CURVES ON VARIETIES

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These are (rough) notes from a 4-part lecture series during the RTG/SCGP "Graduate Workshop on the Birational Complexity of Algebraic Varieties" held at the Simons Center, December 5-9, 2022.

Part 1. Lecture I

1. Introduction

Variety = irreducible, separated, over an algebraically closed field k

Curve = usually a variety of dimension 1 – but sometimes easier to allow several components. But always separated, and reduced, over a field k.

We are interested in studying the implication:

$$\begin{array}{c}\text{Geometry of}\\\text{the set of curves on } X\end{array} \rightsquigarrow \begin{array}{c}\text{Geometry of}\\ X\end{array}$$

e.g., amount of rational curves vs. "birational complexity"

As a *set*, this is not particularly interesting: It's an infinite set with the same cardinality as $\mathbb{R}$.

Issues:

- It seems hopelessly hard to study this set.
- Main issue 1: curves come in families
- Main issue 2: we care mostly about special curves, e.g., rational curves.
  E.g., X is uniruled, rationally connected, .. etc if there are many rational curves. This puts restrictions on the topology on X, e.g., rationality, fundamental group, Hodge numbers,..

Various other ways of studying the curves on X:

- $CH_1(X)$, quotients by equivalence relations, ..
- Hilbert schemes, Chow variety, Morphism schemes, ..
- Cone of curves (Mori cone)
- ..

The same can of course be done for higher dimensional varieties, but: (i) curves are simpler; (ii) close link with divisors.

2. Curves on surfaces

Theorem 2.1 (Cayley, Salmon, 1840s)**.** *There are 27 lines on a cubic surface.*

Two proofs:

- Picard group
- Using incidence correspondence

2.1. **Proof #1 – Picard group.**

We use the fact that a cubic surface is the blow-up of $\mathbb{P}^2$ along 6 general points.

This in turn uses lines, but only to a limited degree (e.g., to prove the existence of two disjoint lines, which is elementary).

The answer:

$$\mathrm{Pic}(X) = \mathbb{Z}^7 = \mathbb{Z}h + \mathbb{Z}e_1 + \ldots + \mathbb{Z}e_6$$

Intersection numbers:

$$h^2 = 1, he_i = 0, e_i^2 = -1$$

Canonical divisor $-K_X = 3h - e_1 - \ldots - e_6$.

This is also the divisor that gives the embedding $X \to \mathbb{P}^3$.

If $L \subset X$ is a line, then $h \cdot L = 1 = -K_X L$ and $L^2 = -1$ (by adjunction formula). So if we write $[L] = ah - b_1 e_1 - \ldots - b_6 e_6$, we must have

- $3a - b_1 - \ldots - b_6 = 1$
- $a^2 - b_1^2 - \ldots - b_6^2 = -1$

This gives that

$$\big((a^2+1)/6\big)^{1/2} = \big((b_1^2 + \ldots + b_6^2)/6\big)^{1/2} \geq (b_1 + \ldots + b_6)/6 = (3a-1)/6$$

or, $0 \geq 3a^2 - 6a - 5 = 3(a-1)^2 - 8$.

This can only happen for $a = 0, 1, 2$. Going through the short list of possibilities, we find, up to permutation,

- e_1 (6 exceptional divisors)
- $h - e_1 - e_2$ (15 strict transforms of lines through pairs of points)
- $2h - e_1 - \ldots - e_5$ (6 conics though 5 out of 6).

Thus we have all the $6 + 15 + 6 = 27$ lines in total.

2.2. **Proof #2 – Incidence correspondnce.**

Let $G = Gr(2,4)$ denote the variety of lines in $\mathbb{P}^3$, and let

$$\mathbb{P}^{19} = \{\sum a_{ijkl} x_0^i x_1^j x_2^k x_3^l\}$$

denote the linear system of cubic surfaces on $\mathbb{P}^3$. Let $U \subset \mathbb{P}^{19}$ denote the open set of smooth cubic surfaces.

There is a *universal cubic hypersurface*

$$\{\sum a_{ijkl} x_0^i x_1^j x_2^k x_3^l\} \subset \mathbb{P}^{19} \times \mathbb{P}^3$$

Let $\mathbf{L}$ denote the following variety:

$$\mathbf{L} = \{([l], f) | l \text{ is a line on } Z(f)\} \subset G \times U$$

One checks that

- $p : \mathbf{L} \to G$ is a projective bundle, so $\mathbf{L}$ is smooth irreducible of dimension 19.
- This means that $q : U \to \mathbb{P}^{19}$ is generically finite.

  In fact, an easy computation using the jacobian criterion shows that q is in fact *etale* of some degree d. (the differential of q has full rank everywhere, so this really looks like a finite covering map).

This means that if we can show that *some* cubic has 27 lines, then *every* smooth cubic has 27 lines (because q is etale, so the same number of preimages everywhere). Then we can just check, for the Fermat cubic

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$$

that the number of lines is 27 (X has $x_0 + x_1 = x_2 + x_3 = 0$, plus lots of cube roots, which give all of the lines..)

2.3. **Conclusion.** The first proof uses hinges on the fact that the group $\mathrm{Pic}(X) = \mathbb{Z}^7$ is easy to study. In general, Chow groups are very complicated.

The second proof uses the notion of a parameter space. It is a sort of "homotopy continuation" argument. This is often used in proofs involving lines or more general rational curves.

Example 2.2. The generic quartic surface contains no lines.

To prove this, we can either use the Picard group or an incidence correspondence.

Exercise 1. *Prove this statement.*

Theorem 2.3 (Noether-Lefschetz theorem)**.** *Picard group of a very general quartic is $\mathbb{Z}$.*

There is a Hodge-theoretic proof of this (Lefschetz pencils).

Indicence correspondence proof: The family of quartics is of dimension

$$\binom{4+3}{3} - 1 = \binom{7}{3} - 1 = 7 * 5 - 1 = 34$$

What about lines? The variety of lines in $\mathbb{P}^3$ is 4-dimensional (=dim Gr(2,4)). For each line, there is a projective space of quartic surfaces containing L.

If $L = Z(x_0, x_1)$, the sequence

$$0 \to \mathcal{O}(-2) \to \mathcal{O}(-1)^2 \to I_L \to 0$$

shows that $H^0(\mathbb{P}^3, I_L(4))$ is of dimension

$$2\binom{3+3}{3} - \binom{2+3}{3} = 20 + 20 - 10 = 30$$

So in all, we obtain a family of dimension 4+30-1=33. □

3. Lines on a fixed hypersurface

Set up:

$X \subset \mathbb{P}^n$ a smooth hypersurface of degree d.

$F = F(X)$=variety of lines on X.

Note that F is a closed subset of the Grassmannian $G = Gr(2, n+1)$. As such, it is the zero set of a section of the vector bundle

$$S^d(U^*)$$

where U is the universal rank 2 sub-bundle on G.

$$0 \to U \to \mathcal{O}^{n+1} \to Q \to 0$$

This all fits into a diagram, similar to that in the first part:

$$\begin{array}{ccc} \mathbf{L} & \xrightarrow{q} & X \\ \downarrow{\scriptstyle p} & & \\ F & & \end{array}$$

Here $\mathbf{L} \subset F \times X$ is given by the universal line

$$\mathbf{L} = \{([l], x) | t \text{ is a line with } l \ni x\}$$

Note that p is a $\mathbb{P}^1$-bundle; $q([l], x) = x$ forgets the line l. Explicitly, $\mathbf{L} = \mathbb{P}(U^*)$.

Lemma 3.1. *The* expected dimension *of F is given by*

$$\dim G - (d+1) = 2(n+1-2) - (d+1) = 2n - d - 3.$$

If X is general, then this indeed counts the right dimension, and F is non-singular by a Bertini theorem for zero sets of vector bundles (see Altman–Kleiman).

If X is non-general, e.g., $x_0^d = 0$, or less drastic, a cone over a hypersurface in $\mathbb{P}^{n-1}$, then the dimension of F can certainly be bigger than the above bound.

We expect a more precise statement for hypersurfaces:

Conjecture 1 (Debarre-de Jong)**.** *F has the correct codimension, provided X is smooth and $n \geq d$.*

Proof. Note that the rank of $S^d(U^*)$ is $d+1$.

Now, the bundle U^*, hence $S^d(U^*)$ is generated by global sections, hence a general section has zero set with expected codimension (Bertini). □

Example 3.2. The generic quartic surface contains no lines, but *some* smooth quartic surfaces do contain lines, e.g., the Fermat. This shows that we need $d \leq n$ in the above theorem.

If F has the correct codimension, we can ask about more information about F. As a first question:

Q: What is the class of F in $CH^{d+1}(G)$?

A: It is given by the top Chern class $c_{d+1}(S^d(U^*))$.

Example 3.3 (Quadrics)**.** In the case $d = 2$, the variety of lines is a smooth of dimension $2n - 5$.

$n = 3$: F=union of two conics in $Gr(2, 4)$. The class is $c_3(S^2U*) = 4c_1 * c_2$.

$n = 4$: $F = \mathbb{P}^3$.

$n \geq 4$: $F = OG(2, n + 1)$ - "orthogonal grassmannian" of 2-planes in an $n + 1$-dimensional vector space.

Here is another way to see the dimension count:

First fix a point p on the quadric. Any line through p lies in the projective tangent space $\mathbf{T}_pQ$.The intersection $\mathbf{T}_pQ \cap Q$ is a cone over a smooth quadric Q' of dimension $n - 3$, so the lines on Q through p are in bijection with the points of Q'. Varying p, we get a space of dimension $(n - 1) + (2n - 3) = 2n - 4$. But we overcounted: for each line, there is a 1-parameter family of points p for which it appears. So the correct count is $2n - 4 - 1 = 2n - 5$.

Example 3.4 (Cubic surfaces again)**.** $\dim G = \dim Gr(2, 4) = 4$, rank $S^3(U^*) = 4$, hence the general cubic surface contains only finitely many lines. A Chern class computation shows that $[F]$ is given by

$$18c_1^2c_2 + 9c_2^2$$

where the c_1, c_2 are the Chern classes of U^*. This has degree $18 + 9 = 27$, giving yet another proof that a cubic surface contains exactly 27 lines, at least for the general cubic.

In higher dimensions there are interesting cases where there is a interplay between the hypersurface and its variety of lines. This has been particularly important in rationality questions; creating both obstructions to irrationality, as well as unirational parameterizations.

Example 3.5 (Cubic threefolds)**.** $\dim G = \dim Gr(2, 5) = 6$; rank $S^3(U^*) = 4$. Thus we expect F to be a surface.

Griffiths–Harris (1969): F is a smooth surface of general type, with invariants $T_F = U$, $K_F = 3C_s$ (C_s = curve of lines meeting a fixed line), $K^2 = 45$.

$$\begin{array}{ccccc} & & 1 & & \\ & 5 & & 5 & \\ 10 & & 25 & & 10 \end{array}$$

The diagram above induces an Abel–Jacobi map

$$H^3(F) \to H^3(\mathbf{L}) = H_3(\mathbf{L}) \to H_3(F) = H^3(F)$$

GH showed that this is an isomorphism. In particular, the Abel–Jacobi map induces an isomorphism between the Albanese variety of F and the intermediate jacobian of X. This fact was crucial in GH's famous proof of the irrationality of a cubic threefold.

Example 3.6 (Cubic fourfolds)**.** $\dim G = \dim Gr(2,5) = 8$; rank $S^3(U^*) = 4$. Thus we expect F to be a fourfold.

Beauville–Donagi (1985): F is a smooth fourfold of hyperKahler type, deformation equivalent to $K3^{[2]}$.

Once again, there is an Abel-Jacobi map

$$H^4(X) \to H^4(\mathbf{L}) = H_6(\mathbf{L}) \to H_6(F) = H^2(F)$$

which is an isomorphism by [BD].

Example 3.7 (Cubic 5-folds)**.** $\dim G = \dim Gr(2,5) = 10$; rank $S^3(U^*) = 4$. Thus we expect F to be a sixfold.

Here less is known. Whenever X is smooth, F is a Fano variety of dimension 6.

Any rational curve on F sweeps out a family of lines on X. Thus X is completely covered by rational surfaces. This is in contrast to cubics of lower dimensions, where F contain fewer rational curves.

Example 3.8. The family of lines contained in a Fermat hypersurface in $\mathbb{P}^n$ has dimension at least $n-3$, which is larger than the expected dimension for $n < d$. Thus the requirement $n \geq d$ is neccessary in the Debarre–de Jong conjecture.

By the way, it would be enough to prove this conjecture for $n = d$, by taking hyperplane sections.

This conjecture has been proved in many cases, e.g., $n \geq 5$ by Debarre, and $d \leq 8$ by Roya Beheshti ($d \leq 6$ also by Landsberg-Tommasi, and Landsberg-Robles). For $n \leq 2d-4$ by Beheshti–Riedl.

Example 3.9 (Quartic threefolds)**.** The variety of lines on a smooth quartic threefold has dimension at least $2*4-4-3=1$. In fact, it is always a curve in char $\neq 2,3$ (Collino).

Notice that there are smooth quartic threefolds in P4 (for instance, the Fermat quartic) which contain cones over curves. So the Fano varieties need not be irreducible.

Quartic fourfold? F is a threefold..

One can also ask about curves of higher degree and genus, e.g., conic curves. For quadrics and cubics this is not so interesting (for cubics: each conic is contained in a plane, and intersecting the cubic hypersurface with this plane gives the conic plus a residual conic).

Example 3.10. The variety of conics on a quartic threefold is a smooth surface F. There is an abel jacobi map

$$H_3(F)$$

Letizia, The Abel-Jacobi mapping for the quartic threefold]

Example 3.11. The variety of twisted cubics H in a cubic fourfold X is also interesting [cf. CH. LEHN, M. LEHN, CH. SORGER, D. VAN STRATEN]. (We assume that X does not contain a plane). Then H is a smooth projective 10-fold. Moreover, there is a $\mathbb{P}^2$-bundle

$$H \to Z$$

where Z is hyperkahler 8-fold of $K3^{[4]}$-type.

Part 2. Lecture II

References: [Mumford: Pathologies in AG II, R. Harthshorne: Deformation theory]

4. Curves in projective space

Any curve C can be embedded in $\mathbb{P}^3$.

(Explain generic projection argument).

Example 4.1. We get many examples by taking curves lying on small degree surfaces. If $Q = \mathbb{P}^1 \times \mathbb{P}^1$ is a quadrics surface, then a curve of bidegree (a, b) has degree $d = a + b$ and genus $(a-1)(b-1)$.

If $C \sim ah - b_1 e_1 - \ldots - b_6 e_6$, then

$$d = 3a - b_1 - \ldots - e_6$$

and

$$g = \binom{a-1}{2} - \sum \binom{b_i}{2}$$

However, given the curve and its genus g, there may be restrictions on the degree d of the embedded curve C.

One such restriction is given by

Proposition 4.2 (Castelnuovo bound)**.** *Let $C \subset \mathbb{P}^3$ be a curve of degree d and genus g, which is not contained in a plane. Then ≥ 3, and*

$$g \leq \begin{cases} 1/4d^2 - d + 1 & \textit{if } d \textit{ is even} \\ 1/4(d^2-1) - d + 1 & \textit{if } d \textit{ is even} \end{cases}$$

Equality is attained for every $d \geq 3$, and any curve for which we have $=$ is contained in a quadric surface.

Many such (d, g) arise on surfaces of degree 2 and 3.

Example 4.3. There is no curve of degree 9 and genus 11 (such a curve would need to lie on a quadric surface, but this turns out not to be possible).

5. Notation

If $Y \subset X$ is a closed subscheme given by ideal sheaf I, the normal sheaf is defined by

$$N_{Y|X} = \mathrm{Hom}_Y(I/I^2, \mathcal{O}_Y).$$

This is a vector bundle of rank = codimension Y if $Y \subset X$ is smooth or lci.

Example 5.1. If Y is a complete intersection of r divisors $D_1, \ldots, D_r$, then

$$N_Y = \bigoplus_{i=1}^{r} \mathcal{O}(D_i)|_Y$$

Example 5.2. Y = twisted cubic in $\mathbb{P}^3$, then $Y \simeq \mathbb{P}^1$, and

$$N_Y \simeq \mathcal{O}(5) \oplus \mathcal{O}(5)$$

If $Y \subset \mathbb{P}^n$ is a projective variety, the *Hilbert polynomial*

Recall that this is the polynomial which for large values agrees with

$$m \mapsto \chi(\mathcal{O}_Y(m))$$

where $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}^n}(1)|_Y$.

6. Hilbert schemes

Here is a basic fact on the existence of Hilbert schemes:

Let Y be a nonsingular closed subvariety of a nonsingular projective variety $X \subset \mathbb{P}^n$ over a field k. Then

(1) There exists a scheme H, called the Hilbert scheme, parametrizing closed subschemes of X with the same Hilbert polynomial P as Y.

There exists a universal subscheme

$$W \subseteq X \times H,$$

flat over H, such that the fibres of W over points $h \in H$ are all closed subschemes of X with the same Hilbert polynomial P.

W is universal in the sense that if T is any other scheme, if $W_0 \subseteq X \times T$ is a closed subscheme, flat over T, all of whose fibres are subschemes of X with the same Hilbert polynomial P, then there exists a unique morphism $\Phi : T \to H$, such that $W_0 = W \times_H T$.

(2) The Zariski tangent space to H at the point $y \in H$ corresponding to Y is given by

$$H^0(Y, N_Y)$$

If

$$H^1(Y, N_Y) = 0,$$

then H is nonsingular at the point y, of dimension equal to

$$h^0(Y, N_Y) = dim_k H^0(Y, N_Y)$$

.

In any case, the dimension of H at y is at least

$$h^0(Y,N) - h^1(Y,N).$$

Hilbert schemes are used all over the place in algebraic geometry. One spectacular application is in Mori's proof of the Hartshorne conjecture, that a variety with an ample tangent bundle must be projective space. This used the lower bound on the dimension of the Hilbert scheme in a very clever way.

Example 6.1. If $Y = D$ is a divisor, then $N_Y = \mathcal{O}_X(D)|_D$. The Hilbert scheme of Y is non-singular: It is the projective space $|D|$ associated to $H^0(X,D)$. Note that it is not always the case that $\dim Hilb = \dim H^0(D, \mathcal{O}_X(D)|_D)$ – there may be h^1.

Example 6.2. For a projective variety X, an interesting special case is the Hilbert scheme $X^{[n]}$ which parameterizes length n 0-dimensional subschemes of X.

Here $P(d) = n$, the constant polynomial.

This is closely related to the symmetric product $X^{(n)} = X^n/S_n$; there is the Hilbert-Chow morphism

$$X^{[n]} \to X^{(n)}$$

which associates a length subscheme $Z \subset X$ to the 0-cycle $[Z]$; that is, the support of Z decorated with multiplicities.

When X is a smooth surface, then $X^{[n]}$ is a smooth variety of dimension $2n$ [Fogarty].

However, the Hilbert schemes $X^{[n]}$ can in general be very complicated, be non-reduced and have many different components, even if X is $\mathbb{A}^3$.

Example 6.3. In the first lecture, we studied Hilbert schemes of lines on hypersurfaces. That is, associated to the Hilbert polynomial

$$P(d) = d + 1.$$

In many cases, these were smooth varieties. I

Example 6.4. The Hilbert scheme H_{2d+1} of conics $C \subset \mathbb{P}^3$ has dimension 8. The normal bundle of a conic C equals

$$N_C = \mathcal{O}_{\mathbb{P}^1}(h) \oplus \mathcal{O}_{\mathbb{P}^1}(2h) = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(4)$$

which has $h^0(N_C) = 3 + 5 = 8$, and no h^1.

Geometrically, H_{2d+1} is a $\mathbb{P}^5$-bundle over $Gr(3,4) = \mathbb{P}^3$.

Example 6.5 (Piene–Schlessinger, 1985)**.** The Hilbert scheme $Hilb_{3d+1}(\mathbb{P}^3)$ consists of two irreducible components H_{12} and H_{15} of dimension 12 and 15. Both of these are smooth, and rational. $H_{12} \cap H_{15}$ is transversal, and = a smooth rational 11-fold.

Here H_{12} = maps $\mathbb{P}^1 \to \mathbb{P}^3$ given by 4 cubic polynomials in s, t.

H_{15} consists of (plane cubic)∪(point).

(draw picture)

Example 6.6. Let $C \subset X$, be a smooth rational curve with normal bundle

$$N_C = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_{n-1})$$

If $a_i \geq 1$, then X is rationally connected. We can see this because C has a very large Hilbert scheme. In fact, C deforms in a family of dimension

$$\dim H^0(N_C) = \sum(a_i + 1) = \deg \det N + (n-1) \geq n$$

In fact, if we fix a point $p \in X$, then $N_C(-p)$ is globally generated. Sections of $H^0(N_C(-p))$ now correspond to the deformations of C fixing the point p. There are at least $n-1$ of these, and they cover X, so X is rationally connected.

Actually, X is rationally connected if and only if there is a smooth rational curve with $a_i \geq 1$ in the normal bundle (the normal bundle is "ample"). I

7. Mumford's example

We saw that the varieties parameterizing lines on hypersurfaces were often quite nice (smooth projective varieties). This is not the case in general, even for $X = \mathbb{P}^3$!

Theorem 7.1 (Mumford, 1962)**.** *There is an irreducible component of the Hilbert scheme of smooth irreducible curves in $\mathbb{P}^3$ of degree 14 and genus 24 that is generically non-reduced.*

Proof. Three steps:

a) We construct an irreducible family U of smooth curves of degree 14 and genus 24, and show that the dimension of the family is 56.

b) For any curve C in the family, we show that $H^0(C, N_C)$ has dimension 57.

Thus $\dim T_{[C]}H = 57 > \dim U$ for $[C] \in H$.

c) We finally show that the family U is not contained in any other irreducible family of curves with the same degree and genus, of dimension > 56.

Step c) shows that the family U is actually an open subset of an irreducible component of the Hilbert scheme, of dimension 56. Hence the scheme U_{red} is integral, and therefore non-singular on some open subset $V \subset U_{red}$. Since U has dimension 56, we get that U is non-reduced.

Here is the construction.

Let X be a non-singular cubic surface in $\mathbb{P}^3$.

We view X as a blow up of $\mathbb{P}^2$ at 6 points in general position

$$X = Bl_6\mathbb{P}^2$$

In the usual basis for $\mathrm{Pic}(X)$, we consider the the divisor class

$$12h - 4e_1 - 4e_2 - 4e_3 - 4e_4 - 4e_5 - 2e_6 = -4K_X + 2e_6$$

This class is very ample, so the linear system contains irreducible nonsingular curves C (Bertini).

The hyperplane divisor H of X equals $H = -K_X$, so

$$\deg C = H \cdot C = -K_X C = 14$$

and the genus is $g = 24$, from the adjunction formula $2g - 2 = C(C + K)$.

Define

$$U = \text{set of all non-singular curves } C \text{ in the above linear system, for all choices of } X \text{ a smooth cubic surface and } L \text{ a line on X.}$$

The cubic surfaces move in an irreducible family of dimension 19,

As they move, the lines on them are permuted transitively, so that U is an irreducible family of curves. Since $Pic(X)$ is discrete, the only algebraic families of curves on X are the linear systems.

This means that

$$\dim U = 19 + \dim |C| = 18 + h^0(X, \mathcal{O}_X(C))$$

Note that since $d > 9$, each of our curves C is contained in a *unique* cubic surface.

Use the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_C(C) \to 0.$$

We have $h^0(\mathcal{O}_X) = 1$, $h^1(\mathcal{O}_X) = 0$.

The linear system $\mathcal{O}_C(C)$ on C has degree $C^2 = 60$. This is greater than $2g - 2$, so there is no h^1. Riemann–Roch gives $h^0(\mathcal{O}_C(C) = 60 + 1 - 24 = 37$.

Hence

$$\dim U = 18 + 37 + 1 = 56$$

Step b). Computation of $h^0(C, N_C)$. There is an exact sequence of normal bundles

$$0 \to N_{C/X} \to N_C \to N_X|_C \to 0.$$

Now $N_{C/X} = \mathcal{O}_C(C)$, which has $h^0 = 37$ and no h^1.

Since X is a cubic surface, $N_X = \mathcal{O}_X(3)$, so we find

$$h^0(N_C) = 37 + h^0(\mathcal{O}_C(3)).$$

By Riemann-Roch,

$$h^0(\mathcal{O}_C(3)) = 3 \cdot 14 + 1 - 24 + h^1(\mathcal{O}_C(3)) = 19 + h^1(\mathcal{O}_C(3)).$$

By Serre duality on C:

$$h^1(C, \mathcal{O}_C(3)) = h^0(\omega_C(-3))$$

By adjunction , on X,

$$\omega_C = \mathcal{O}_C(C + K_X) = \mathcal{O}_C(C - H) = \mathcal{O}_C(3H + 2e_6)$$

. Thus $h^1(\mathcal{O}_C(3)) = h^0(\mathcal{O}_C(2e_6))$. Now we use the sequence

$$0 \to O_X(2e_6 - C) \to \mathcal{O}_X(2e_6) \to \mathcal{O}_C(2e_6) \to 0.$$

Note that $2e_6 - C = -4H$, which has $h^0 = h^1 = 0$ by Kodaira vanishing. Hence

$$h^0(\mathcal{O}_C(2e_6)) = h^0(\mathcal{O}_X(2e_6)) = 1$$

(e_6 is effective, and no multiple moves on X).

Thus

$$h^1(\mathcal{O}_C(3)) = 1, \quad h^0(\mathcal{O}_C(3)) = 20$$

and

$$h^0(N_C) = 37 + 20 = 57.$$

Step c). To show that U is not contained in a larger family of dimension > 56, we proceed by contradiction.

If $D \in W$ was a general curve in this supposed larger family W , then D would be smooth, still of degree 14 and genus 24, but would not be contained in any cubic surface, because our family U contains all those curves that can be obtained by varying X and varying C on X.

From the exact sequence

$$0 \to I_D(4) \to \mathcal{O}_{\mathbb{P}^3}(4) \to \mathcal{O}_D(4) \to 0$$

we find

$$0 \to H^0(I_D(4)) \to H^0(\mathcal{O}_{\mathbb{P}^3}(4)) \to H^0(\mathcal{O}_C(4)) \to ..$$

The dimension of the middle term is 35; that of the term on the right, by Riemann–Roch, 33. Hence $h^0(I_D(4)) \geq 2$.

Take two independent quartic surfaces F_0, F_1 containing D.

Since D is not contained in a cubic (or lesser degree) surface, F_0, F_1 are irreducible and distinct, so their intersection has dimension 1, and provides us with a linkage from D to the residual curve $Q = F_0 \cap F_1 - D$.

Q has degree $16 - 14 = 2$, so Q is a possibly reducible conic in $\mathbb{P}^3$.

Note that each C determines a Q, and vice versa: given F, F_0 containing a conic Q, we can recover C by taking the residual curve $F \cap F_0 - Q$.

We can study the set of triples (F_0, F_1, Q):

We saw that the Hilbert scheme H_{2d+1} of conics in $\mathbb{P}^3$ has dimension 8.

By the exact sequence

$$0 \to \mathcal{O}(1) \to \mathcal{O}(2) \oplus \mathcal{O}(3) \to I_Q(4) \to 0$$

we get that For each conic Q,

$$h^0(Q, I_Q(4)) = 10 + 20 - 4 = 26.$$

The choice of the pencil $< F_0, F_1 >$, is then parameterized by a Grassmannian $Gr(2, 26)$, which has dimension $2 \cdot (24) = 48$.

This means that the dimension of the family of curves D as above is $\leq 8 + 48 = 56$. This completes the proof. $\square$

7.1. Further pathologies.

Theorem 7.2 (Vakil)**.** *The following Hilbert schemes satisfy Murphy's Law for moduli spaces:*

- *The Hilbert scheme of nonsingular surfaces in P5*
- *the Hilbert scheme of surfaces in P4*

8. The morphism scheme

For two varieties X and Y there is a scheme $Mor(X, Y)$ which parameterizes the set of morphisms

$$f : Y \to X$$

This generalizes the Hilbert scheme which parameterizes closed subschemes $Y \to X$.

But it is in fact also a special case of the Hilbert scheme construction, because one can associate a morphism its *graph* Γ_f which is a subvariety of $\Gamma_f \subset X \times X$ isomorphic to Y.

When X is smooth along the image of f, the normal bundle of Γ_f equals f^*T_X.

In this setting, the tangent space to $Mor(Y, X)$ at $f : Y \to X$ is given by

$$T_{[f]}Mor(Y, X) = H^0(Y, f^*T_X).$$

More generally: If X and Y are projective varieties, the tangent space to $Mor(Y, X)$ at $f : Y \to X$ is given by

$$T_{[f]}Mor(Y, X) = Hom(f^*\Omega_X, \mathcal{O}_Y).$$

Theorem 8.1. *Let X and Y be projective varieties and let*

$$f : Y \to X$$

be a morphism such that X is non-singular along $f(Y)$. Then, locally around $[f]$ the scheme $Mor(Y, X)$ can be defined by $h^1(Y, f^*T_X)$ equations in a non-singular variety of dimension $h^0(Y, f^*T_X)$.

In particular, any irreducible component of $Mor(Y, X)$ through $[f]$ has dimension at least

$$h^0(Y, f^*T_X) - h^1(Y, f^*T_X)$$

Example 8.2 (Twisted cubics again)**.** We consider the maps $f : \mathbb{P}^1 \to \mathbb{P}^3$ defining twisted cubics.

We pull back the Euler sequence to $\mathbb{P}^1$:

$$0 \to f^*\mathcal{O} \to f^*\mathcal{O}(1)^4 \to f^*T_{\mathbb{P}^3} \to 0$$

(exact because everything is locally free). And we find

$$h^0(f^*T_{\mathbb{P}^3}) = 4 \cdot 4 - 1 = 15$$

To get the Hilbert scheme of twisted cubic, we factor out by the action of $PGL(2)$, to get

$$\dim H_{12} = 15 - \dim PGL(2) = 15 - 3 = 12,$$

as we expected.

Corollary 8.3. *Let $f : \mathbb{P}^1 \to X$ be a (non-constant) morphism, then*

$$\dim Mor(\mathbb{P}^1, X) = -K_X \cdot C + (1-g)\dim X = -K_X \cdot C + \dim X$$

Part 3. Curves on K3 surfaces

Reference: [D. Huybrechts: Lectures on K3 surfaces].

The Fermat quartic

$$x_0^4 + \ldots + x_3^4 = 0$$

contains 48 lines: $x_0 = \omega x_1$, $x_2 = \omega' x_3$ for $\omega^4 = -1$.

However, most quartic surfaces do not contain lines.

In fact, K3 surfaces typically contain no $\mathbb{P}^1$s at all: If $C \simeq \mathbb{P}^1$, then

$$C^2 = -2,$$

thus this cannot happen whenever $\mathrm{Pic}(X) = \mathbb{Z}$ for instance.

Remark 8.4. The property "does not contain a $\mathbb{P}^1$" is very much a "very general property" in the family of K3 surfaces; non-isotrivial 1-dimensional families $\mathcal{X} \to T$ of K3 surfaces contain infinitely many fibers X_t where there exists a smooth rational curve.

However, it turns out that there is always lots of *singular* rational curves.

Example 8.5. For a quartic surface, there can be singular hyperplane sections which are rational (e.g., quartic curves with three nodes, or perhaps a triple point.)

These curves are of interest in number theory; rational curves give us many rational points.

Here is a sample theorem:

Theorem 8.6 (Bogomolov–Mumford, Mori–Mukai, ..)**.**

a) Any polarized K3 surface (X, H) contains at least one rational curve $C \in |H|$.
b) The very general polarized K3 surface (X, H) contains a nodal integral rational curve $C \in |H|$.
c) For any $n > 0$, the very general (X, H) contains an integral rational curve $C \in |nH|$.

In particular the very general K3 surface contains an infinite (hence dense) set of rational curves.

Part c) was improved recently by Chen–Gounelas–Liedke who proved that in fact any K3 surface contains infinitely many rational curves.

The aim of today's talk is to explain a) in this theorem.

But the same ideas basically also lead to b) and c).

9. Proof of the theorem (i) and (ii)

Here is the main idea for the proof:

(I) Construct a special K3 surface X_0, together with two smooth rational curves C_1, C_2 that intersect transversely in a certain way
 Explicit construction.

(II) Prove that the pair $(X_0, C_1 + C_2)$ deforms to (X, C) where X is a very general K3 surface, and C is an irreducible nodal rational curve.
 Requires some defomation theory.

(III) Deduce that *any* K3 surface contains a rational curve.
 This step is actually easy: Take any K3 X, and a family $\mathcal{X} \to \Delta$ of K3s specializing to X, i.e., $X = \mathcal{X}_0$. By (II), each $\mathcal{X}_t$ has an irreducible rational normal curve C_t.
 By the "Hilbert scheme argument", these fit into a family $\mathscr{C} \to \mathcal{X}$.
 This means that also $X = \mathcal{X}_0$ has some 1-cycle $\mathscr{C}_0$ of the same arithmetic genus p_a (by invariance of Hilbert polynomials). But this means that also X has a rational curve.

9.1. **Step (I).** Many possible constructions possible here: one uses Kummer surfaces:

Take two elliptic curves E_1, E_2 with an isogeny

$$\phi : E_1 \to E_2$$

of degree $2d + 5$, e.g.,

$$\mathbb{C}/(\mathbb{Z}(2d + 5)) + i\mathbb{Z}) \to \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$$

Let $\Gamma = \Gamma_\phi \subset E_1 \times E_2$ be the graph.

Γ contains 4 of the 16 fixed points.

We take $S_0 := E_1 \times E_2/\pm$, and let X_0 denote the blow-up of S_0 along the 16 fixed points.

$C_1 \subset X_0$ – the strict transform of C_1.

$C_2 \subset X_0$ – the strict transform of $E_1 \times 0$.

Then one checks:

- C_1 and C_2 are smooth rational curves; $C_1^2 = C_2^2 = -2$
- They intersect transversally in $d + 2$ points
- $C_1 + C_2$ is nef and big on X_0
 Nef because $(C_1 + C_2)C_i = d > 0$.
 Big because $(C_1 + C_2)^2 = 2d$.
- The class is primitive:
 $C_1 + C_2$ intersects the fibers of $X_0 \to E_1/\pm$ and $X_0 \to E_2/\pm$ in 2 and $2d + 5$ respectively.

9.2. **Step (II).** Take a family of K3 surfaces

$$\pi : \mathcal{X} \to S$$

over a smooth connected base S.

For instance, we can take the entire deformation space $S = \mathrm{Def}(X)$. (Warning: Then some of the fibers can be non-algebraic K3s!). Perhaps a better choice: $S = \mathrm{Def}(X, \mathcal{O}_X(C))$ pameterizing deformations of X together with the line bundle.

Let $C \subset X := \mathcal{X}_0$ be a possibly reducible curve (e,g., $C_1 + C_2$).

$H^1(X, \mathcal{O}(C)) = H^2(X, \mathcal{O}(C)) = 0$, by Kawamata-Viehweg. This implies that any curve in $|C|$ is the specialization of a divisor on $\mathcal{X}$.

More precisely, there is a line bundle $\mathbf{L}$ on $\mathcal{X}$ such that $\mathbf{L}|_X = \mathcal{O}(C)$, and any section of $H^0(X, \mathcal{O}(C))$ comes from a section of $\mathbf{L}$ by restriction.

Deformations of C in the family $\pi : \mathcal{X} \to S$ are parameterized by a projective bundle

$$P := \mathbb{P}(\pi_* \mathbf{L}) \to S$$

(Again, by the vanishing above, the fibers are given by the linear systems $|\mathbf{L}_{\mathcal{X}_t}|$.)

Note that P has dimension

$$\dim P = \dim S + g$$

where $g = h^0(X, \mathcal{O}(C)) - 1 = \frac{C^2}{2} + 1$.

C is not a smooth curve, so it does not define a point on the moduli space of curves of genus g, $\mathscr{M}_g$.

However, it is a *stable curve*, i.e., nodal, and with only finitely many automorphisms.

$\rightsquigarrow$ $[C]$ defines a point in $\overline{\mathscr{M}}_g$, the moduli space of *stable curves.*

This is a compactification of $\mathscr{M}_g$. Here are some facts about this moduli space:

- $\overline{\mathscr{M}}_g$ is a smooth DM stack of dimension $3g - 3$
- It has a coarse moduli space $\overline{M}_g$, which is a projective variety with terminal singularities. The closed points of this variety are indeed in bijection with the set of stable curves up to isomorphism.
- $\overline{\mathscr{M}}_g - \mathscr{M}_g$ is a snc divisor parameterizing singular curves of genus g.
  It decomposes into strata

  $$\partial \overline{\mathcal{M}}_g = \coprod_{0 \le h \le (g/2)} \Delta_h$$

  wherea
  Δ_0 is the set of irreducible curves with exactly one node.
  Δ_h is the set of cuves which are unions of two smooth rational curves meeting at a point.
- The locus $W_\nu \subset \overline{\mathscr{M}}_g$ of stable curves C with $\ge \nu$ nodes is a closed subset of codimension ν in $\overline{\mathscr{M}}_g$.

Let $U \subset P$ denote the open set of curves which are stable. (Stability is an open property)

We get a classifying morphism $\phi : U \to \overline{\mathscr{M}}_g$.

Note that $\phi[C]$ is contained in the locus $W_g \subset \overline{\mathscr{M}}_g$ of curves with at least g nodes. We remarked above that this has codimension g in $\overline{\mathscr{M}}_g$.

Hence we find a subvariety $T \subset U$, which parameterizes curves with at least g nodes, and such that $\dim T \geq \dim U - g = \dim S$.

[DRAW PICTURE]

Now the kicker: these deformations do not deform within $X = \mathcal{X}_0$ (the curves we have constructed are all unions of rational curves). Therefore they must deform out of the special fiber to curves $C_t \subset \mathcal{X}_t$.

This means that $T \to S$ is dominant.

Now take $\pi : \mathcal{X} \to S$ such that $\rho(\mathcal{X}_t) = 1$ for very general t. Since C is primitive, the curves constructed must be irreducible. Hence we get integral nodal rational curves in the very general K3 surface $\mathcal{X}_t$.

Remark 9.1. To get c) in the theorem, we

10. More fun facts

Theorem 10.1. *A K3 surface contains a 1-dimensional family of elliptic curves. In particular, any K3 is dominated by an elliptic surface.*

Theorem 10.2. *Any hyperkahler manifold of dimension $2n$ of type, $K3^{[n]}$, GK, or OG contain rational curves that move in famililes of dimension $2n-2$.*

Theorem 10.3 (Mongardi–O.)**.** *Let X be a projective holomorphic symplectic variety of $K3^{[n]}$-type or of generalized Kummer type. Then the semigroup of effective curve classes is generated (over $\mathbb{Z}$) by classes of rational curves.*

This is related to the *Integral Hodge Conjecture*, to be discussed tomorrow.

Part 4. The Integral Hodge conjecture for 1-cycles

10.1. The "standard Hilbert scheme argument".

The following situation is typical:

Given a family of varieties

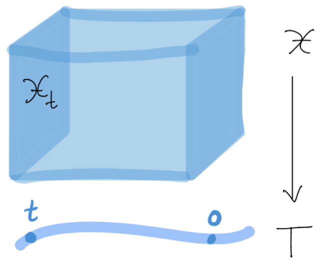
$\rightsquigarrow$ want to deduce something on $\mathcal{X}_t$ for (very) general t from something on $\mathcal{X}_0$.

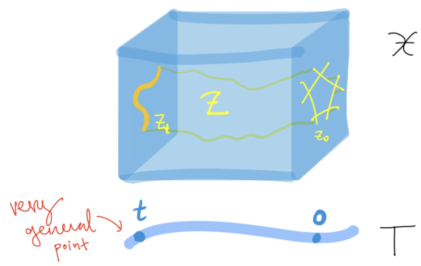
Key point: If t is very general, then *any* subvariety of $\mathcal{X}_t$ extends to all other fibers.

Why is this?

We have the relative Hilbert scheme $H = Hilb(X/T)$ which parameterizes closed subschemes in the fibers of $X \to T$. H has countably many components.

Let $U \to H$ denote the universal family.





$$\begin{array}{ccc} U & \xrightarrow{i} & \mathcal{X} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \pi} \\ H & \longrightarrow & T \end{array}$$

There is a projection map $U \to \mathcal{X} \to T$.

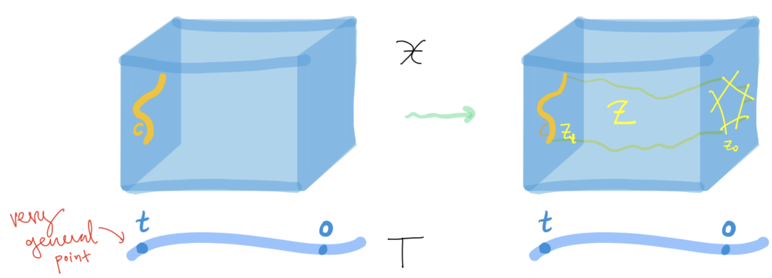
Let H' denote the union of the components of H so that $f \circ i$ is *not* dominant.

Then

$$T' = (f \circ i)(U_{H'} \subsetneq T$$

is a countable union of closed subsets.

If $t \in T - T'$ is any point, then any subscheme in $\mathcal{X}_t$ deforms out of $\mathcal{X}_t$:



Precisely, for $\Gamma \subset \mathcal{X}_t$

$\rightsquigarrow$ $\exists$ component $H_0 \subset H$ plus universal family $Z \to H_0$ such that $f \circ i : Z \to T$ is dominant, and $Z|_t = \Gamma$.

The *specialization map*

$$CH^p(\mathcal{X}_t) \to CH^p(\mathcal{X}_0)$$

is compatible with intersection products.

Why does the Hilbert scheme have only countably many components? If we fix the Hilbert polynomial P, the corresponding Hilbert scheme H_P is a scheme of finite type over k.

Thus since there are only countably many Hilbert polynomials P, the total Hilbert scheme $H = \bigcup_P H_P$ consists of at most countably many components.

The same holds for the morphism schemes $Mor(Y, X)$.

This fact is very important in specialization arguments.

Example 10.4. Let $\mathcal{X} \to T = \mathbb{A}^1_{\mathbb{C}}$ denote a flat family of projective varieties. Let $s \in \mathbb{A}^1_{\mathbb{C}}$ be a very general point. Then if $X = \mathcal{X}_s$ contains a rational curve, then *every* $\mathcal{X}_t$ contains a rational curve, and these curves are deformations of eachother.

Example 10.5. Let $\mathcal{X} \to T = \mathbb{A}^1_{\mathbb{C}}$ denote a flat family of projective varieties. Let $s \in \mathbb{A}^1_{\mathbb{C}}$ be a very general point. Then if $X = \mathcal{X}_s$ is rational, then $\mathcal{X}_t$ is also rational for $t \in T$ very general.

11. The Integral Hodge conjecture

Let $X/\mathbb{C}$ denote a smooth projective variety of dimension n.

We have the Hodge decomposition

$$H^i(X, \mathbb{C}) = \bigoplus_{p+q=i} H^{p,q}(X)$$

We define

$$\begin{array}{rcl} H^{p,p}(X, \mathbb{Z}) & = & \text{classes of } H^{2p}(X, \mathbb{Z}) \text{ which map into} \\ & & H^{p,q}(X) \text{ in the above decomposition} \end{array}$$

In particular, torsion classes lie in this group.

The Integral Hodge conjecture $H^{p,p}(X, \mathbb{Z})$ is generated by algebraic classes.

This means that the cycle class map $CH^p(X) \to H^{p,p}(X, \mathbb{Z})$ is surjective.

The integral Hodge conjecture is not a conjecture: The first counterexamples were given by Atiyah–Hirzebruch in the 60s, using projective approximations to classifying spaces BG.

We will now present a simpler counterexample, due to Kollár .

12. Kollár 's counterexample

Theorem 12.1 (Kollár). *Let $X \subset \mathbb{P}^4$ be a very general hypersurface of degree $p \geq 5$ is a prime number. Then*

$$H^4(X, \mathbb{Z}) = \mathbb{Z}l$$

and any curve has degree divisible by p.

Thus $p^3 \cdot l$ is algebraic, but l is not.

Very general = outside a countable union of Zariski closed subsets.

One can improve this to degree 48, and show that any curve has degree divisible by 2.

Proof. Take $p = 5$ for simplicity.

Let $C \subset X$ be a curve. We want to prove that

$$h \cdot C = 0 \mod 5 \tag{*}$$

Specialization method: If there exists *some* hypersurface $X_0 \subset \mathbb{P}^4$ such that the claim (*) holds, then it holds also on X.

X_0 will be a very singular hypersurface.

Take five general polynomials of degree 5 $f_0, \dots, f_4$.

These define a morphism

$$\phi : \mathbb{P}^3 \to \mathbb{P}^4$$

and the image X_0 is a degree $5^3 = 125$.

We can view ϕ as a sequence of generic projection $\mathbb{P}^3 \to \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-1} \dashrightarrow \cdots \dashrightarrow \mathbb{P}^4$.

By the theory of generic projections, $\phi : \mathbb{P}^3 \to X_0$ is generically injective, and

- 2:1 on a surface $S \subset \mathbb{P}^3$
- 3:1 on a curve $\Gamma \subset \mathbb{P}^3$
- 4:1 on a finite set of points.

By these properties, there is a cycle D on $\mathbb{P}^3$ such that $\phi_*(D) \in \{C, 2C, 3C\}$, and so $\deg \phi_* D \mid 6 \deg C$. On the other hand,

$$\begin{aligned} \deg \phi_* D &= h \cdot \phi_* D \\ &= \phi^* h \cdot D \\ &= \mathcal{O}(5) \cdot D \\ &= 0 \mod 5 \end{aligned}$$

But then also $\deg C = 0 \mod 5$, because 6 and 5 are coprime. ◻

13. More on the IHC

Definition 13.1. We define the *Voisin group* as

$$Z^{2r}(X) = \frac{H^{r,r}(X,\mathbb{Z})}{< \text{algebraic classes} >}$$

The Integral Hodge conjecture (IHC) is particularly interesting for curves. I.e., we ask whether Z^{2n-2} is zero or not.

Proposition 13.2. *$Z^4(X)$ and $Z^{2n-2}(X)$ are birational invariants among smooth projective varieties.*

Proof. Easy to see using Weak factorization; we know how the cohomology groups change after smooth blow ups. ☐

In Kollár 's example $Z^4(X) \to \mathbb{Z}/p \to 0$, so $\mathbb{Z}^4(X) \neq 0$.

But we do not know whether $Z^4(X)$ equals $\mathbb{Z}/p$, $\mathbb{Z}/p^2$, or $\mathbb{Z}/p^3$.

13.1. IHC for 3-folds.

For 3-folds, the IHC holds in many cases:

- $\kappa = -\infty$ (=uniruled). Voisin: IHC holds for uniruled 3-folds.
- $\kappa = 0$. Voisin: IHC holds for 3-folds with $K_X = \mathcal{O}_X$ and $H^2(X, \mathcal{O}_X) = 0$.
- $\kappa = 0$. Totaro: IHC holds for 3-folds with $K_X = \mathcal{O}_X$ and $H^0(X, K_X) \neq 0$. E.g., all abelian 3-folds.

Example 13.3 (Benoist-O.)**.** The IHC can fail on products

$$X = S \times E$$

where S is an Enriques surface and E is an elliptic curve.

These have $\kappa = 0$ and $H^0(K_X) = 0$, so Totaro's result is essentially optimal.

Which class is non-algebraic? By Kunneth,

$$\begin{aligned} H^4(X,\mathbb{Z}) &= H^4(S) \otimes H^0(E) \\ &\quad \oplus H^3(S) \otimes H^1(E) \\ &\quad \oplus H^2(S) \otimes H^2(E) \end{aligned}$$

Example 13.4 (Hassett–Tschinkel, Totaro)**.** The IHC fails for many hypersurfaces of bidegree $(3,4)$ in $\mathbb{P}^1 \times \mathbb{P}^3$.

The proof degrenerates to

$$x_0^4 + tx_1^4 + t^2x_2^4 + t^3x_3^4 = 0$$

Here there is a clever valuation-theoretic argument.

Example 13.5 (Hassett–Tschinkel, Totaro)**.** The IHC fails for certain Enriques surface fibrations

$$X \to \mathbb{P}^1$$

This has some implication for questions in arithmetic flavour.

The proof involves a quite tricky degeneration argument!

13.2. Higher dimensions.

Example 13.6. The IHC can fail for uniruled 4-fold ($\mathbb{P}^1 \times$Kollár 's example).

Example 13.7 (Schreieder)**.** There exists a unirational 4-fold so that $Z^4(X) \neq 0$.

Theorem 13.8 (Mongardi–O.)**.** *Let X be a projective holomorphic symplectic variety of $K3^{[n]}$-type or of generalized Kummer type. Then the semigroup of effective curve classes is generated (over $\mathbb{Z}$) by classes of rational curves.*

Corollary 13.9. *The Integral Hodge conjecture holds for cubic fourfolds.*

Why? Use the Abel-Jacobi isomorphism of Beauville–Donagi

$$H_2(F, \mathbb{Z}) \to H_4(X, \mathbb{Z})$$

(This isomorphism is compatible with the Hodge structures).

13.3. IHC and rationality.

There are in fact deeper connections betwween the IHC and rationality.

Example 13.10. $X \subset \mathbb{P}^4$ cubic threefold.

$$\begin{aligned} J^3(X) &= \text{intermediate jacobian of } X \\ &= H^3(X, \mathbb{C})/F^1H^3(X, \mathbb{C}) + H^3(X, \mathbb{Z}) \end{aligned}$$

Then $(J^3(X), \Theta)$ is a ppav of dimension g.

CG: X is rational $\implies$ $J^3(X)$ is a product of jacobians of curves

$\iff \frac{\Theta^{g-1}}{(g-1)!} = [C_1] + \dots [C_r]$, where C_i are curves on $J^3(X)$.

$\frac{\Theta^{g-1}}{(g-1)!}$ is called the minimal class; it lies in $H^{2g-2}(J^3, \mathbb{Z})$.

If $J = JC$ is a Jacobian of C, then $C \to JC$ has fundamental class $\frac{\Theta^{g-1}}{(g-1)!}$ by the Poincare formula.

Theorem 13.11 (Voisin $\sim$ 2015)**.** *X stably rational $\implies$ $\frac{\Theta^{g-1}}{(g-1)!}$ is algebraic.*

Theorem 13.12 (C. Gabrowski)**.** *$Z^{2g-2}(A)$ for all abelian g-folds A $\iff$ $\frac{\Theta^{g-1}}{(g-1)!}$ is algebraic for all (A, Θ).*

The proof uses the fourier-Mukai transform

13.4. **Open questions.** Notice that Kollár's example constructed above is a hypersurface very high degree, hence is of general type. We can ask:

Conjecture 2. *Let X be a Fano (or more generally, rationally connected) variety of dimension n. Does the integral Hodge conjecture hold for cohomology classes of degree $2n-2$? I.e., is it true that*

$$Z^{2n-2}(X) = 0$$

The motivation here is that rationally connected varieties contain a lot of rational curves – enough to generate all of $H_2(X, \mathbb{Z})$?

Conjecture 3 (Griffiths–Harris question)**.** *Let $X \subset \mathbb{P}^4$ be a very general hypersurface of degree $d \geq 6$. Then the degree of every curve $C \subset X$ is divisible by d.*

Theorem 13.13 (Paulsen)**.** *Let $n \geq 3$ be an integer. Then there exists a set of degrees d with positive density such that Conjecture 2 is true in degree d.*

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