Fano varieties with torsion in H^3

(joint work with Jørgen Vold Rennemo)

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X = smooth projective variety of dimension n / $\mathbb{C}$

Main object of the talk:

$$\operatorname{Tors} H^3(X,\mathbb{Z})$$

This is a (stable) birational invariant, introduced by Artin-Mumford in the 1970s.

Michael Artin

David Mumford

Example

$$H^*(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}, \ldots$$

Example

Any rational X has no torsion in $H^3(X, \mathbb{Z})$.

Reason: If $\widetilde{X} \to X$ is a blow-up in a smooth center $Z \subset X$, then

$$H^3(\widetilde{X}, \mathbb{Z}) = H^3(X, \mathbb{Z}) \oplus H^1(Z, \mathbb{Z}) \cdot E$$

No torsion in $H^1(Z, \mathbb{Z})$ $\rightsquigarrow$ $\text{Tors}\, H^3(\widetilde{X}, \mathbb{Z}) = \text{Tors}\, H^3(X, \mathbb{Z})$.

Theorem (Artin–Mumford ($\sim$ 1970))

There exist double covers

$$X \to \mathbb{P}^3$$

branched along certain singular quartic surfaces $S \subset \mathbb{P}^3$, such that a *desingularization*

$$\widetilde{X} \to X$$

has torsion in $H^3(\widetilde{X}, \mathbb{Z})$.

These 3-folds are unirational, but not (stably) rational.

Constructing such examples is difficult.

Relation to Brauer group: For X smooth projective rationally connected, we have

$$\operatorname{Tors} H^3(X,\mathbb{Z}) = Br(X) = \frac{\left\{\mathbb{P}^n\text{-fibrations}\right\}}{\left\{\text{projectivized vector bundles}\right\}}.$$

Question (Beauville)

Is there a *Fano variety* with non-trivial torsion in $H^3(X, \mathbb{Z})$?

Example ($\dim X = 2$)

X is a Del Pezzo surface $\rightsquigarrow$ rational.

Example ($\dim X = 3$)

105 families of Fanos.
They all have no torsion in $H^3(X, \mathbb{Z})$.

Main Theorem

Theorem (O.-Rennemo)

There are Fano 4-folds with
$$\operatorname{Tors} H^3(X, \mathbb{Z}) = \mathbb{Z}/2.$$

The examples have Picard number 1.

Also examples in higher dimensions.

It is easy to make Fano varieties torsion in other cohomology groups, e.g., using blow-ups, products, etc.

$\mathbb{P}^{14} = \mathbb{P}(S^2 V^\vee)$ the space of quadrics in $V = \mathbb{C}^5$.

Definition

$$\begin{aligned} Z_r &= \Big\{ \text{quadrics of rank } \le r \Big\} \\ &= \text{zero locus of } (r+1)\times(r+1) \text{ minors of } \begin{pmatrix} u_0 & u_1 & u_2 & u_3 & u_4 \\ u_1 & u_5 & u_6 & u_7 & u_8 \\ u_2 & u_6 & u_9 & u_{10} & u_{11} \\ u_3 & u_7 & u_{10} & u_{12} & u_{13} \\ u_4 & u_8 & u_{11} & u_{13} & u_{14} \end{pmatrix} \text{ in } \mathbb{P}^{14} \end{aligned}$$

- Z_4 is a quintic hypersurface (dimension 13)
- Z_3 is a subvariety of degree 20 (dimension 11)
- $Z_2 = \mathrm{Sym}^2(\mathbb{P}^4)$ (dimension 8)
- Z_1 is the 2nd Veronese embedding of $\mathbb{P}^4$ (dimension 4)

Note: $\mathrm{sing}(Z_r) = Z_{r-1}$ for each $r = 2, 3, 4$.

$$Z_4 \;=\; \text{hypersurface defined by } \det \begin{pmatrix} u_0 & u_1 & u_2 & u_3 & u_4 \\ u_1 & u_5 & u_6 & u_7 & u_8 \\ u_2 & u_6 & u_9 & u_{10} & u_{11} \\ u_3 & u_7 & u_{10} & u_{12} & u_{13} \\ u_4 & u_8 & u_{11} & u_{13} & u_{14} \end{pmatrix} = 0.$$

Z_4 parameterizes rank 4 quadrics in $\mathbb{P}^4$, e.g.,

$$Q = x_0x_3 - x_1x_2$$

$\leadsto$ a quadric of rank 4 in $\mathbb{P}^4$ contains two families of 2-planes.

We will use these to define a double cover

$$W_4 \longrightarrow Z_4.$$

Define

$$U = \left\{ \text{ pairs } ([\Pi],[Q]) \text{ where } \mathbb{P}(\Pi) \subset Q \text{ is a 2-plane on } Q \right\} \subset Gr(3,V) \times \mathbb{P}(S^2 V^\vee).$$

The first projection is a projective bundle over $Gr(3,V)$.
(fibers = projective space of all quadrics containing a given 2-plane.)

If Q is a quadric with $(\Pi, Q) \in U$, then $\operatorname{rank} Q \leq 4$.
(because it contains a 2-plane.)

$\rightsquigarrow$ The second projection maps into Z_4.

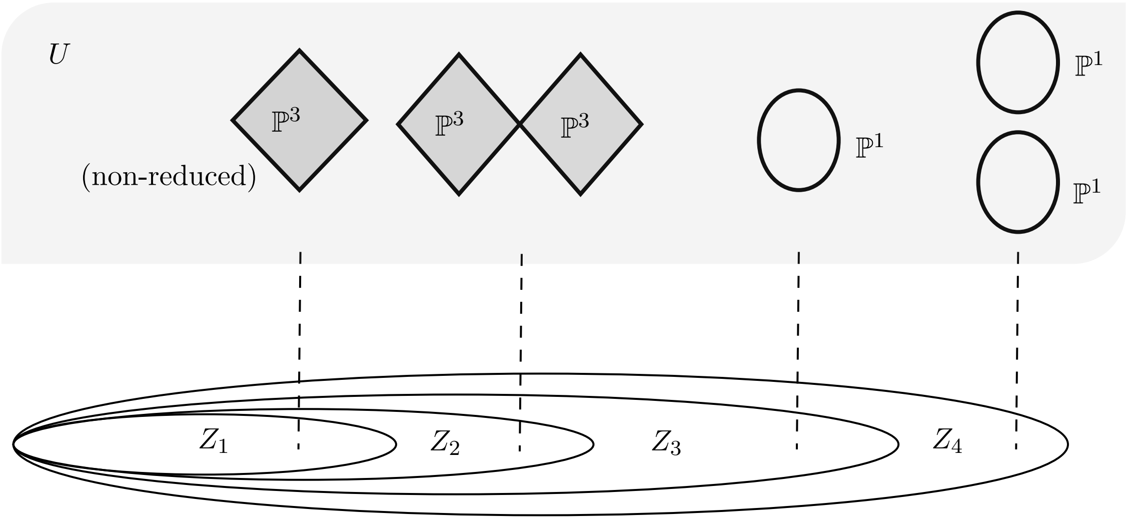
$$U = \left\{ \text{ pairs } ([\Pi],[Q]) \text{ where } \mathbb{P}(\Pi) \subset Q \text{ is a 2-plane on } Q \right\} \subset Gr(3,V) \times \mathbb{P}(S^2 V^\vee).$$

We define W_4 via the Stein factorization

$$U \xrightarrow{\ \tau\ } W_4 \xrightarrow{\ \sigma\ } Z_4$$

Then:

- σ is finite of degree 2
- τ is generically a $\mathbb{P}^1$-bundle:
  A fiber of pr_1 over a rank 4 quadric $Q \in Z_4$ consists two copies of $\mathbb{P}^1$.



U
$\mathbb{P}^3$
(non-reduced)
$\mathbb{P}^3$
$\mathbb{P}^3$
$\mathbb{P}^1$
$\mathbb{P}^1$
$\mathbb{P}^1$
Z_1
Z_2
Z_3
Z_4

Some facts:

1. W_4 has canonical singularities, Gorenstein, $\mathbb{Q}$-factorial, and

$$\mathrm{Pic}(X) = \mathbb{Z}\, H$$

   where $H = \tau^* \mathcal{O}_{Z_4}(1)$.

2. $\tau : W_4 \to Z_4$ is quasi-etale $\rightsquigarrow$ W_4 is Fano with

$$K_{W_4} = \tau^* K_{Z_4} = -10H$$

3. "Miracle": W_4 has a smaller singular locus than Z_4:

$$\mathrm{sing}\, W_4 = \tau^{-1}(Z_2)$$

   which has dimension 8.

4. $U \to W_4$ restricts to a $\mathbb{P}^1$-fibration over $W_4^\circ = W_4 - \tau^{-1}(Z_2)$. There is no rational section.

Complete intersections in W_4

Definition

Let

$$X = W_4 \cap H_1 \cap \ldots \cap H_9$$

where $H_i \in |H|$ are generic divisors.

- X has dimension
$$13 - 9 = 4$$
- X is Fano:
$$K_X = K_{W_4} + 9H = -H.$$
- X avoids $\tau^{-1}(Z_2)$ (which has dimension 8) $\rightsquigarrow$ X is smooth.
- Restricting the $\mathbb{P}^1$-fibration to X $\rightsquigarrow$ a non-trivial torsion class $\sigma \in H^3(X, \mathbb{Z})$.

Hodge diamond:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 9 & & 67 & & 9 & & 0
\end{array}
$$

If we do the same thing with $V = \mathbb{C}^n$ for $n \geq 5$, we get a double cover

$$\sigma : W_4 \to Z_4$$

and these varieties have dimension $4n - 7$.

Z_2 has dimension $2n - 2$.

$$X = W_4 \cap H_1 \cap \ldots \cap H_{2n-1}$$

is a smooth Fano manifold of index one of dimension $2n - 6$ with

$$H^3(X, \mathbb{Z}) = \mathbb{Z}/2.$$

Application / Motivation

Two "coniveau" filtrations on $H^l(X,\mathbb{Z})$:

$N^cH^l(X,\mathbb{Z})$= classes supported on proper subvarieties $Y \subset X$ of codimension $\geq c$.

$\widetilde{N}^cH^l(X,\mathbb{Z})$= classes $\widetilde{j}_*\beta$ where $\widetilde{j}$ is a composition $\widetilde{Y} \xrightarrow{desing} Y \hookrightarrow X$.

We always have $\widetilde{N}^cH^l(X,\mathbb{Z}) \subset N^cH^l(X,\mathbb{Z})$ and

$$N^1H^l(X,\mathbb{Z})/\widetilde{N}^1H^l(X,\mathbb{Z})$$

is a stable birational invariant.

Question (Voisin)

Is there a rationally connected variety where these two filtrations are different?

Theorem (O.-Rennemo)

Yes, there are Fano examples in any dimension ≥ 6.

Proposition (Colliot-Thélène–Voisin, Bloch-Srinivas, Voevodsky,..)

For X rationally connected, we have

$$H^l(X,\mathbb{Z}) = N^1(X,\mathbb{Z})$$

for all $l > 0$.

On the other hand:

Proposition (Benoist-O.)

If $\sigma \in H^3(X,\mathbb{Z})$ is a class with

$$\sigma^2 \mod 2 \neq 0 \in H^6(X,\mathbb{Z}/2),$$

then

$$\sigma \notin \widetilde{N}^1 H^3(X,\mathbb{Z}).$$

We check that this indeed happens.

We check that $\sigma^2 \neq 0 \mod 2$ in $H^6(X, \mathbb{Z}/2)$.

$$\begin{aligned} GO(4) &= \text{orthogonal similtude group} \\ &= \left\{ g \in GL(4) \;\middle|\; \langle gx, gy \rangle = \chi(g)\langle x, y\rangle \right\} \end{aligned}$$

$GO(4)^\circ$ = connected component of id.

$$\begin{aligned} \mathrm{Hom}(\mathbb{C}^5, \mathbb{C}^4) &\to Sym^2(\mathbb{C}^4)^\vee \\ M &\mapsto q(x,y) = \langle Mx, My \rangle \end{aligned}$$

induces

$$\begin{aligned} \mathrm{Hom}(\mathbb{C}^5, \mathbb{C}^4) /\!\!/ GO(4) &\simeq Z_4 \\ \mathrm{Hom}(\mathbb{C}^5, \mathbb{C}^4) /\!\!/ GO(4)^\circ &\simeq W_4 \end{aligned}$$

$\rightsquigarrow$ W_4 is an "algebraic approximation" to $BGO(4)^\circ$.

$\therefore$ Can use topological arguments to compute H^3.

- The exact sequence
$$1 \to SO(4) \to GO(4)^\circ \to \mathbb{C}^* \to 1.$$
gives a fibre bundle $\pi : BSO(4) \to BGO(4)^\circ$ with fiber $\mathbb{C}^*$, and Gysin sequence
$$\cdots \to H^i(BGO(4)^\circ, \mathbb{Z}) \xrightarrow{\pi^*} H^i(BSO(4), \mathbb{Z}) \xrightarrow{\pi_*} H^{i-1}(BGO(4)^\circ, \mathbb{Z}) \to \cdots$$

- The cohomology of $BSO(4)$:

H^0	H^1	H^2	H^3	H^4	H^5	H^6
$\mathbb{Z}$	0	0	$\mathbb{Z}/2 \cdot \nu$	$\mathbb{Z}p \oplus \mathbb{Z}e$	0	$\mathbb{Z}/2 \cdot \nu^2$

- This gives:
$$H^1(BGO(4)^\circ, \mathbb{Z}) = 0,\ H^2(BGO(4)^\circ, \mathbb{Z}) = \mathbb{Z},\ H^3(BGO(4)^\circ, \mathbb{Z}) = \mathbb{Z}/2.$$

- Lefschetz theorems give the same cohomology groups for X.