$ and the spectral sequence degenerates to the two exact sequences

$$0\to E_2^{1,p}\to \varprojlim_{\Gamma_1\cap\Gamma_2}{}^{(p+1)} HA^q(\hat{-})\to E_2^{0,p+1}\to 0$$

and

$$0\to E_2^{0,p}\to \varprojlim_{\Gamma_1}{}^{(p)} HA^q(\hat{-})\times \varprojlim_{\Gamma_2}{}^{(p)} HA^q(\hat{-})$$

$$\to \varprojlim_{\Gamma_1\cap\Gamma_2}{}^{(p)} HA^q(\hat{-})\to E_2^{1,p}\to 0.$$

Putting them together, we obtain the long-exact sequence of the proposition. ∎

To state and prove the next proposition we need some technical notation and definitions.

For $\lambda_0\in\Lambda$ and $L\subset\Lambda$ we let $C_q(L,\lambda_0)$ be the vector space on the set

$$S_q(L,\lambda_0)=\left\{(\eta_1,\dots,\eta_q)\in L^q \,\middle|\, \sum_{j=1}^{q}\eta_j={}^\circ_0\right\}$$

and consider the complex $(C_{\bullet}(L,\lambda_0),\partial)$, where the differential is the homogeneous part of the inhomogeneous differential δ_n of Section 2, denoted ∂_n. We recall the definition

$$\partial_n : C_n(L,\lambda_0)\to C_{n-1}(L,\lambda_0)$$

with

$$\partial_n(\lambda_1,\dots,\lambda_n)=\sum_{i=1}^{n-1}(-1)^i(\lambda_1,\dots,\lambda_i+\lambda_{i+1},\dots,\lambda_n).$$

It is easy to see that $\partial_{n-1}\partial_n=0$. Denote by

$$U_q=\{\lambda_0\in\Lambda \mid H_q(C_{\bullet}(\Lambda,\lambda_0),\partial)=0\}$$

the subset of Λ where the qth homology of the given comple vanish.

PROPOSITION 3.2. *The map $HA^q(\hat{\lambda}_2;k)\to HA^q(\hat{\lambda}_1;k)$, $q\geqslant 1$, induced by the inclusion $\lambda_1\leqslant\lambda_2$ is an isomorphism whenever $\hat{\lambda}_2-\hat{\lambda}_1\subset U_q$,* where U_q is defined as above.

Proof. It is enough to show the proposition for $L\subset L'\subset U_q$, where

$L' - L = \{u\}$ is a one-element set and u is minimally greater than L, that is if $u \geqslant u'$ for some $u' \in U_q$, then $u' \in L$. This is because we have a filtration

$$\hat{\lambda}_1 = L_1 \subset L_2 \subset \cdots \subset L_s = \hat{\lambda}_2,$$

where $L_{i+1} - L_i = \{u_i\}$, u_i is minimally greater than L_i and u_i belongs to U_q.

Consider the exact sequence of complexes

$$0 \to D_S^q(L, u; k) \to C_S^q(L'; k) \to C_S^q(L; k) \to 0,$$

where

$$D_S^q(L, u; k) = \{\xi : C_q(L, u) \to k \mid \xi(x \bullet y) = 0\}$$

and the differential is the dual of the homogeneous differential given above. $x \bullet y$ is the shuffle-product extended by linearity.

We must show that the complex $(D_S^q(L, u; k), \partial)$ is acyclic (except for degree zero). Dualizing the problem we are led to the study of the short-exact sequence of complexes

$$0 \to Sh_{\bullet}(L, u) \to C_{\bullet}(L, u) \to C_{\bullet}^S(L, u) \to 0, \qquad (*)$$

where $Sh_q(L, u)$ is the subspace of $C_q(L, u)$ consisting of all shuffle-products $x \bullet y$ with $x \in C_p(L, u)$ and $y \in C_{q-p}(L, u)$, and

$$C_q^S(L, u) = C_q(L, u)/Sh_q(L, u).$$

The differentials are the homogeneous ∂ defined above.

The question is whether a homotopy for $C_{\bullet}(L, u)$ will induce a homotopy for the subcomplex $Sh_{\bullet}(L, u)$. We are working over a field of characteristic zero, and the following lemma gives an answer.

LEMMA 3.3. *Let g be a homotopy for $C_{\bullet}(L, u)$. The map*

$$h : Sh_{\bullet}(L, u) \to Sh_{\bullet}(L, u)$$

defined by

$$h(x \bullet y) = \tfrac{1}{2}(g(x) \bullet y + (-1)^p x \bullet g(y))$$

for $x \in C_p(L, u)$ and $y \in C_{q-p}(L, u)$ is a homotopy for the subcomplex $Sh_{\bullet}(L, u)$.

Proof.

$$
\begin{aligned}
(dh+hd) &= d(\tfrac{1}{2}(g(x)\bullet y)) + d(\tfrac{1}{2}(-1)^p(x\bullet g(y)))\\
&\quad + h(d(x)\bullet y) + (-1)^p h(x\bullet d(y))\\
&= \tfrac{1}{2}(d(g(x))\bullet y + (-1)^{p-1} g(x)\bullet d(y)\\
&\quad + (-1)^p d(x)\bullet g(y) + (-1)^{2p} x\bullet d(g(y)))\\
&\quad + \tfrac{1}{2}(g(d(x))\bullet y + (-1)^{p-1} d(x)\bullet g(y)\\
&\quad + (-1)^p g(x)\bullet d(y) + (-1)^p x\bullet g(d(y)))\\
&= \tfrac{1}{2}((dg+gd)(x)\bullet y) + ((-1)^{p-1} + (-1)^p)\, g(x)\bullet d(y)\\
&\quad + ((-1)^p + (-1)^{p-1})\, d(x)\bullet g(y) + x\bullet (dg+gd)(y))\\
&= \tfrac{1}{2}(x\bullet y + x\bullet y)\\
&= x\bullet y. \quad \blacksquare
\end{aligned}
$$

The assumption in the proposition ensures that the complex $C_{\bullet}(L, u)$ is acyclic and since working over a field, dualizing of the complexes in (*) will give an inclusion of acyclic cochain complexes

$$D_S^{\bullet}(L, u; k) \subset D^{\bullet}(L, u; k). \quad \blacksquare$$

The assumption in Proposition 3.2 is that the set $\hat{\lambda}_2 - \hat{\lambda}_1$ sits inside U_q. So we must study the set U_q, or better, the complex $C_{\bullet}(L, u)$ with differential ∂.

A basis for $C_q(L, u))$ consists of all tuples $(\eta_1, \ldots, \eta_q) \in L^q$ with $\sum \eta_j = u$. They may also be written as ordered tuples:

$$\eta_1 < \eta_1 + \eta_2 < \cdots < \eta_1 + \eta_2 + \cdots + \eta_{q-1} < \eta_1 + \cdots + \eta_q = u.$$

Observe that all tuples have u as their maximal element. Removing this top element we obtain an ordered tuple of the ordered set $Lk(u) = \{\lambda \in L \mid \lambda < u\}$, the link of $\hat{u}$. It is easy to see that this sets up a bijection between $\bigcup_q S_q(L, u)$ and the simplicial set associated to the link $Lk(u)$.

The homogeneous differential coincide through the bijection with the usual differential of the ordered set, the alternating sum of the face maps. The homology of the simplicial sets $Lk(u)$ are studied in [L–S] and we state, without proof, one result from his paper.

Let α be the right-most minimal element of Λ_+ excepting the generator of the face γ_1, and β the left-most minimal element excepting the generator of the face γ_2. Denote by U the subset of Λ_+ given by

$$U = \{m \cdot \beta + n \cdot \gamma_2, m \cdot \alpha + n \cdot \gamma_1 \mid m, n \in \mathbb{Z}_+, n > 1\}.$$

LEMMA 3.4. *For all $q \geqslant 1$ we have the inclusion $U \subset U_q$.*

Proof. See Lemma 2.5 of [L–S]. ∎

COROLLARY 3.5. *The morphism $HA^q(\lambda_1 < \lambda_2)$ is an isomorphism for all $q \geqslant 1$ whenever $\hat{\lambda}_2 - \hat{\lambda}_1 \subset U$.*

Proof. Combine Proposition 3.2 and Lemma 3.4. ∎

To calculate the graded algebra cohomology groups we have seen that we need information about invers limits of the pre-sheaves $HA^q(\hat{-})$ over various ordered subsets of the monoid Λ. In [L2], it is shown that these calculations can be made over an even smaller subset under the assumption of cofinality.

Let $\Gamma_0 \subset \Gamma$ be ordered sets. If $\gamma \in \Gamma$ we put $B_\gamma(\Gamma_0) = \{\gamma' \in \Gamma_0 \mid \gamma' > \gamma\}$.

DEFINITION 3.6. A subset $\Gamma_0 \subset \Gamma$ is called cofinal if the following two conditions are satisfied:

(i) For every $\gamma \in \Gamma$, we have $B_\gamma(\Gamma_0) \neq \varnothing$.

(ii) For every finite family $\gamma_1, \gamma_2, ..., \gamma_s$ of elements of $B_\gamma(\Gamma_0)$ there exists a $\gamma_0 \in B_\gamma(\Gamma_0)$ such that for every $i = 1, 2, ..., s$ we have either $\gamma_i > \gamma_0$ or $\gamma_i < \gamma_0$.

Using the theory for cofinal subsets it is rather easy to prove the next proposition.

PROPOSITION 3.7. *For the subsets Γ_1 and Γ_2 of the monoid Λ, defined above, we have the equation*

$$\varprojlim_{\Gamma_i}{}^{(p)} HA^q(\hat{-}) = 0 \qquad \text{for} \quad i = 1, 2,\ p \geqslant 1,\ q \geqslant 1.$$

Proof. In the ordered sets $\Gamma_1 \cap U$ and $\Gamma_2 \cap U$ there are cofinal subsets $\{\gamma_1, \gamma_2, ...\}$ isomorphic to $\mathbb{Z}_+$ and such that $\hat{\gamma}_j - \hat{\gamma}_{j-1} \subset U$. Equipe these with the constant presheaf $HA^q(\hat{-})$. It is well known that the higher derivatives vanish. Using the cofinality of the subset and Theorem 1.2.4 of [L2] the proposition follows. ∎

The following is also true:

PROPOSITION 3.8. *For a two-dimensional torus embedding $A = k[\Lambda]$ we have*

$$HA^q(\Lambda_+) = 0 \qquad \text{for} \quad q \geqslant 2.$$

Proof. An immediate consequence of Corollary 2.12 and the fact that A

is an isolated singularity (see, for instance, [P]) and therefore has finite dimensional cohomology groups. ∎

Remark 3.9. If we put $L = \Lambda_+$ and $L_0 = \Lambda_+ - \Lambda(\lambda)$ into Proposition 2.5 and use Proposition 3.8 we get an exact sequence

$$\begin{aligned} 0 &\to HA^1(\Lambda_+ - \Lambda(\lambda), \Lambda_+; k) \\ &\to HA^1(\Lambda_+; k) \to HA^1(\Lambda_+ - \Lambda(\lambda); k) \\ &\to HA^2(\Lambda_+ - \Lambda(\lambda), \Lambda_+; k) \to 0 \end{aligned}$$

and isomorphisms

$$\begin{aligned} 0 &\to HA^n(\Lambda_+ - \Lambda(\lambda); k) \\ &\to HA^{n+1}(\Lambda_+ - \Lambda(\lambda), \Lambda_+; k) \to 0 \qquad n \geq 2. \end{aligned}$$

We come back to the use of this in the next section.

The last theorem of this section gives the algebra cohomology groups of a monoid algebra in terms of Harrison cohomology of finite monoid-like subsets of the monoid.

THEOREM 3.10. *Let Λ be as above and let $\lambda \in \Lambda$. There is a spectral sequence $E_2^{p,q}$ converging to $HA^{\cdot}(\Lambda_+ - \Lambda(\lambda); k)$, where*

$$\begin{aligned} 0 &\to E_2^{0,q} \to \varprojlim_{\Gamma_1} HA^q(\hat{-}) \times \varprojlim_{\Gamma_2} HA^q(\hat{-}) \\ &\to \varprojlim_{\Gamma_1 \cap \Gamma_2} HA^q(\hat{-}) \to E_2^{1,q} \to 0 \end{aligned}$$

and

$$E_2^{p,q} = \varprojlim_{L'\lambda}{}^{(p-1)} HA^q(\hat{-}; k)$$

for $p \geq 2$.

Proof. Using Proposition 3.7, the Mayer–Vietoris sequence of Proposition 3.1 splits up into the following exact sequences:

$$\begin{aligned} 0 &\to \varprojlim_{\Gamma_1 \cup \Gamma_2} HA^q(\hat{-}) \to \varprojlim_{\Gamma_1} HA^q(\hat{-}) \times \varprojlim_{\Gamma_2} HA^q(\hat{-}) \\ &\to \varprojlim_{\Gamma_1 \cap \Gamma_2} HA^q(\hat{-}) \to \varprojlim_{\Gamma_1 \cup \Gamma_2}{}^{(1)} HA^q(\hat{-}) \to 0 \\ 0 &\to \varprojlim_{\Gamma_1 \cap \Gamma_2}{}^{(p)} HA^q(\hat{-}) \to \varprojlim_{\Gamma_1 \cup \Gamma_2}{}^{(p+1)} HA^q(\hat{-}) \to 0 \end{aligned}$$

Combine this with the spectral sequence of Theorem 2.7, and the theorem follows immediately. ∎

4. Products

In this section we define products in Harrison cohomology. First, we do it for algebras, and then in the graded case, for monoid-like sets. In the last part of the section we consider the case of two-dimensional torus embeddings where we concentrate on the cup-products. Throughout this section the $k[\Lambda]$-module M is $k[\Lambda]$ itself and we start by defining the products in the complex $\mathrm{Mor}_S(\Lambda^{\bullet}, k[\Lambda])$.

Definition 4.1. Let $\xi \in \mathrm{Mor}_S(\Lambda^n, k[\Lambda])$ and $\eta \in \mathrm{Mor}_S(\Lambda^m, k[\Lambda])$. Define the composition product $\xi \circ \eta \in \mathrm{Mor}_S(\Lambda^{n+n-1}, k[\Lambda])$ by

$$\xi \circ \eta = \sum_{i=1}^{n} (-1)^{(i-1)(m-1)} \xi \circ_i \eta,$$

where the ith composition product is given formally as

$$\begin{aligned} &\xi \circ_i \eta(\lambda_1, ..., \lambda_{n+m-1}) \\ &\quad = \xi(\lambda_1, ..., \lambda_{i-1}, \eta(\lambda_i, ..., \lambda_{i+n-1}), \lambda_{i+n}, ..., \lambda_{n+m-1}). \end{aligned}$$

By formally, we mean that we extend the composition by linearity.

It is easily seen that the composition product of two cocycles not necessarily again is a cocycle. But if we anti-symmetrize the product the cocyle property survives.

Definition 4.2. The graded commutator product of the complex $\mathrm{Mor}_S(\Lambda^{\bullet}, k[\Lambda])$ is defined by

$$[\xi, \eta] = \xi \circ \eta - (-1)^{(m-1)(n-1)} \eta \circ \xi,$$

where ξ and η are as above.

Lemma 4.3. *Let ξ and η be as in Definition 4.1. Then we have*

$$d[\xi, \eta] = [\xi, d\eta] + (-1)^{(m-1)}[d\xi, \eta].$$

Proof. See Theorem 4 of [G]. ∎

In Lemma 1.7, we define the element s_n, the sum of all $(i, n-i)$-shufflings, and we gave its action on $\mathrm{Mor}(\Lambda^n, M)$. We extend the action to the products.

Lemma 4.4. *Let ξ and η be as above. If $\xi \cdot s_n = \eta \cdot s_m = 0$, then $(\xi \circ \eta) \cdot s_{n+m-1} = 0$ and therefore $[\xi, \eta] \cdot s_{n+m-1} = 0$.*

Proof. See Theorem 5.7 of [G–S]. ▮

A consequence of the Lemmas 4.3 and 4.4 is that the graded commutator product is a product in Harrison cohomology, i.e., it defines a graded product

$$[-,-]:\mathrm{Harr}^n(\Lambda, k[\Lambda])\times \mathrm{Harr}^m(\Lambda, k[\Lambda]) \to \mathrm{Harr}^{n+m-1}(\Lambda, k[\Lambda]). \qquad (*)$$

The grading to which we refer is the $\mathbb{N}$-grading of the cohomology groups.

Remember from Chapter 2 that the semi-group Λ is a concellative monoid, contained in some $\mathbb{Z}^r_+$ and with no non-trivial subgroups. Such monoids have natural structure as ordered sets defined by the monoid. The monoid algebra $k[\Lambda]$, where $\Lambda\subset\mathbb{Z}^r_+$, has a natural $\mathbb{Z}^r$-grading, and the same is true for the Harrison complex $\mathrm{Mor}_S(\Lambda^\bullet, k[\Lambda])$.

We denote by $\mathrm{Mor}^\lambda_S(\Lambda^\bullet, k[\Lambda])$ the subcomplex of $\mathrm{Mor}_S(\Lambda^\bullet, k[\Lambda])$ consisting of all morphisms of degree $\lambda\in\mathbb{Z}^r$. The $\mathbb{N}$-graded product is obviously $\mathbb{N}$–$\mathbb{Z}^r$-bigraded and we may write the $\mathbb{Z}^r$-graded version of (*):

$$[-,-]:\mathrm{Harr}^{n,\lambda_1}(\Lambda, k[\Lambda])\times \mathrm{Harr}^{m,\lambda_2}(\Lambda, k[\Lambda]) \to \mathrm{Harr}^{n+m-1,\lambda_1+\lambda_2}(\Lambda, k[\Lambda]).$$

Remember that Theorem 2.10 gives an isomorphism between $\mathbb{Z}^r$-graded Harrison cohomology of monoid-algebras and Harrison cohomology of monoid-like sets. Using this correspondance we get the following important theorem (where we again have skipped the ground field):

THEOREM 4.5. *Let* $\Lambda\subset\mathbb{Z}^r_+$ *and* $\Lambda(\lambda)=(\lambda+\Lambda)\cap\Lambda_+$. *There is a* $\mathbb{Z}^r$*-graded product in Harrison cohomology of monoid-like sets*

$$[-,-]: HA^n(\Lambda_+-\Lambda(\lambda_1),\Lambda_+)\otimes HA^m(\Lambda_+-\Lambda(\lambda_2),\Lambda_+) \to HA^{n+m-1}(\Lambda_+-\Lambda(\lambda_1+\lambda_2),\Lambda_+)$$

defined on complex level by

$$[\xi,\eta]=\xi\circ\eta-(-1)^{(m-1)(n-1)}\eta\circ\xi,$$

where

$$\xi\circ\eta=\sum_{i=1}^{n}(-1)^{(i-1)(m-1)}\xi\left(\gamma_1,\dots,\gamma_{i-1},-\lambda_1+\sum_{k=0}^{i+n-1}\gamma_k,\gamma_{i+n},\dots,\gamma_{n+m-1}\right)\cdot\eta(\gamma_i,\dots,\gamma_{i+n-1})$$

and $\xi\in C^n_S(\Lambda_+-\Lambda(-\lambda_1),\Lambda_+)$, $\eta\in C^m_S(\Lambda_+-\Lambda(-\lambda_2),\Lambda_+)$.

Proof. Just straightforward, but explicitly use the isomorphism of Theorem 2.10. ▮

We study the product of Theorem 4.5 in the case $n = m = 1$. This corresponds to the cup-product of algebra cohomology and is the main tool in the study of liftings of algebras.

Remember that in the torus case we have certain vanishing theorems and consequently the following relations between relative and non-relative Harrison cohomology (Remark 3.9).

In the lowest degree an exact sequence

$$0 \to HA^1(\Lambda_+ - \Lambda(\lambda), \Lambda_+) \to HA^1(\Lambda_+)$$

$$\to HA^1(\Lambda_+ - \Lambda(\lambda)) \xrightarrow{\beta_1} HA^2(\Lambda_+ - \Lambda(\lambda), \Lambda_+) \to 0$$

and for the higher degrees, $n \geqslant 2$, an isomorphism

$$0 \to HA^n(\Lambda_+ - \Lambda(\lambda)) \xrightarrow{\beta_n} HA^{n+1}(\Lambda_+ - \Lambda(\lambda), \Lambda_+) \to 0$$

(Note: It is of course not necessary that the algebra is a torus embedding of dimension 2. What is required is that the Harrison cohomology $HA^n(\Lambda_+)$ vanish for $n \geqslant 2$. The above sequences exist and the rest of this section work just as well if we more generally take $A = k[\Lambda]$ to be a monoid algebra with this vanishing property.)

We have a diagram of maps

$$\begin{array}{ccc} HA^1(\Lambda_+ - \Lambda(\lambda_1)) \otimes HA^1(\Lambda_+ \Lambda(\lambda_2)) & & \\ \downarrow & & \\ HA^2(\Lambda_+ - \Lambda(\lambda_1), \Lambda_+) \otimes HA^2(\Lambda_+ - \Lambda(\lambda_2), \Lambda_+) & & HA^2(\Lambda_+ - \Lambda(\lambda_1 + \lambda_2)) \\ & & \downarrow \\ & \xrightarrow[{[-,-]}]{} & HA^3(\Lambda_+ - \Lambda(\lambda_1 + \lambda_2), \Lambda_+), \end{array}$$

where the leftmost vertical arrow is the tensorproduct of two β_1's for the values $\lambda = \lambda_1$ and $\lambda = \lambda_2$. The horizontal arrow is the product of Theorem 4.5 and the rightmost vertical arrow is β_2 for another choice of λ.

If we use the fact that the isomorphism β_2 has an invers, we get a map

$$\psi : HA^1(\Lambda_+ - \Lambda(\lambda_1)) \otimes HA^1(\Lambda_+ - \Lambda(\lambda_2)) \to HA^2(\Lambda_+ - \Lambda(\lambda_1 + \lambda_2)).$$

This is a bilinear map which is symmetric and we call it the ($\mathbb{Z}^r$-)graded cohomology product of Λ_+.

Now let us look at the product from another point of view. There is a natural cup-product defined on the complex level. We consider the diagram

$$\begin{array}{ccc} 0 & & \\ \uparrow & & \\ C^1(\Lambda_+ - \Lambda(\lambda_1))\otimes C^1(\Lambda_+ - \Lambda(\Lambda_2)) & & C^2(\Lambda_+ - \Lambda(\lambda_1+\lambda_2)), \\ \uparrow & & \uparrow{\scriptstyle\alpha} \\ C^1(\Lambda_+)\otimes C^1(\Lambda_+) & \xrightarrow[\phi]{\hspace{2cm}} & C^2(\Lambda_+) \end{array}$$

where $\phi(\xi, \eta)(\gamma_1, \gamma_2) = \xi(\gamma_1)\cdot\eta(\gamma_2) + \xi(\gamma_2)\cdot\eta(\gamma_1)$, the symmetric cup-product.

If $\xi \in C^1(\Lambda_+ - \Lambda(\lambda_1))$ and $\eta \in C^1(\Lambda_+ - \Lambda(\lambda_2))$ we make extensions $\tilde{\xi}, \tilde{\eta} \in C^1(\Lambda_+)$ by setting $\tilde{\xi}(\lambda) = \tilde{\eta}(\lambda) = 0$ for $\lambda \notin \Lambda_+ - \Lambda(\lambda_1)$, resp. $\lambda \notin \Lambda_+ - \Lambda(\lambda_2)$.

DEFINITION 4.6. With the notation as above we define

$$\xi \cup \eta = \alpha(\phi(\tilde{\xi}, \tilde{\eta})),$$

where $\alpha : C^2(\Lambda_+) \to C^2(\Lambda_+ - \Lambda(\lambda_1 + \lambda_2))$ is induced by the restriction $\Lambda_+ - \Lambda(\lambda_1 + \lambda_2) \subsetneq \Lambda_+$.

We want the cup-product not only to be a product of complexes, but in fact a product in cohomology. The next lemma gives the details.

LEMMA 4.7. *The cup-product of Definition* 4.6 *is a product of cohomology, i.e., if* $d\xi = d\eta = 0$ *then also* $d(\xi \cup \eta) = 0$.

Proof. Let $\gamma_1, \gamma_2, \gamma_3 \in \Lambda_+$ satisfy $\gamma_1 + \gamma_2 + \gamma_3 \leqslant \lambda_1 + \lambda_2$. Then we have

$$\begin{aligned} d(\xi \cup \eta)&(\gamma_1, \gamma_2, \gamma_3) \\ &= \xi \cup \eta(\gamma_2, \gamma_3) - \xi \cup \eta(\gamma_1 + \gamma_2, \gamma_3) \\ &\quad + \xi \cup \eta(\gamma_1, \gamma_2 + \gamma_3) - \xi \cup \eta(\gamma_1, \gamma_2) \\ &= \tilde{\xi}(\gamma_2)\,\tilde{\eta}(\gamma_3) + \tilde{\xi}(\gamma_3)\,\tilde{\eta}(\gamma_2) - \tilde{\xi}(\gamma_1 + \gamma_2)\,\tilde{\eta}(\gamma_3) - \tilde{\xi}(\gamma_3)\,\tilde{\eta}(\gamma_1 + \gamma_2) \\ &\quad + \tilde{\xi}(\gamma_1)\,\tilde{\eta}(\gamma_2 + \gamma_3) + \tilde{\xi}(\gamma_2 + \gamma_3)\,\tilde{\eta}(\gamma_1) - \tilde{\xi}(\gamma_1)\,\tilde{\eta}(\gamma_2) - \tilde{\xi}(\gamma_2)\,\tilde{\eta}(\gamma_1) \\ &= d\tilde{\xi}(\gamma_1, \gamma_2)\cdot\tilde{\eta}(\gamma_3 + \tilde{\xi}(\gamma_3)\cdot d\tilde{\eta}(\gamma_1, \gamma_2) - \tilde{\xi}(\gamma_1)\cdot d\tilde{\eta}(\gamma_2, \gamma_3) \\ &\quad - d\tilde{\xi}(\gamma_2, \gamma_3)\cdot\tilde{\eta}(\gamma_1). \end{aligned}$$

Now suppose $d\xi = d\eta = 0$. We either have $\gamma_1 \leqslant \lambda_1$ and then necessarily $\gamma_2 + \gamma_3 \leqslant \lambda_2$ or we have $\gamma_1 \nleqslant \lambda_1$. In either case,

$$\tilde{\xi}(\gamma_1) \cdot d\tilde{\eta}(\gamma_2, \gamma_3) = 0,$$

in the first case because $d\tilde{\eta}(\gamma_2, \gamma_3) = d\eta(\gamma_2, \gamma_3) = 0$ and in the second case because $\tilde{\xi}(\gamma_1) = 0$.

The same procedure can be carried out for the relations $\gamma_1 \leqslant \lambda_2$, $\gamma_3 \leqslant \lambda_1$ and $\gamma_3 \leqslant \lambda_2$. ∎

Using the lemma we know that the cup-product $\bigcup$ induces a product in cohomology, which we also denote by $\bigcup$;

$$\bigcup : HA^1(\Lambda_+ - \Lambda(\lambda_1)) \otimes HA^1(\Lambda_+ - \Lambda(\lambda_2)) \to HA^2(\Lambda_+ - \Lambda(\lambda_1 + \lambda_2)).$$

Note the analogy with the product ψ defined above. It is in fact the same product, as proved in the next theorem.

THEOREM 4.8. *The two products ψ and $\bigcup$ of Harrison cohomology coincide, i.e.,*

$$\psi(x, y) = x \cup y,$$

whenever $x \in HA^1(\Lambda_+ - \Lambda(\lambda_1))$ and $y \in HA^1(\Lambda_+ - \Lambda(\lambda_2))$.

Proof. Let $\xi \in C^1(\Lambda_+ - \Lambda(\lambda_1))$ with $d\xi = 0$ represent x and $\eta \in C^1(\Lambda_+ - \Lambda(\lambda_2))$, $d\eta = 0$ represent y. ξ and η are extended to whole Λ_+ by $\tilde{\xi}$, resp. $\tilde{\eta}$, where $\tilde{\xi}(\lambda_0) = \tilde{\eta}(\lambda_0)$ for $\lambda_0 \in \Lambda(\lambda_1)$, resp. $\lambda_0 \in \Lambda(\lambda_2)$.

The product $\psi(\xi, \eta)$ is represented in $C^3(\Lambda_+ - \Lambda(\lambda_1 + \lambda_2), \Lambda_+)$ by $[d\tilde{\xi}, d\tilde{\eta}]$. On the other hand, $x \cup y$ is represented in the same group by $d(\phi(\tilde{\xi}, \tilde{\eta}))$. We must show that the two elements are cohomologous, i.e., are equal modulo boundaries in $C^{\bullet}(\Lambda_+ - \Lambda(\lambda_1 + \lambda_2), \Lambda_+)$.

Let $\gamma_1, \gamma_2, \gamma_3 \in \Lambda_+$. Then we have

$$\begin{aligned}
[d\tilde{\xi}, d\tilde{\eta}](\gamma_1, \gamma_2, \gamma_3) &= (d\tilde{\xi} \circ d\tilde{\eta} + d\tilde{\eta} \circ d\tilde{\xi})(\gamma_1, \gamma_2, \gamma_3) \\
&= d\tilde{\xi}(\gamma_1 + \gamma_2 + \lambda_2, \gamma_3) \cdot d\tilde{\eta}(\gamma_1, \gamma_2) \\
&\quad - d\tilde{\xi}(\gamma_1, \gamma_2 + \gamma_3 + \lambda_2) \cdot d\tilde{\eta}(\gamma_2, \gamma_3) \\
&\quad + d\tilde{\eta}(\gamma_1 + \gamma_2 + \lambda_1, \gamma_3) \cdot d\tilde{\xi}(\gamma_1, \gamma_2) \\
&\quad - d\tilde{\eta}(\gamma_1, \gamma_2 + \gamma_3 + \lambda_1) \cdot d\tilde{\xi}(\gamma_2, \gamma_3) \\
&= d(\phi(\tilde{\xi}, \tilde{\eta}))(\gamma_1, \gamma_2, \gamma_3) \\
&\quad - d(d\tilde{\xi} \cdot \tilde{\eta})(\gamma_1, \gamma_2, \gamma_3) - d(d\tilde{\eta} \cdot \tilde{\xi})(\gamma_1, \gamma_2, \gamma_3),
\end{aligned}$$

where $d\tilde{\xi} \cdot \tilde{\eta}(\gamma_1, \gamma_2) = d\tilde{\xi}(\gamma_1, \gamma_2) \cdot \tilde{\eta}(\gamma_1 + \gamma_2 + \lambda_1)$.

Thus $d\tilde{\xi} \in C^2(\Lambda_+ - \Lambda(\lambda_1 + \lambda_2), \Lambda_+)$ implies $d\tilde{\xi} \cdot \tilde{\eta} \in C^2(\Lambda_+ - \Lambda(\lambda_1 + \lambda_2), \Lambda_+)$. ∎

The theorem gives a nice way of computing the cup-product via the ordinary cup-product rather than the less elegant composition product.

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