

Simplifyable Modules

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The A -module E is **infinitesimally simplifyable** if there exists a lifting E' of E to $k[\epsilon]$ such that E' is a simple $A \otimes k[\epsilon]$ -module.

Theorem (1)

Let E be an infinitesimally simplifyable A -module and let

$$0 \rightarrow M \xrightarrow{f} E \xrightarrow{g} N \rightarrow 0$$

*be a non-split short exact presentation of E ($\text{Ext}_A^1(N, M) \neq 0$).
Then $\text{Ext}_A^1(M, N) \neq 0$.*

Example 1.

Let $A = k\langle x, y \rangle / (x^2 - 1 + y^2 + \beta[x, y])$. Let

$P : (x, y) \mapsto (a, b)$ be a 1-dimensional module. Then

$\text{Ext}_A^1(P, Q) \neq 0$ if and only if $Q : (x, y) \mapsto -t(a, b) + s(-b, a)$,
where $t = \frac{1-\beta^2}{1+\beta^2}$ and $s = \frac{2\beta}{1+\beta^2}$.

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Thus, for $M(a, b) \neq N(c, d)$, $\text{Ext}_A^1(N, M) \neq 0$ (existence of an
non-trivial extension) and $\text{Ext}_A^1(M, N) \neq 0$ (simplifyability) if
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Notice also that there exists a two-dimensional simple
 A -module if and only if $\beta = 0$.

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Let $A = k\langle x, y \rangle / (x^2 - 1 + [x, y]^2 y)$. Then A has no simple 2-dimensional modules, but for any $P : (x, y) \mapsto (\pm 1, b)$ and $Q : (x, y) \mapsto (\mp 1, c)$ we have $\text{Ext}_A^1(P, Q) \neq 0$ and $\text{Ext}_A^1(P, Q) \neq 0$.

Let E be a non-simple, indecomposable A -module with a finite composition series.

$$E = E_r \rightarrow E_{r-1} \rightarrow \dots \rightarrow E_1 \rightarrow E_0 = 0$$

Define **support** of E :

$$\text{Supp}(E) := \{V_1, \dots, V_r\}$$

where

$$V_i = \ker\{E_i \rightarrow E_{i-1}\} \quad i = 1, 2, \dots, r$$

The A -module E is an **iterated extension** of $\text{Supp}(E)$

Main Results

Presheaves on
ordered setsDeformations
of presheavesSketch of
proof

Definition

Let A be a k -algebra and $\mathcal{V} = \{V_1, \dots, V_r\}$ a finite set of simple A -modules. The directed extension graph $Q\mathcal{V}$ of $\mathcal{V}$ has the modules V_i as its vertices and there is an arrow from V_i to V_j if and only if $\text{Ext}_A^1(V_i, V_j) \neq 0$.

By a **cycle** in the extension graph $Q\mathcal{V}$ we shall mean a **path**, starting and ending at the same vertex. A cycle is said to be **complete** if it contains all the vertices of $Q\mathcal{V}$.

Theorem (2)

Let A be an associative k -algebra and E a Schur-module. Let $\mathcal{V} = \text{Supp}(E) = \{V_1, \dots, V_r\}$ and let $Q\mathcal{V}$ be the extension graph. Suppose E is infinitesimally simplifiable. Then there exists a complete cycle in $Q\mathcal{V}$.

Let Λ be a poset, considered as a category with the inclusion relations as the only morphisms.

Let A be a k -algebra over a field k , and let $\mathcal{F}, \mathcal{G} : \Lambda \rightarrow A\text{-}\mathbf{mod}$ be presheaves of left A -modules on Λ , i.e. contravariant functors on Λ with values in the category of left A -modules.

The morphism category $Mor(\Lambda)$ is a poset with relations $\lambda' \leq \lambda$ as its objects and a relation $(\lambda' \leq \lambda) \leq (\gamma' \leq \gamma)$ if and only if

$$\gamma' \leq \lambda' \leq \lambda \leq \gamma$$

Define a covariant functor

$$Hom_k(\mathcal{F}, \mathcal{G}) : Mor(\Lambda) \longrightarrow A\text{-}\mathbf{bimod}$$

by

$$Hom_k(\mathcal{F}, \mathcal{G})(\lambda' \leq \lambda) = Hom_k(\mathcal{F}(\lambda), \mathcal{G}(\lambda'))$$

Introduce a double complex

$$K^{p,q} = \prod_{\lambda_0 \leq \dots \leq \lambda_p} C^q(A, \operatorname{Hom}_k(\mathcal{F}, \mathcal{G})(\lambda_0 \leq \lambda_p)) \quad p, q \geq 0$$

where $C^q(A, \mathcal{H})$ is the Hochschild cochain complex of A with values in an A -bimodule $\mathcal{H}$.

Differentials

$$d : K^{p,q} \longrightarrow K^{p,q+1} \quad (\text{Hochschild differential})$$

$$\delta : K^{p,q} \longrightarrow K^{p+1,q}$$

$$(\delta\psi)_{\lambda_0 \leq \dots \leq \lambda_{p+1}} = \mathcal{G}(\lambda_0 \leq \lambda_1) \psi_{\lambda_1 \leq \dots \leq \lambda_{p+1}}$$

$$+ \sum_{i=1}^p (-1)^i \psi_{\lambda_0 \leq \dots \leq \hat{\lambda}_i \leq \dots \leq \lambda_{p+1}} + (-1)^{p+1} \psi_{\lambda_0 \leq \dots \leq \lambda_p} \mathcal{F}(\lambda_p \leq \lambda_{p+1})$$

Let $Tot^\bullet(\mathcal{F}, \mathcal{G}) := Tot(K^{\bullet\bullet})$ be the total complex of the double complex $K^{\bullet\bullet}$, with total differential $\partial = d + (-1)^p \delta$.

There is a one-to-one correspondance between isomorphism classes of presheaf extensions of $\mathcal{F}$ by $\mathcal{G}$ and elements of $H^1(Tot^\bullet(\mathcal{F}, \mathcal{G}))$.

Define

$$Ext_\Lambda^n(\mathcal{F}, \mathcal{G}) := H^n(Tot(\mathcal{F}, \mathcal{G})) \quad n \geq 0$$

There exists a spectral sequence

$$E_2^{p,q} = \lim_{\substack{\leftarrow \\ Mor(\Lambda)}}^{(p)} Ext_A^q(\mathcal{F}(t), \mathcal{G}(s)) \Rightarrow Ext_\Lambda^{p+q}(\mathcal{F}, \mathcal{G})$$

where $s, t : Mor(\Lambda) \rightarrow \Lambda$ are the source and target functors.

- A **lifting** of the presheaf $\mathcal{F} : \Lambda \rightarrow A\text{-mod}$ to a pointed local artinian k -algebra (R, π) in $\underline{a}$ is a presheaf $\mathcal{F}_R$ on Λ of left, R -flat $A \otimes_k R^{op}$ -modules, together with a morphism of $A \otimes_k R^{op}$ -presheaves $\eta : \mathcal{F}_R \rightarrow \mathcal{F}$ inducing an isomorphism $\eta \otimes_R k : \mathcal{F}_R \otimes_R k \xrightarrow{\cong} \mathcal{F}$. The right R -structure of $\mathcal{F}$ is given by π . By flatness we have $\mathcal{F}_R \simeq \mathcal{F} \otimes_k R$ as presheaves of right R -modules.

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- Two liftings $\mathcal{F}'_R$ and $\mathcal{F}''_R$ are said to be **equivalent** if there exists an isomorphism

$$\phi : \mathcal{F}'_R \longrightarrow \mathcal{F}''_R$$

of presheaves such that $\eta'' \circ \phi = \eta'$.

Define deformation functor

$$Def_{\mathcal{F}} : \underline{a} \longrightarrow \underline{sets}.$$

by

$$Def_{\mathcal{F}}(R) = \{ \text{liftings of } \mathcal{F} \text{ to } R \} / \sim$$

Theorem

Let $\pi : R \rightarrow k$ be an object in $\underline{a}$ and $\mathcal{F}_R \in \text{Def}_{\mathcal{F}}(R)$ a class of liftings of $\mathcal{F}$ to R . Let $u : S \rightarrow R$ be a surjective morphism in $\underline{a}$, such that $\mathfrak{m}_S \cdot \ker u = 0$. There exists an obstruction

$[\psi] \in \text{Ext}_{\Lambda}^2(\mathcal{F}, \mathcal{F} \otimes \ker u)$ such that $[\psi] = 0$ in

$\text{Ext}_{\Lambda}^2(\mathcal{F}, \mathcal{F} \otimes \ker u)$ is equivalent to the existence of a lifting of $\mathcal{F}_R$ to S . If $[\psi] = 0$, then the set of deformations of $\mathcal{F}_R$ to S is given as a principal homogenous space over $\text{Ext}_{\Lambda}^1(\mathcal{F}, \mathcal{F} \otimes \ker u)$.

Applying $\text{Def}_{\mathcal{F}}$ to $u : R = k[x]/(x^2) \rightarrow k$, with $\ker u \simeq k$, we see that the tangent space of the deformation functor is given by $\text{Ext}_{\Lambda}^1(\mathcal{F}, \mathcal{F})$.

A sketch of proof for Theorem (1):

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A short exact sequence

$$\xi : \quad 0 \rightarrow M \xrightarrow{f} E \xrightarrow{g} N \rightarrow 0$$

of left A -modules can be considered as a presheaf on the poset $\Lambda_2 = \{0 \leq 1 \leq 2\}$ with $\xi(2) = M$, $\xi(1) = E$, $\xi(0) = N$, $\xi(0 \leq 1) = g$ and $\xi(1 \leq 2) = f$, with the additional exactness data, f injective, g surjective and $\text{im}(f) = \text{ker}(g)$.

Definition

Let ξ be an exact sequence of A -modules, considered as a presheaf on the ordered set Λ_2 . Let (R, π) be a local artinian k -algebra. An **exact lifting** ξ_R of ξ to R is a lifting which again is an exact sequence. Put,

$$\text{Def}_{\xi,e}(R) = \{\text{exact liftings of } \xi \text{ to } R\} / \sim .$$

$\text{Def}_{\xi,e}$ is obviously a subfunctor of Def_{ξ} .

Let ξ_R represent an isomorphism class of exact liftings. Let $u : S \rightarrow R$ be a surjective morphism in $\underline{a}$, such that $\mathbf{m}_S \cdot \ker u = 0$ and suppose ξ_S is a lifting of ξ_R . The lifting ξ_S is represented by a cohomology class $[\beta] \in \text{Ext}_{\Lambda_2}^1(\xi, \xi \otimes \ker u)$, defined by a cocycle $\beta = (\beta^{0,1}, \beta^{1,0})$ in the total complex $\text{Tot}^\bullet(\xi, \xi \otimes \ker u)$.

Lemma

The lifting ξ_S of the exact lifting ξ_R , represented by $\beta = (\beta^{1,0}, \beta^{0,1})$ is exact if and only if $\beta^{1,0}$ satisfies the condition

$$\beta_{0 \leq 1}^{1,0} \circ \xi(1 \leq 2) + \xi(0 \leq 1) \circ \beta_{1 \leq 2}^{1,0} = 0$$

For an exact presheaf ξ on Λ_2 we have

$$E_2^{p,q} = \lim_{\leftarrow}^{(p)} Ext_A^q(\xi(t), \xi(s)) = 0, \quad p \geq 2$$
$$Mor(\Lambda_2)$$

The spectral sequence degenerates and we have

$$0 \rightarrow \lim_{\leftarrow}^{(1)} Ext_A^q(\xi(t), \xi(s)) \longrightarrow Ext_{\Lambda_2}^{q+1}(\xi, \xi)$$
$$\longrightarrow \lim_{\leftarrow} Ext_A^{q+1}(\xi(t), \xi(s)) \rightarrow 0$$
$$Mor(\Lambda_2)$$

for $q \geq 0$.

Let the extension E be given by an exact sequence

$$\xi : \quad 0 \rightarrow M \xrightarrow{f} E \xrightarrow{g} N \rightarrow 0$$

of left A -modules, considered as a presheaf on the poset Λ_2 as described above. There is a natural forgetful morphism of deformation functors

$$g : \text{Def}_\xi \longrightarrow \text{Def}_E$$

inducing a k -linear map of tangent spaces

$$g(k[\epsilon]) : \text{Ext}_{\Lambda_2}^1(\xi, \xi) \longrightarrow \text{Ext}_A^1(E, E)$$

The image of the map is denoted

$$\text{Ext}_A^1(E, E)_\xi := \text{im}(g(k[\epsilon])) \subset \text{Ext}_A^1(E, E)$$

Main observation:

Let E be infinitesimally simplifiable. Then for every non-trivial presheaf ξ , presenting E as an extension module, the inclusion

$$\mathrm{Ext}_A^1(E, E)_\xi \subset \mathrm{Ext}_A^1(E, E)$$

is strict.

Let E be a non-simple Schur-module and let

$$\xi : \quad 0 \rightarrow M \xrightarrow{f} E \xrightarrow{g} N \rightarrow 0$$

be a non-split short exact sequence of A -modules, presenting E . Then

$$\mathrm{Ext}_A^1(E, E)_\xi \simeq \ker\{f^* \circ g_* : \mathrm{Ext}_A^1(E, E) \rightarrow \mathrm{Ext}_A^1(M, N)\}$$

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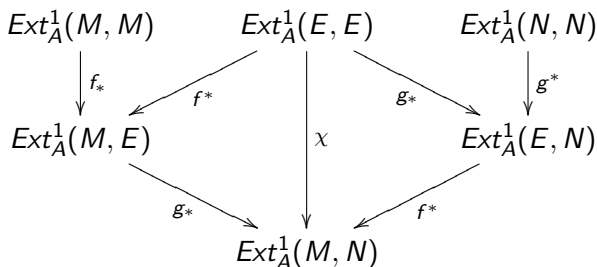
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$$\mathrm{Ext}_A^1(E, E)_\xi \simeq \ker\{f^* \circ g_* : \mathrm{Ext}_A^1(E, E) \rightarrow \mathrm{Ext}_A^1(M, N)\}$$

Why?

$$\begin{array}{ccccc} \text{Ext}_A^1(M, M) & & \text{Ext}_A^1(E, E) & & \text{Ext}_A^1(N, N) \\ \downarrow f_* & \swarrow f^* & \downarrow \chi & \searrow g^* & \downarrow g^* \\ \text{Ext}_A^1(M, E) & & & & \text{Ext}_A^1(E, N) \\ & \searrow g_* & & \swarrow f^* & \\ & \text{Ext}_A^1(M, N) & & & \end{array}$$



An element $\alpha \in \text{Def}_\xi(k[\epsilon]) = \text{Ext}_{\Lambda_2}^1(\xi(t), \xi(s))$ is given by a triple $\alpha = (a, b, c)$, where

$$a \in \text{Ext}_A^1(E, E) \quad b \in \text{Ext}_A^1(M, M) \quad c \in \text{Ext}_A^1(N, N)$$

and such that $f_*(a) = f^*(b)$ and $g^*(c) = g_*(b)$.

Thus $g_*f^*(b) = g_*f^*(a) = 0 \in \text{Ext}_A^1(M, N)$ and the image of $E_2^{0,1}$ lies in the given kernel.

$$\begin{array}{ccccc}
 \text{Ext}_A^1(M, M) & & \text{Ext}_A^1(E, E) & & \text{Ext}_A^1(N, N) \\
 \downarrow f_* & \swarrow f^* & \downarrow \chi & \searrow g_* & \downarrow g^* \\
 \text{Ext}_A^1(M, E) & & & & \text{Ext}_A^1(E, N) \\
 & \searrow g_* & & \swarrow f^* & \\
 & \text{Ext}_A^1(M, N) & & &
 \end{array}$$

On the other hand, for $b \in \ker\{\text{Ext}_A^1(E, E) \rightarrow \text{Ext}_A^1(M, N)\}$, we have $f^*(b) \in \ker g_*$ and $g_*(b) \in \ker f^*$. By exactness there exists $a \in \text{Ext}_A^1(M, M)$ such that $f_*(a) = f^*(b)$ and $c \in \text{Ext}_A^1(N, N)$ such that $g^*(c) = g_*(b)$. This proves that $E_2^{0,1}$ maps surjectively onto the kernel.

Infinitesimally simplifyability implies that

$$\mathrm{Ext}_A^1(E, E)_\xi \subset \mathrm{Ext}_A^1(E, E)$$

is a strict inclusion for all ξ and it follows that $\mathrm{Ext}_A^1(M, N) \neq 0$ is a necessary condition.