The Integral Hodge conjecture

Lecture 1

Introduction

Chow groups

Known results on the IHC

Specialization methods

Kollár's example

Each lecture will be quite independent from the others

Lecture 2

Topological obstructions

The examples of Atiyah-Hirzebruch

Two coniveau filtrations

Lecture 3

Application to arithmetic questions

Enriques surface fibrations

Lecture 4

Positive results

Main goals:

- Explain the IHC and why it is interesting / important
  - Hodge conjecture: How do we detect whether or not a cohomology class is algebraic?
  - Rationality problem: Birational invariants
  - Arithmetic questions: Existence of K-rational points
- Survey the various techniques of producing counterexamples
  - Construction problems in algebraic geometry
  - topological arguments (Atiyah – Hirzebruch, Totaro, ..)
  - Degeneration techiques (Kollár, ..)
- Explain some of the positive results (3-folds, cubic 4-folds, ..)

Introduction

X = a smooth projective variety $/\mathbb{C}$ of dim n

$H^p(X, \mathbb{Z})$ = p-th singular cohomology group

Hodge decomposition

De Rham classes of complex (p,q)-forms

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

properties

$$H^{p,q}(X) \simeq H^q(X, \Omega^p_X)$$

$$H^{q,p}(X) = \overline{H^{p,q}(X)} \quad \text{(Hodge symmetry)}$$

$$H^{n-p,\, n-q}(X) \simeq H^{p,q}(X) \quad \text{(Serre duality)}$$

W.V.D. Hodge (1903–1975)

ex $X \subset \mathbb{P}^5$ smooth cubic 4-fold

$$H^0(X,\mathbb{C}) = H^2(X,\mathbb{C}) = H^6(X,\mathbb{C}) = H^8(X,\mathbb{C}) = \mathbb{C}$$

$$H^1(X,\mathbb{C}) = H^3(X,\mathbb{C}) = H^5(X,\mathbb{C}) = 0$$

$$H^4(X,\mathbb{C}) = H^{3,1}(X) \oplus H^{2,2}(X) \oplus H^{1,3}(X) = \mathbb{C} \oplus \mathbb{C}^{21} \oplus \mathbb{C}$$

$\rightsquigarrow$ "Hodge diamond"

$$h^{pq} = \begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 1 & & 21 & & 1 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

Chow groups

$$CH^r(X) = r\text{-th Chow group}$$

$$= \frac{\left\{ \begin{array}{l} \text{free abelian group generated by} \\ \text{codim } r \text{ subvarieties } Z \subset X \end{array} \right\}}{\text{rational equivalence}}$$

$$CH_r(X) := CH^{n-r}(X)$$

Wei-Liang Chow (1911-1995)

ex $\quad CH^0(X) = \mathbb{Z} \cdot [X]$

$\quad CH^1(X) = \mathrm{Pic}(X)$

ex $\quad X \subset \mathbb{P}^3$ surface of degree ≥ 4

$\rightsquigarrow \quad CH^1(X) = \mathbb{Z}^\rho \qquad \rho =$ Picard number of X

$\quad CH^2(X)$ is enormous.

(Mumford)

Cycle class map

$Y \hookrightarrow X$ a subvariety of codim r

$\rightsquigarrow$ fundamental class $[Y] \in H^{2r}(X,\mathbb{Z})$

$$\begin{array}{c} [Y] \\ \| \\ j_* 1 \end{array}$$

where j is the composition

$$\begin{tikzcd} \tilde{Y} \arrow[d, "\text{desing.}"'] \arrow[dr, "j"] & \\ Y \arrow[r, hook] & X \end{tikzcd}$$

$$\rightsquigarrow \quad cl : CH^r(X) \longrightarrow H^{2r}(X,\mathbb{Z})$$

Easy fact: The image of cl lies in

$$H^{r,r}(X,\mathbb{Z}) = H^{2r}(X,\mathbb{Z}) \cap H^{r,r}(X,\mathbb{C})$$

the subgroup of integral Hodge classes

$=$ classes in $H^{2r}(X,\mathbb{Z})$ which map to $H^{r,r}(X,\mathbb{C})$ via the map $H^{2r}(X,\mathbb{Z}) \longrightarrow H^{2r}(X,\mathbb{C})$

ex $X \subset \mathbb{P}^5$ cubic 4-fold

$CH^0(X) = \mathbb{Z}$

$CH^1(X) = \mathbb{Z}$ (Lefschetz hyperplane theorem)

$CH^2(X) = H^{2,2}(X,\mathbb{Z})$ (Voisin)

$\cong \mathbb{Z}^h$ for some $1 \leq h \leq 21$

$CH^3(X) =$ non-f.g. abelian group (non-trivial)

$CH^4(X) = \mathbb{Z}$ (X is rationally connected)

The Integral Hodge conjecture (IHC)

$$CH^r(X) \xrightarrow{cl} H^{r,r}(X,\mathbb{Z}) \quad \text{is surjective}$$

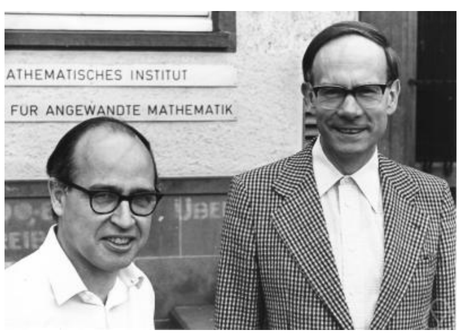
Hodge conjecture (HC)

$$CH^r(X) \otimes \mathbb{Q} \longrightarrow H^{r,r}(X,\mathbb{Q}) \quad \text{is surjective}$$

The IHC is not really a conjecture:

Theorem (Atiyah – Hirzebruch 1962)
$\exists$ smooth projective 7-fold X with a non-algebraic torsion class.

(More on this later.)



M. F. Atiyah F. Hirzebruch

Known results

The IHC holds for $r=0$ and $r=n$ (trivial).

For $r=1$:

Lefschetz (1,1)-theorem

$H^{1,1}(X,\mathbb{Z})$ is generated by Chern classes of (algebraic) line bundles

Solomon Lefschetz

The usual proof:

1. $\mathrm{Pic}(X) \simeq H^1(X, \mathcal{O}_X^\times)$

2. Exponential sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \longrightarrow 0$$

(sheaves in the analytic topology!)

$$\begin{tikzcd} \rightsquigarrow \; \arrow[r] & H^1(\mathcal{O}_X) \arrow[r] & H^1(\mathcal{O}_X^\times) \arrow[r, "c_1"] & H^2(X,\mathbb{Z}) \arrow[r] \arrow[d] & H^2(X,\mathcal{O}_X) \\ & & & H^2(X,\mathbb{C}) \arrow[ur, "\text{projection onto } H^{0,2}(X)\text{-factor}"'] & \end{tikzcd}$$

$\rightsquigarrow \; H^{1,1}(X,\mathbb{Z}) = \ker \mathrm{pr}$
$\qquad\qquad\qquad\;\; = \mathrm{image}\; c_1$

$\therefore$ Any integral $(1,1)$-class is the Chern class of a complex analytic line bundle on X.

3. By GAGA, any complex analytic line bundle is algebraic
$\rightsquigarrow$ done. $\blacksquare$

In addition, the HC holds for $r=n-1$ (curve classes):

Hard Lefschetz Theorem

$h = [\text{ample divisor}] \in H^2(X,\mathbb{Z})$

$$\rightsquigarrow \quad H^2(X,\mathbb{Q}) \xrightarrow{\cdot h^{n-2}} H^{2n-2}(X,\mathbb{Q})$$

is an isomorphism

This map respects the Hodge decomposition

$\Rightarrow$ any class in $H^{2n-2}(X,\mathbb{Q}) \cap H^{n-1,n-1}(X,\mathbb{C})$ is the product of $(n-1)$ divisors $\Rightarrow$ algebraic.

$\Rightarrow$ HC holds in degree $2n-2$.

Note: This result is strictly for $\mathbb{Q}$-coefficients.

There are 3-folds X where HC holds in degree 4 but IHC does not

Theorem (Kollár ~92)

Let $X \subset \mathbb{P}^4$ be a very general hypersurface of degree p^3 where $p \geq 5$ is a prime number.

$\ell \cdot h = 1$

Then

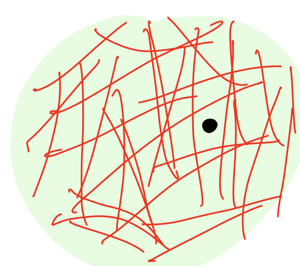
$$H^4(X, \mathbb{Z}) = \mathbb{Z} \cdot \ell$$

and any curve on X has degree $\equiv 0 \mod p$

$\therefore$ $p^3 \cdot \ell$ is algebraic, but ℓ is not.

János Kollár

very general = outside a countable union of Zariski closed subsets.



Defn We define the Voisin group of degree $2r$ as

$$Z^{2r}(X) = \frac{H^{r,r}(X,\mathbb{Z})}{\langle \text{algebraic classes} \rangle} \qquad \text{"}\frac{HC}{IHC}\text{"}$$

Note: $HC \Rightarrow Z^{2r}(X)$ is a finite group

Claire Voisin

ex In Kollár's example,

$$Z^4(X) \longrightarrow \mathbb{Z}/p \longrightarrow 0 \quad \rightsquigarrow \quad Z^4(X) \neq 0$$

(but we don't know whether it is $\mathbb{Z}/p$, $\mathbb{Z}/p^2$ or $\mathbb{Z}/p^3$ ..)

The two cases

- $Z^4(X)$ (codimension 2)
- $Z^{2n-2}(X)$ (dimension 1)

are the most interesting.

Prop The groups $Z^4(X)$ and $Z^{2n-2}(X)$ are birational invariants for smooth projective varieties.

By Weak Factorization, we may factor any birational map $f : X \dashrightarrow X'$ as

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "f"] & & X' \end{tikzcd}$$

where p, q are compositions of blowups in smooth centers

For $W \to X$ the blow up of a smooth $Z \subset X$

$$H^4(W, \mathbb{Z}) = H^4(X, \mathbb{Z}) \oplus H^2(Z, \mathbb{Z})[E]$$

$$\cup$$

$$H^{2,2}(W, \mathbb{Z}) = H^{2,2}(X, \mathbb{Z}) \oplus H^{1,1}(Z, \mathbb{Z}) \cdot [E]$$

and everything here is algebraic.

Similar argument for $H^{2n-2}(X, \mathbb{Z})$.

There are in fact deeper connections between IHC and rationality:

X smooth rationally connected 3-fold

$J^3(X) =$ intermediate jacobian of X

$$= \frac{H^3(X,\mathbb{C})}{F^1H^3(X,\mathbb{C}) + H^3(X,\mathbb{Z})}$$

$(J^3(X), \Theta)$ is a principally polarized abelian variety of dim g

Clemens-Griffiths Let $X \subset \mathbb{P}^4$ be a cubic 3-fold.

X rational $\Longrightarrow$ $J^3(X)$ is a jacobian of curves

$$\Longleftrightarrow \frac{\theta^{g-1}}{(g-1)} = [C_1] + \dots + [C_r]$$

C_i curves on $J^3(X)$

H. Clemens and P. Griffiths

$\dfrac{\Theta^{g-1}}{(g-1)!}$ is called the minimal class; it lies in $H^{2g-2}(J^3, \mathbb{Z})$.

ex

If $J^3(X) = JC$ is a jacobian, then $C \hookrightarrow J^3(X)$

with

$$[C] = \frac{\Theta^{g-1}}{(g-1)!} \qquad \leftarrow \text{Poincaré formula}$$

Voisin ($\sim$ 2015)

X stably rational $\Longrightarrow$ $\dfrac{\Theta^{g-1}}{(g-1)!}$ is algebraic

C. Gabrowski

$Z^{2g-2}(A) = 0$ for all abelian g-folds A $\Longleftrightarrow$ $\dfrac{\Theta^{g-1}}{(g-1)!}$ is algebraic for all (A, Θ)

The proof uses the Fourier–Mukai transform.

Results for 3-folds

For 3-folds X, the question is whether $Z^4(X) = 0$.

Q: Kollár's examples are (very) of general type.
Is this a coincidence? $(\kappa = \dim X)$

$\kappa = -\infty$ (= uniruled)

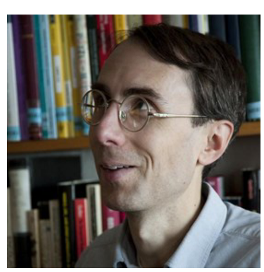
Voisin: IHC holds for uniruled 3-folds

$\kappa = 0$

Voisin: IHC holds for 3-folds with
$K_X = \mathcal{O}_X$ and $H^2(X, \mathcal{O}_X) = 0$

Totaro: IHC holds for 3-folds with
$\kappa = 0$ and $H^0(X, K_X) \neq 0$

all abelian 3-folds



Burt Totaro

Q: Does the IHC hold for all 3-folds with $\kappa = 0$?

A. No!

Kodaira dim 0 for surfaces:

- K3
- abelian
- Enriques

Prop (Benoist - O.) The IHC can fail on products

$$X = S \times E$$

where S is an Enriques surface and E is an elliptic curve.

these have $\kappa = 0$ and $H^0(K_X) = 0$

⟿ The above results are essentially optimal.

Olivier Benoist

Which classes are non-algebraic on $X = S \times E$?

By Künneth,

$$
\begin{aligned}
H^4(X,\mathbb{Z}) \;=\;& H^4(S,\mathbb{Z}) \otimes H^0(E,\mathbb{Z}) && pt \times E \\
&+ \\
& H^3(S,\mathbb{Z}) \otimes H^1(E,\mathbb{Z}) && \leftarrow \mathbb{Z}/2 \otimes \mathbb{Z}^2 \\
&+ \\
& H^2(S,\mathbb{Z}) \otimes H^2(E,\mathbb{Z}) && D \times pt
\end{aligned}
$$

Let $\alpha \in H^1(S,\mathbb{Z}/2) \simeq \mathbb{Z}/2$ be the class associated to the K3 cover

$$T \xrightarrow{2:1} S$$

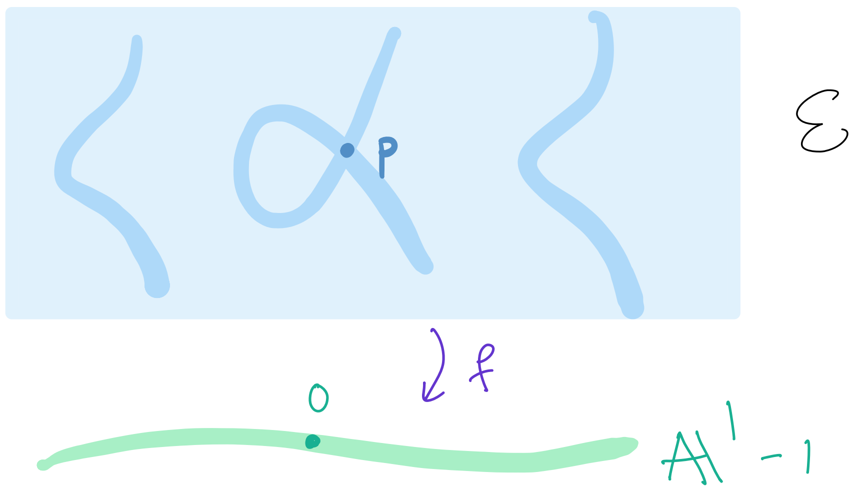
Topological computation:

The IHC holds on $X = S \times E$ $\iff$ $\forall \beta \in H^1(E, \mathbb{Z}/2) \Rightarrow \exists Z \in CH^2(E \times S)$ such that $Z^* \alpha = \beta$ $(*)$

$\therefore$ every class in $H^1(E,\mathbb{Z}/2)$ is obtained from α via some correspondence

One then argues by contradiction:
Assume $(*)$ holds for <u>every</u> genus 1 curve E.

One considers a degeneration

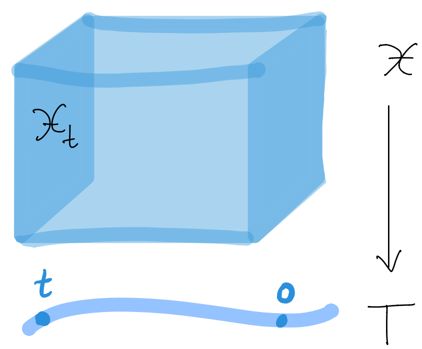


$\rightsquigarrow$ Reach a contradiction on the special fiber.

Specialization method

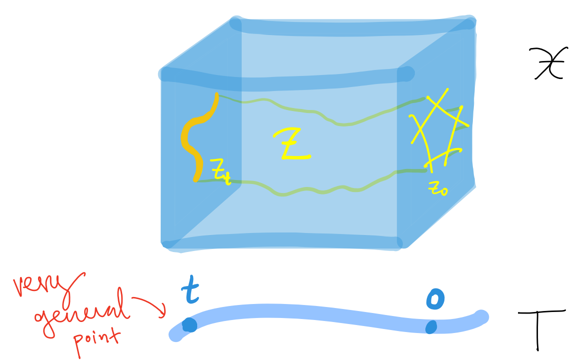
The following situation is typical:

Given a family



$\rightsquigarrow$ want to disprove the IHC on the very general fiber $\mathcal{X}_t$

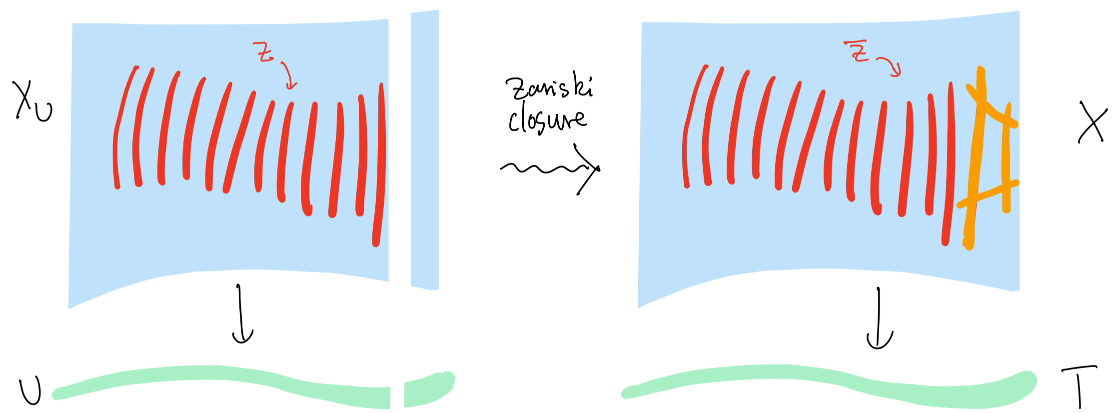
Key point Subvarieties in $\mathcal{X}_t$ extend to all other fibers



$\rightsquigarrow$ can try to get a contradiction by analyzing the special fiber $\mathcal{X}_0$

$\mathcal{X}_0$ may be very singular!

Extending cycles from an open set



$U \subseteq T$ open set

$Z \in CH^p(\mathcal{X}_U) \rightsquigarrow \bar{Z} \in CH^p(X)$ extending Z

Take Zariski closure of the components of Z.

Rmk This may look trivial, but it is a key difference between algebraic geometry and topology:

$$CH^p(X) \longrightarrow CH^p(U) \longrightarrow 0 \qquad \text{always surjective}$$

$$H^p(X) \to H^p(U) \to H^{p+1}(X, U) \to \cdots$$

Extending cycles from a very general fiber

Set-up

$$\begin{array}{c} \mathcal{X} \\ f \big\downarrow \\ T \end{array} \quad \text{projective family}$$

Relative Hilbert scheme: $H = \mathrm{Hilb}(\mathcal{X}/T)$ parametrizes closed subschemes Z in the fibers $\mathcal{X}_t$.

$\exists$ universal family

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{i} & \mathcal{X} \\ \big\downarrow \pi & & \\ H & & \end{array}$$

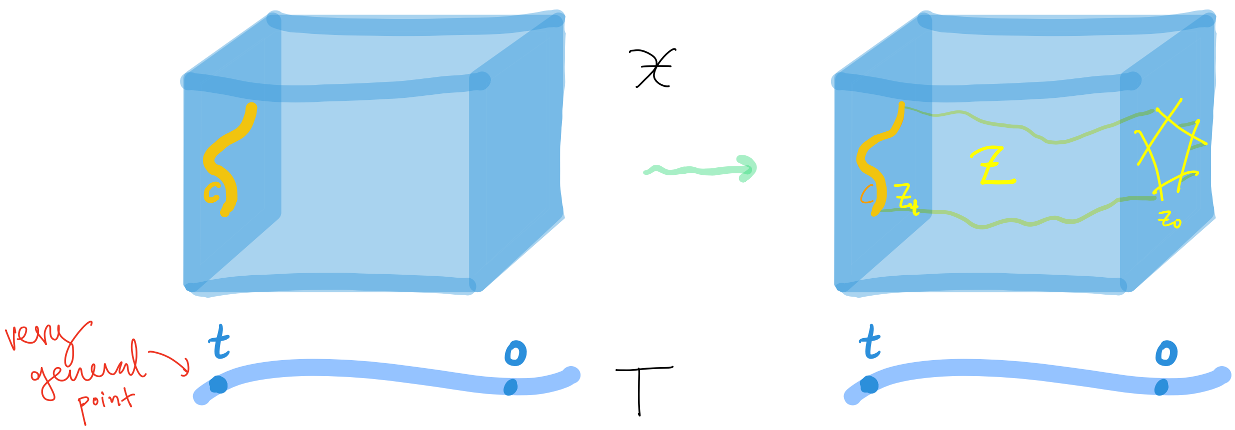
H has countably many components (only countably many Hilbert polynomials)

H' = union of components such that $f \circ i$ is not surjective

$\rightsquigarrow$ $T' = (f \circ i)(\mathcal{Z}_{H'}) \subsetneq T$

is a countable union of closed subsets.

If $t \in T - T'$, then any subscheme in $\mathcal{X}_t$ deforms out of $\mathcal{X}_t$:



Precisely: For $\Gamma \subset \mathcal{X}_t$

$\rightsquigarrow$ $\exists$ component $H_0 \subset H$ + universal family $\begin{matrix} Z \\ \downarrow \\ H_0 \end{matrix}$

such that $f \circ i : Z \to T$ is dominant and

$$Z\big|_t = \Gamma \qquad \text{(base change of } Z \text{ equals } \Gamma\text{)}$$

The specialization map

$$CH^p(\mathcal{X}_t) \longrightarrow CH^p(\mathcal{X}_0)$$

is compatible with intersection products.

Remark

If $\begin{array}{c} \mathcal{X} \\ \downarrow f \\ T \end{array}$ is a smooth family of proj. varieties $/\mathbb{C}$,

we may identify

$$H^k(\mathcal{X}_s, \mathbb{Z}) = H^k(\mathcal{X}_0, \mathbb{Z}) \quad \forall s \in T$$

$\therefore$ If $\alpha_s \in H^k(\mathcal{X}_s, \mathbb{Z})$ is algebraic, then so is

$\alpha_0 \in H^k(\mathcal{X}_0, \mathbb{Z})$.

Kollár's counterexample

$X \subset \mathbb{P}^4$ a smooth hypersurface of degree 125 (125 can be reduced to 48)

Lefschetz' hyperplane theorem gives

János Kollár

h = hyperplane divisor

$$H^2(X,\mathbb{Z}) = \mathbb{Z}\, h$$

$$h \cdot \ell = 1$$

$$H^4(X,\mathbb{Z}) = \mathbb{Z}\, \ell$$

Suppose now X is **very general**

$\rightsquigarrow$ X contains no line

In fact:

Claim If $C \subset X$ is a curve, then

$$h \cdot [C] \equiv 0 \mod 5 \qquad (*)$$

Specialization method: If there exists <u>some</u> hypersurface $X_0 \subset \mathbb{P}^4$ such that $(*)$ holds, then it holds also on X.

X_0 will be a (very) singular hypersurface.

Construction: Pick $f_0, \dots, f_4 \in k[x_0, x_1, x_2, x_3]$ general, of degree 5.

$\rightsquigarrow$ morphism $\mathbb{P}^3 \xrightarrow{\varphi} \mathbb{P}^4$

$X_0 := \varphi(\mathbb{P}^3)$ is a hypersurface of degree $5^3 = 125$.

May view X_0 as the image of veronese + generic projections:

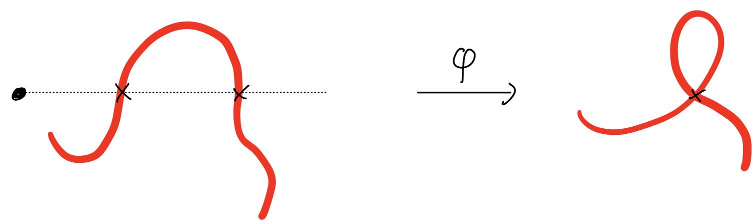
$$\begin{tikzcd}
 & V \arrow[r, hook] & \mathbb{P}^{55} \arrow[d, "\text{generic projection}"] \\
\mathbb{P}^3 \arrow[ur, hook, "\text{veronese embedding}"] \arrow[rr, "\varphi"'] & & X_0 \subset \mathbb{P}^4
\end{tikzcd}$$

The morphism $\varphi : \mathbb{P}^3 \longrightarrow X_0$ is finite; and

$2:1$ on a surface $S \subset \mathbb{P}^3$

$3:1$ on a curve $\Gamma \subset \mathbb{P}^3$ ← by the theory of generic projections

$4:1$ on a finite set of points



$\therefore$ If $C \subset X_0$ is a curve, then there is a curve $D \subset \mathbb{P}^3$ such that

$$\boxed{\varphi_* D = 6 \cdot C}$$

$\rightsquigarrow$ $\deg \varphi_* D = h \cdot \varphi_* D$

$= \varphi^* h \cdot D$ (projection formula)

$= \mathcal{O}_{\mathbb{P}^3}(5) \cdot D$

$\equiv 0 \mod 5$

But then also $\deg C \equiv 0 \mod 5$. ∎

Conjecture (Griffiths – Harris)
For a very general hypersurface $X \subset \mathbb{P}^4$ of degree $d \geq 6$, we have

$$\deg C \equiv 0 \mod d$$

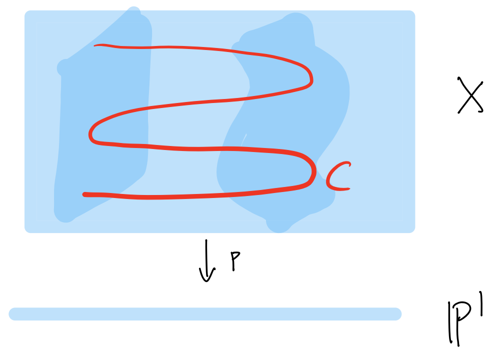
for every curve $C \subset X$.

Rmk Degree 5 hypersurfaces $X \subset \mathbb{P}^4$ contain lines (at least 2875)

The example of Hassett–Tschinkel and Totaro

Thm Let $X \subset \mathbb{P}^1 \times \mathbb{P}^3$ be a very general $(3,4)$-divisor. Then every curve $C \subset X$ has even degree over $\mathbb{P}^1$.

X is a K3-fibration over $\mathbb{P}^1$



Lefschetz: $H^2(X,\mathbb{Z}) = \mathbb{Z}F \oplus \mathbb{Z}H$

$H^4(X,\mathbb{Z}) = \mathbb{Z}f \oplus \mathbb{Z}\sigma$

$F \cdot f = 0$

$F \cdot \sigma = 1$

$H \cdot f = 1$

$H \cdot \sigma = 0$

$\rightsquigarrow$ the class σ is not algebraic.

Brendan Hassett

Yuri Tschinkel

Burt Totaro

"On the integral Hodge and Tate conjectures over a number field"

Specialization: Specialize to $X_0 \subset \mathbb{P}^1 \times \mathbb{P}^3$ given by

$$x_0^3 y_0^4 + x_0^2 x_1 y_1^4 + x_0 x_1^2 y_2^4 + x_1^3 y_3^4 = 0$$

Claim Every curve on X_0 has even degree / $\mathbb{P}^1$

Specialization map

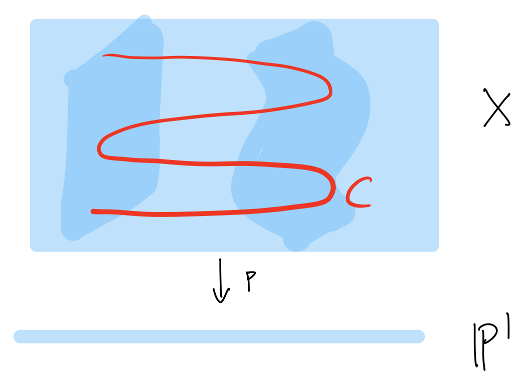
$$CH_1(X_{\overline{\mathbb{Q}}}) \longrightarrow CH_1(X_{\overline{\mathbb{F}_p}})$$

Enough to show: every curve in $X_{\overline{\mathbb{F}_p}}$ has even degree / $\mathbb{P}^1$.

Just need the degree of

$$C \longrightarrow \mathbb{P}^1_{\overline{\mathbb{F}_p}}$$

restricted to the generic point of $\mathbb{P}^1_{\overline{\mathbb{F}_p}}$



Restrict further to $\overline{\mathbb{F}_p}((t))$: $\leftarrow$ Laurent series around $[0,1]$

Enough to show:

$$t_0^4 + t x_1^4 + t^2 x_2^4 + t^3 x_3^4 = 0 \qquad (*)$$

has no rational point over any odd-degree extension $\overline{\mathbb{F}_p}((s))$ over $\overline{\mathbb{F}_p}((t))$

Suppose such a point exists

$\Rightarrow$ $\exists$ Laurent series in $\overline{\mathbb{F}_p}((s))$

$t(s)$, $x_0(s)$, ..., $x_3(s)$ satisfying $(*)$

s.t. $r = \mathrm{ord}_s t$ is odd.

$$x_0^4 + t x_1^4 + t^2 x_2^4 + t^3 x_3^4 = 0$$

$\downarrow$ valuation mod 4

$0 \qquad r \qquad 2r \qquad 3r$

r odd $\Rightarrow$ $0, r, 2r, 3r$ all different

$\Rightarrow$ $(*)$ implies that

$$x_0^4 = t x_1^4 = t^2 x_2^4 = t^3 x_3^4 = 0$$

$$\Rightarrow \quad X_0(s) = \dots = X_3(s) = 0$$

$\Rightarrow$ not a point in $\mathbb{P}^3_{\overline{\mathbb{F}_p((s))}}$ $\blacksquare$

Rmk This gives examples defined over $\overline{\mathbb{Q}}$.

$$x_0^3 y_0^4 + x_0^2 x_1 y_1^4 + x_0 x_1^2 y_2^4 + x_1^3 y_3^4 + p \cdot G(x_0, x_1, y_0, \dots y_3) = 0.$$

Topological obstructions

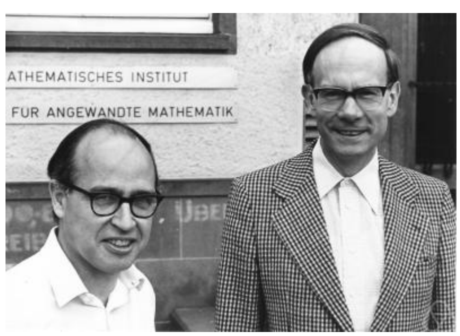
Q Given a cohomology class $\alpha \in H^{2p}(X, \mathbb{Z})$
$\rightsquigarrow$ how do we prove that α is <u>not</u> algebraic?

Hodge-theoretic obstruction: α must be of type (p,p).

Atiyah & Hirzebruch discovered a much more subtle obstruction:

Prop (Atiyah – Hirzebruch)
If $\alpha \in H^{2p}(X, \mathbb{Z})$ is algebraic, then all the odd Steenrod operations vanish:

$$\mathrm{Sq}^i(\underbrace{\bar{\alpha}}_{\text{mod } 2 \text{ reduction of } \alpha}) = 0 \qquad \text{for all } i = 3, 5, 7, \ldots$$



Steenrod operations

Sq^i is a cohomology operation on $H^m(-,\mathbb{Z}/2)$ of degree i

$$Sq^i : H^m(X,\mathbb{Z}/2) \longrightarrow H^{m+i}(X,\mathbb{Z}/2)$$

properties

(1) $Sq^0 = id$

(2) $Sq^m(\alpha) = \alpha \cup \alpha$ if $\alpha \in H^m(X,\mathbb{Z}/2)$

$Sq^{>m}(\alpha) = 0$

(3) $Sq = Sq^0 + Sq^1 + Sq^2 + \dots$ satisfies

$$Sq(\alpha \cup \beta) = Sq(\alpha) \cup Sq(\beta)$$

Cartan's formula: $Sq^i(\alpha \cup \beta) = \sum_{j=0}^{i} Sq^j \alpha \cup Sq^{i-j} \beta$

(4) $f : X \longrightarrow Y$

$$\rightsquigarrow \begin{tikzcd} H^m(X,\mathbb{Z}/2) \arrow[r, "Sq^i"] & H^{m+i}(X,\mathbb{Z}/2) \\ H^m(Y,\mathbb{Z}/2) \arrow[u, "f^*"] \arrow[r, "Sq^i"] \arrow[ur, phantom, "\circlearrowleft"] & H^{m+i}(Y,\mathbb{Z}/2) \arrow[u, "f^*"'] \end{tikzcd}$$

Relative Wu formula

If $f: Y \longrightarrow X$ is a C^∞-map of compact manifolds

$$Sq(f_* \alpha) = f_*\big(Sq(\alpha) \cup w(N_f)\big)$$

$N_f = f^* T_X - T_Y$ virtual normal bundle

$w(N_f) = w(f^* T_X) \cdot w(T_Y)^{-1}$ $\qquad w =$ Stiefel-Whitney class

If $Y \subset X$ is a closed complex subvariety, then

$[Y] = f_* 1$ where $1 \in H^0(\tilde{Y}, \mathbb{Z})$

$$\begin{tikzcd} \tilde{Y} \arrow[r, "\text{desing.}"'] \arrow[rr, bend left, "f"] & Y \arrow[r, hook] & X \end{tikzcd}$$

and

$$w_{2k}(N_f) = c_k(N_f) \quad (\text{mod } 2)$$

$$w_{2k-1}(N_f) = 0 \quad (\text{mod } 2)$$

$\rightsquigarrow$ no terms in odd degree.

$$\rightsquigarrow \quad Sq([Y] \bmod 2) = Sq(f_* 1) \qquad 1 \in H^0$$

$$\overset{\text{Wu}}{=} f_*(1 \cup w(N_f))$$

$$\Rightarrow \quad Sq^i([Y] \bmod 2) = 0 \quad \text{for all odd } i.$$

$\therefore$ If α is algebraic, then

$$Sq^3(\bar{\alpha}) = Sq^5(\bar{\alpha}) = Sq^7(\bar{\alpha}) = \cdots = 0$$

Now we just need to find examples where this obstruction actually takes place.

Examples via classifying spaces

G a (finite) group

BG = classifying space of principal G-bundles

$= EG/G$ $\qquad$ EG = any contractible space on which G acts freely.

$\rightsquigarrow \quad H^q(BG, \mathbb{Z}) = H^q(G, \mathbb{Z})$ (Group cohomology)

ex
$$B\mathbb{Z} = S^1 \qquad (E\mathbb{Z} = \mathbb{R})$$
$$B\mathbb{Z}^n = S^1 \times \dots \times S^1 \qquad (E\mathbb{Z} = \mathbb{R}^n)$$
$$B\mathbb{Z}/2 = \mathbb{RP}^\infty \qquad (E\mathbb{Z}/2 = S^\infty)$$
$$B\mathbb{C}^* = \mathbb{CP}^\infty$$

Main example: $G = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$.

Note $H^*(\mathbb{Z}/2, \mathbb{Z}/2) = H^*(\mathbb{RP}^\infty, \mathbb{Z}/2) = \mathbb{Z}/2[x]$ $\quad \deg x = 1$.

By Künneth,

$$H^*(G, \mathbb{Z}/2) = \mathbb{Z}/2[x_1, x_2, x_3] \qquad \deg x_i = 1$$

Consider the element $x_1 x_2 x_3 \in H^3(G, \mathbb{Z}/2)$:

$$Sq^3 Sq^1 (x_1 x_2 x_3)$$
$$= Sq^3 (Sq^0(x_1 x_2) Sq^1(x_3) + Sq^1(x_1 x_2) Sq^0(x_3))$$
$$= Sq^3 (x_1 x_2 \cdot x_3^2 + x_1 \cdot x_2^2 + x_1^2 x_2 \cdot x_3)$$

$$Sq^3(x_1 x_2 x_3^2) = Sq^0(x_3) \cdot Sq^3(x_1 x_2 x_3) + Sq^1(x_3) \cdot Sq^2(x_1 x_2 x_3)$$
$$= x_3 x_1^2 x_2^2 x_3^2 + x_3^2 (x_1^2 x_2^2 x_3 + Sq^1(x_1 x_2) x_3^2)$$
$$= x_1^2 x_2^2 x_3^3 + x_1^2 x_2^2 x_3^3 + x_3^4 (x_1^2 x_2 + x_1 x_2^2)$$
$$= x_1^2 x_2 x_3^4 + x_1 x_2^2 x_3^4$$

$\rightsquigarrow$ sum $\neq 0$.

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

$\rightsquigarrow$ $\alpha = \beta_2 (x_1 x_2 x_3) \in H^4(G, \mathbb{Z})$

is an element satisfying

$$Sq^3(\bar{\alpha}) = Sq^3(Sq^1(x_1 x_2 x_3)) \neq 0$$

Problem BG is not a smooth projective variety (an infinite dimensional stack)

Godeaux-Serre varieties

G a finite group

Serre: Given any $r \geq 1$, one can find:

- a representation of G on $\mathbb{P}^N$
- smooth G-invariant complete intersection $Y \subset \mathbb{P}^N$

such that

(1) G acts freely on Y
(2) $\dim_{\mathbb{C}} Y = r$

and $X = Y/G$ is a smooth projective variety

and $H^m(G,\mathbb{Z}) \hookrightarrow H^m(X,\mathbb{Z}) \quad m \leq r$

We have a (torsion) class $\alpha \in H^4(G,\mathbb{Z})$ s.t.

$$Sq^3(\bar{\alpha}) \neq 0 \quad \text{in} \quad H^7(G,\mathbb{Z}).$$

Pick $r = 7$ and an X as above.

$\rightsquigarrow$ $H^{\leq 7}(G,\mathbb{Z}) \subset H^{\bullet}(X,\mathbb{Z})$

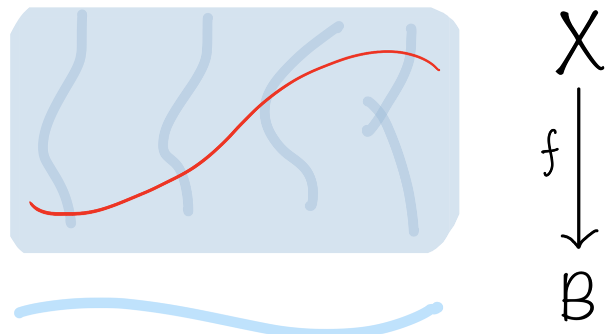
$\rightsquigarrow$ X admits a non-algebraic class in $H^4(X,\mathbb{Z})$.

Rmk X is a smooth complex 7-fold.

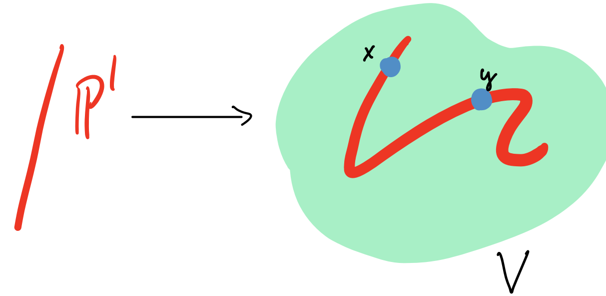
Pushing the proof a little bit $\rightsquigarrow$ can take X to have K_X torsion. (so $\kappa = 0$)

Application to some arithmetic questions

Let $f : X \to B$ be a morphism of complex projective varieties, where B is a curve



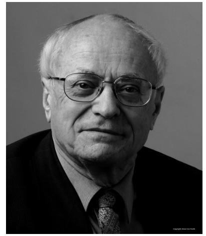
Graber–Harris–Starr theorem: If the general fiber of f is rationally connected, then f has a section.



A variety V is *rationally connected* if any two general points $x, y \in V$ can be joined by a rational curve:

ex $\exists$ polynomials $p_0(t), \ldots, p_4(t)$ s.t. $x_0 = p_0(t), \ldots, x_4 = p_4(t)$ solves

$$x_0^4 + t x_1^4 + t^2 x_2^4 + t^3 x_3^4 + t^5 x_4^4 = 0 \qquad \mathbb{P}^4_{k(t)}$$



J.P. Serre

A. Grothendieck

$K = k(B)$

Serre (1958) (in a letter to Grothendieck):
Is the same conclusion true for varieties X/K with $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$?

Serre adds that it is "sans doute trop optimiste".

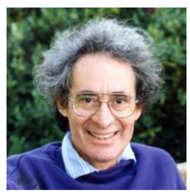
rationally connected varieties satisfy this

Graber–Harris–Mazur–Starr, Lafon, Starr ($\sim$ 2002)

No: There exist Enriques surface fibrations over curves with no section.

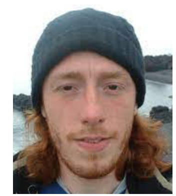
T. Graber

J. Harris



B. Mazur

J. Starr



G. Lafon

A question of Esnault:

If X/K satisfies

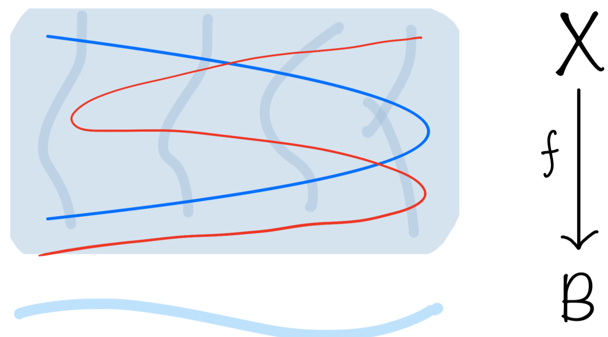
$$H^i(X, \mathcal{O}_X) = 0 \quad \text{for } i > 0,$$

does X/K admit a 0-cycle of degree 1?

More geometrically: If $f : X \to B$ is a fibration with general fiber X_b satisfying $H^i(X_b, \mathcal{O}_{B_b}) = 0$ for $i > 0$: Do we have

$$\gcd\Big(\deg(C/B) \mid C \subset X \text{ a curve} \Big) = 1?$$

H. Esnault


X
f
B

Main result of today:

Theorem (O.-Suzuki)

There exists an Enriques surface fibration

$$X \to \mathbb{P}^1$$

such that every curve $C \subset X$ has even degree over $\mathbb{P}^1$.

F. Suzuki

Relation to the Integral Hodge Conjecture

Colliot-Thélène–Voisin: For $f : X \to B$ with $\mathcal{O}$-acyclic fibers:

$$f_* : H_2(X, \mathbb{Z}) \to H_2(B, \mathbb{Z})$$

is surjective.

Thus there is a homology class $\sigma \in H_2(X, \mathbb{Z})$ which has degree 1 on a fiber.
$\therefore$ "there is no topological obstruction to the existence of sections"

J.-L. Colliot-Thélène

C. Voisin

The class σ is automatically Hodge, so we obtain a new counterexample to the Integral Hodge conjecture.

In the example, 4σ is algebraic, but σ is not.

Enriques surfaces

Surfaces S with

- $\pi_1(S) = \mathbb{Z}/2$
- $2K_S = 0$

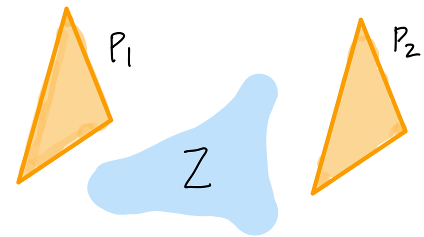
There is a universal cover $\pi : Z \to S$ where Z is a K3 surface

Example
Let $S \subset \mathbb{P}^2 \times \mathbb{P}^2$ be the surface defined by the 2×2 minors of a generic matrix

$$\begin{pmatrix} p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \end{pmatrix} \qquad \begin{array}{rcl} p_i & = & p_i(x_0, x_1, x_2) \\ q_i & = & q_i(y_0, y_1, y_2) \end{array}$$

where $\deg p_i = (2,0)$ and $\deg q_i = (0,2)$. Then S is an Enriques surface.

Here is the K3 cover:
On $\mathbb{P}^5 = \operatorname{Proj} k[x_0, x_1, x_2, y_0, y_1, y_2]$, there is an involution



$$\iota : \mathbb{P}^5 \to \mathbb{P}^5$$

defined by $\iota^*(x_i) = x_i$, $\iota(y_i) = -y_i$.
Consider the quadrics

$$F_i = p_i + q_i \qquad \begin{aligned} p_i &= p_i(x_0, x_1, x_2) \\ q_i &= q_i(y_0, y_1, y_2) \end{aligned}$$

invariant under ι

These define a K3 surface

$$Z = \{F_0 = F_1 = F_2 = 0\} \subset \mathbb{P}^5$$

ι acts freely on Z, as Z is disjoint from

$$\operatorname{Fix}(\iota) = P_1 \cup P_2 \qquad \begin{aligned} P_1 &= V(x_0, x_1, x_2) \simeq \mathbb{P}^2 \\ P_2 &= V(y_0, y_1, y_2) \simeq \mathbb{P}^2 \end{aligned}$$

Hence $S = Z/\iota$ is a smooth Enriques surface.

An Enriques surface fibration

Let $Y \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ be the threefold defined by the 2×2 minors of a generic matrix

$$\begin{pmatrix} p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \end{pmatrix} \qquad \begin{aligned} p_i &= sA_i + tB_i \\ q_i &= sC_i + tD_i \end{aligned}$$

where $\deg p_i = (\mathbf{1}, \mathbf{2}, \mathbf{0})$ and $\deg q_i = (\mathbf{1}, \mathbf{0}, \mathbf{2})$.

Then Y is a smooth threefold, and the first projection defines an Enriques surface fibration

$$p : Y \to \mathbb{P}^1.$$

Properties of Y

- Y has Kodaira dimension 1
- Y is simply connected and $H^*(\overset{Y}{\cancel{X}}, \mathbb{Z})$ has no torsion.
- Hodge diamond

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 26 & & 0 & \\
0 & & 45 & & 45 & & 0
\end{array}
$$

The geometry of Y

Let $F_i = p_i + q_i$, considered as a $(1,2)$ form on $\mathbb{P}^1 \times \mathbb{P}^5$.

$$\begin{tikzcd}
 & Z = Bl_{\mathbb{P}^1 \times (P_1 \cup P_2)} Z_0 \arrow[loop right, "\iota"] \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\
Z_0 = \{F_0 = F_1 = F_2 = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^5 \arrow[loop right, "\iota"] & & Y \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2
\end{tikzcd}$$

K3 fibration

π is the blow-up of the fixed points of ι:

- $(\mathbb{P}^1 \times P_1) \cap Z_0$ ($= 12$ points $p_{1,1}, \ldots, p_{1,12}$); and
- $(\mathbb{P}^1 \times P_2) \cap Z_0$ ($= 12$ points $p_{2,1}, \ldots, p_{2,12}$)

$\rightsquigarrow$ 24 exceptional divisors

$$\begin{matrix} E_{1,1}, & \ldots & E_{1,12} \\ E_{2,1}, & \ldots & E_{2,12} \end{matrix}$$

p is a double cover, ramified along the $E_{i,j}$.

Out of the 24 $E_{i,j}$'s, we single out $E_{1,1}, \ldots, E_{1,12}$ (from the fixed points on P_1).

If Y is defined by $\begin{pmatrix} p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \end{pmatrix}$, the $E_{1,i}$ are the components of

$$E_1 = \{p_0 = p_1 = p_2 = 0\} \subset Y.$$

Claim

For a curve $C \subset Y$ we have

$$\deg(C/\mathbb{P}^1) \equiv C \cdot \left(\sum_{j=1}^{12} E_{1,j} \right) \mod 2.$$

$\therefore$ If $C \subset Y$ is a section of $Y \to \mathbb{P}^1$, then C has to intersect at least one of the $E_{1,j}$'s (!).

This is enough to show that Y fails the IHC:

$$\begin{tikzcd}[column sep=small] & Z = Bl_{\mathbb{P}^1 \times (P_1 \cup P_2)} Z_0 \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\ Z_0 = \{F_0 = F_1 = F_2 = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^5 & & Y \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2 \end{tikzcd}$$

Lefschetz hyperplane theorem $\rightsquigarrow$ $H_2(Z_0, \mathbb{Z}) = \mathbb{Z}f_1 \oplus \mathbb{Z}f_2$
$\rightsquigarrow$ $H_2(Z_0, \mathbb{Z}) \to H_2(\mathbb{P}^1, \mathbb{Z})$ surjective
$\rightsquigarrow$ $H_2(Z, \mathbb{Z}) \to H_2(\mathbb{P}^1, \mathbb{Z})$ surjective
$\rightsquigarrow$ Y admits a class $\gamma \in H_2(Y, \mathbb{Z})$ such that

$$\deg(\gamma/\mathbb{P}^1) = 1 \qquad \gamma \cdot E_{i,j} = 0 \qquad \forall i,j$$

$\rightsquigarrow$ γ is Hodge, but not algebraic.

We consider a degeneration $\mathcal{Y} \to T$ with special fiber $\mathcal{Y}_0$.

If $Y = \mathcal{Y}_t$ is a very general fiber, then there is a specialization map

$$CH_1(Y) \to CH_1(\mathcal{Y}_0)$$

compatible with intersection products.

So it suffices to prove the congruence

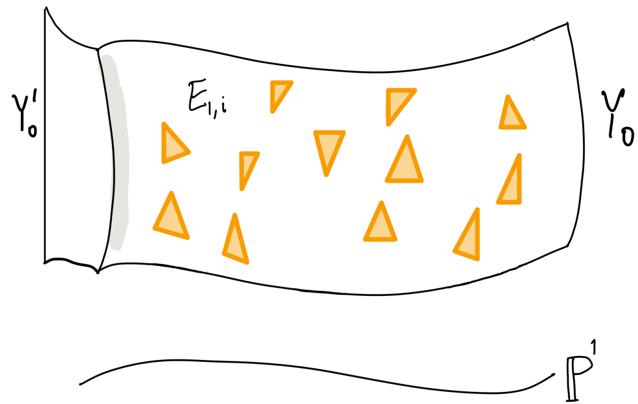
$$\deg(C/\mathbb{P}^1) \equiv C \cdot \left(\sum_{j=1}^{12} E_{1,j} \right) \mod 2.$$

on $\mathcal{Y}_0$.

The degeneration: $\mathcal{Y} \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2 \to \operatorname{Spec} k[\epsilon]$ defined by the minors of

$$M_\epsilon = \begin{pmatrix} p_0 & p_1 & p_2 \\ sy_0^2 + \epsilon r_0 & sy_1^2 + \epsilon r_1 & sy_2^2 + \epsilon r_2 \end{pmatrix}$$

Special fiber over $\epsilon = 0$: $\mathcal{Y}_0 = Y_0 \cup Y_0'$



- $Y_0 \cap Y_0' = \{s = 0\}$= an Enriques surface
- $V(p_0, p_1, p_2) = E_{1,1} \cup \cdots \cup E_{1,12}$ does not intersect Y_0' (hence lies in $(\mathcal{Y}_0)_{\mathrm{reg}}$).

$$\deg(C/\mathbb{P}^1) \equiv C \cdot \left(\sum_{j=1}^{12} E_{1,j}\right) \mod 2$$

Y_0 is defined by the matrix

$$\begin{pmatrix} p_0 & p_1 & p_2 \\ y_0^2 & y_1^2 & y_2^2 \end{pmatrix}$$

Let $D_1 = \{p_0 = 0\}$; this is a divisor of type $(1, 2, 0)$.

For $C \subset Y_0$ a curve,

$$\text{''} \quad C \cdot (1,0,0) = C \cdot pr_1^* \mathcal{O}(1)$$

$$\deg(C/\mathbb{P}^1) \equiv D_1 \cdot C \mod 2$$

On the other hand,

$$D_1 = 2 \cdot V(y_0) + \sum_{j=1}^{12} E_{1,j}$$

This gives the desired congruence.

Main point: some Cartier divisor becomes double on the degeneration Y_0.

The threefold X and proof of the main theorem

Let $X \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ be the threefold defined by the 2×2 minors of a generic matrix

$$\begin{pmatrix} p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \end{pmatrix} \qquad \begin{aligned} p_i &= s^2 A_i + st B_i + t^2 C_i \\ q_i &= s^2 D_i + st E_i + t^2 F_i \end{aligned}$$

where $\deg p_i = (\mathbf{2}, \mathbf{2}, \mathbf{0})$ and $\deg q_i = (\mathbf{2}, \mathbf{0}, \mathbf{2})$.

Theorem

Any curve $C \subset X \to \mathbb{P}^1$ has even degree over $\mathbb{P}^1$.

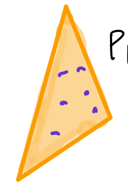
Properties of X

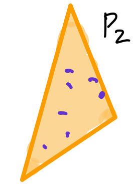
- X has Kodaira dimension 1
- X is simply connected and $H^*(X, \mathbb{Z})$ has no torsion.
- Hodge diamond

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 50 & & 0 & \\
0 & & 99 & & 99 & & 0
\end{array}
$$

On $X \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ there are now $24 + 24 = 48$ exceptional divisors

$$\begin{array}{ccc} E_{1,1}, & \dots & E_{1,24} \\ E_{2,1}, & \dots & E_{2,24} \end{array}$$





We focus on $E_{1,1}, \dots, E_{1,24}$; the components of

$$E_1 = \{p_0 = p_1 = p_2 = 0\}.$$

Basic strategy: Prove the following *key congruence*:

$$\deg(C/\mathbb{P}^1) \equiv C \cdot \left(\sum_{k=1}^{12} E_{1,j_k} \right) \mod 2 \tag{1}$$

for any **12-tuple** $1 \leq j_1 < \dots < j_{12} \leq 24$.

This will imply the theorem: We would get that

$$C \cdot E_{1,1} \equiv \cdots \equiv C \cdot E_{1,24} \mod 2,$$

and hence that $\deg(C/\mathbb{P}^1)$ is even.

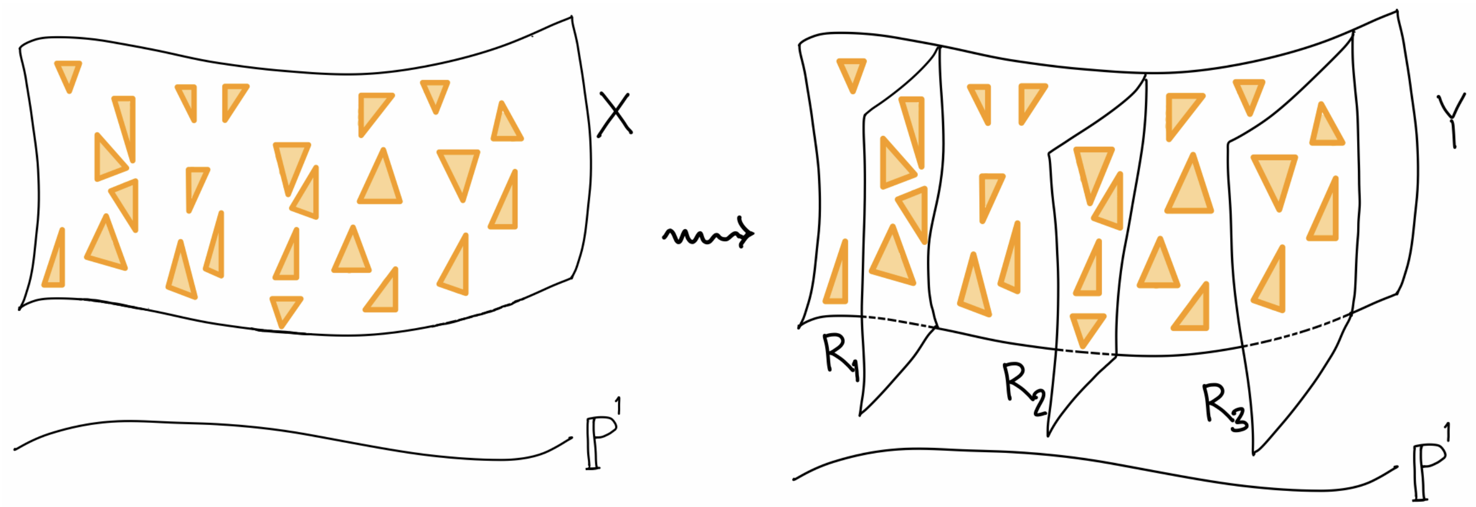
We want to prove that

$$\deg(C/\mathbb{P}^1) \equiv C \cdot \left(\sum_{j=1}^{12} E_{1,j_k} \right) \mod 2 \tag{2}$$

1. **Monodromy argument**: Reduce to proving (2) for *some* 12-tuple $j_1 < \ldots < j_{12}$.
2. **Specialization argument**: Prove (2) for some $(j_1, \ldots, j_{12})$ by analyzing a certain degeneration of X.

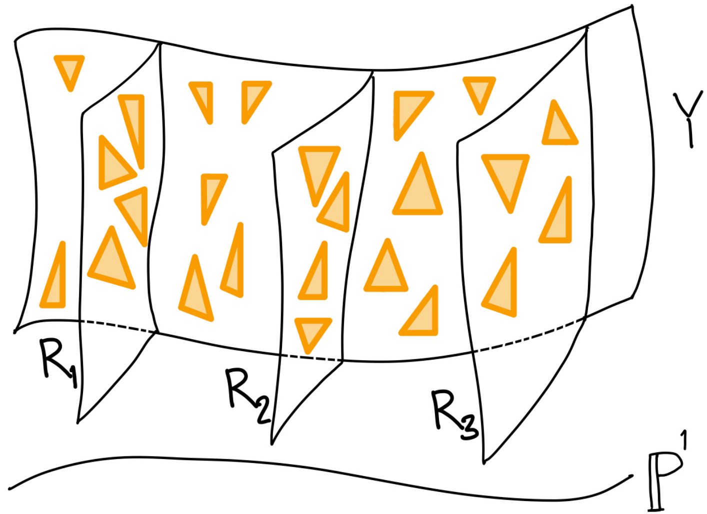
Here is the degeneration:

$$M = \begin{pmatrix} sp_0 + \epsilon r_0 & (s-t)p_1 + \epsilon r_1 & (s+t)p_2 + \epsilon r_2 \\ stq_0 + \epsilon s_0 & t(s-t)q_1 + \epsilon s_1 & t(s+t)q_2 + \epsilon s_2 \end{pmatrix}$$



The special fiber over $\epsilon = 0$ is a union

$$Y \cup R_1 \cup R_2 \cup R_3$$



- Y is the previous Enriques surface fibration with 12 planes $E_{1,j_1}, \ldots, E_{1,j_{12}}$
- On Y, we know that

$$\deg(C/\mathbb{P}^1) \equiv C \cdot \left(\sum_{k=1}^{12} E_{1,j_k} \right) \mod 2 \tag{3}$$

The main congruence (2) follows from this.

Some positive results

Recall:

$Z^4(X) = H^{2,2}(X,\mathbb{Z})/\langle alg \rangle$ are birational invariants
$Z_2(X) = H^{n-1,n-1}(X,\mathbb{Z})/\langle alg \rangle$ (and $=0$ for rational varieties)

Thm (Voisin, Totaro) The IHC holds for

- uniruled 3-folds
- 3-folds satisfying either
  - $K_X = \mathcal{O}_X$ and $H^2(\mathcal{O}_X) = 0$
  - $\kappa = 0$ and $H^0(K_X) \neq 0$.

Basic idea: Let $S \in |mH|$ be smooth + generic H ample, $m \gg 0$

Lefschetz hyperplane theorem:

$$i_*: H_2(S,\mathbb{Z}) \twoheadrightarrow H_2(X,\mathbb{Z}) \quad \text{is surjective}$$

Variation of HS-argument $\Rightarrow$ $\exists$ enough deformations of S such that every class in $H_2(X,\mathbb{Z})$ is i_* of a $(1,1)$-class on S_t which is algebraic!

Q: What about 4-folds?

The IHC already fails on uniruled 4-folds:

$$X = \mathbb{P}^1 \times Y$$

where Y is Kollár's example.

Schreieder: $\exists$ unirational 4-fold X s.t. $Z^4(X) \neq 0$

no known explicit description of a non-algebraic class..

It is not known whether $Z^6(X)$ can be $\neq 0$ for unirational / rationally connected 4-folds.

S. Schreieder

Main result of today:

Theorem The IHC holds for cubic 4-folds

≥ 3 known proofs:

(1) Via Normal functions (Voisin)

(2) Via the variety of lines (Mongardi - O.)

(3) Via derived categories (BLMNP / Perry)

The IHC holds for Calabi-Yau 3-folds.

Q Does the IHC hold on Calabi-Yau varieties in dimension $n \geq 4$?

$Z^{2n-2}(X)$

The answer is probably "No" for $K_X = 0$ in general:

The cubic 3-fold is stably irrational $\Rightarrow \frac{\Theta^4}{4!}$ not algebraic on $J^3(X)$, which is an abelian 5-fold

Theorem (Skauli)

Y = smooth toric variety

$X \subset Y$ smooth complete intersection of ample divisors

$$X = H_1 \cap \dots \cap H_k$$

of dimension ≥ 3, such that $-K_X$ is nef.

Then the IHC holds on X in degree $2n-2$.

In fact $H_2(X, \mathbb{Z})$ is generated by rational curves.

This is the main source of examples of Calabi-Yau varieties.

($\sim$500,000,000 examples in dimension 3)

Here $H_2(X, \mathbb{Z}) \xrightarrow{\sim} H_2(Y, \mathbb{Z})$ by Lefschetz HT

Casagrande: $H_2(Y, \mathbb{Z})$ is generated by classes of contractible classes

$\gamma \in H_2(Y, \mathbb{Z})$ is contractible $\iff$ $\exists$ toric morphism $\pi : Y \longrightarrow Z$ with connected fibers s.t. $\pi(C) = pt \iff [C] \in \mathbb{Q}_{>0} \cdot \gamma$.

Idea One shows that for each contraction $\pi : Y \longrightarrow Z$ there is a curve $C \subset X$ pushing forward to the contractible class on Y.

This uses very geometric arguments:

ex $\ell \subset \mathbb{P}^4$ $\qquad \ell = \{x_0 = x_1 = x_2 = 0\}$ torus-invariant line in $\mathbb{P}^4$

$Y = Bl_\ell \mathbb{P}^4$ $\qquad$ toric 4-fold of Picard number 2.

$$H^2(Y, \mathbb{Z}) = \mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot E \qquad H \cdot h = 1 \qquad H \cdot e = 0$$

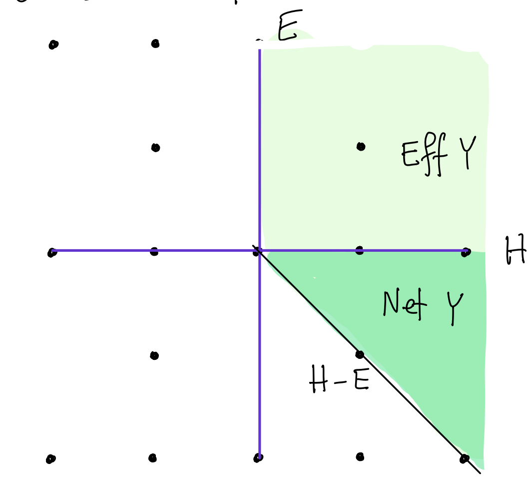
$$H_2(Y, \mathbb{Z}) = \mathbb{Z} \cdot h \oplus \mathbb{Z} \cdot e \qquad E \cdot h = 0 \qquad E \cdot e = -1$$

$-K_Y = 5H - 2E$ $\qquad$ base-point free, ample $\Rightarrow Y$ is Fano

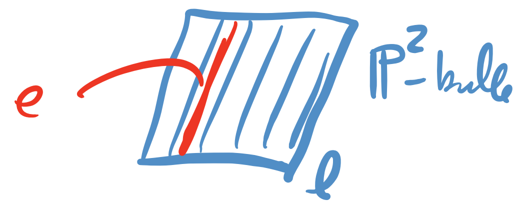
$X \in |-K_Y|$ $\qquad$ smooth Calabi-Yau 3-fold

$X =$ strict transform of a quintic hypersurface containing ℓ with multiplicity 2.

Cones on Y:



$$\begin{tikzcd}[column sep=small] & X \subset Y \arrow[dl, "\text{blow up}"', "H"] \arrow[dr, "\mathbb{P}^2\text{-bundle}", "H-E"'] & \\ \mathbb{P}^4 \arrow[rr, dashed, "\text{projection from } \ell"'] & & \mathbb{P}^2 \end{tikzcd}$$



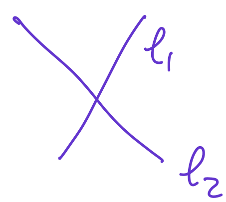
Claim 1 X contains a curve with class e

E = exc. divisor $\cong \mathbb{P}^2 \times \mathbb{P}^1$

Restriction $X|_E$ = $(2,3)$-divisor on $\mathbb{P}^2 \times \mathbb{P}^1$

= conic bundle over $\mathbb{P}^1$

Some fibers of $X|_E \longrightarrow \mathbb{P}^1$ split as



$\rightsquigarrow$ ℓ_1 and ℓ_2 give a curve of class e.

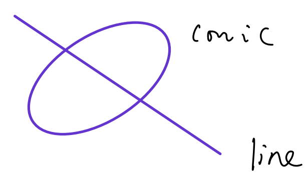
Claim 2 X contains a curve of class $h-e$.

The $\mathbb{P}^2$-bundle $p: Y \longrightarrow \mathbb{P}^2$ restricts to an elliptic fibration

$$p|_X : X \longrightarrow \mathbb{P}^2$$

Ampleness argument $\Rightarrow$ $p|_X$ is non-isotrivial

$\Rightarrow$ some fibers split as



This line has class $h-e$ $\Rightarrow$ DONE.

The general argument uses

- ampleness / positivity
- existence of lines on hypersurfaces
- toric geometry

B. Skauli

See Skauli's paper for more details.

Irreducible symplectic varieties

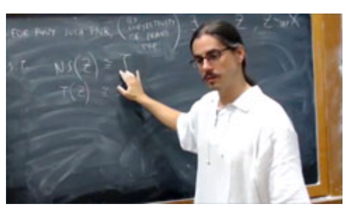
Theorem Let X be a projective irreducible holomorphic symplectic (IHS) variety of $K3^{[n]}$- or generalized Kummer type. Then the integral Hodge conjecture for 1-cycles holds on X, i.e.

$$Z_2(X) = 0.$$

In fact: The semigroup of effective curve classes is generated (over $\mathbb{Z}$) by classes of rational curves.

Corollary The IHC holds for a smooth cubic 4-fold.

In fact, the proof shows that $CH^2(X) = H^{2,2}(X,\mathbb{Z})$ (Bloch – Srinivas) is generated by classes of rational surfaces.



G. Mongardi

Proof of corollary

F is a 4-fold of $K3^{[2]}$-type

$X \subset \mathbb{P}^5$ a cubic 4-fold

$F = F(X) =$ Fano variety of lines on X

$P \subset F \times X$ the incidence correspondence

$\mathbb{P}^1$-bundle

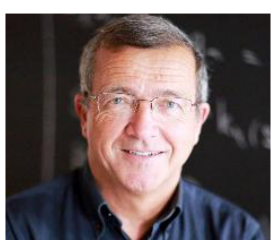
$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"] & X \subset \mathbb{P}^5 \\ F \subset Gr(2,6) & \end{tikzcd}$$

$P = \{ ([\ell], x) \mid x \in \ell \}$

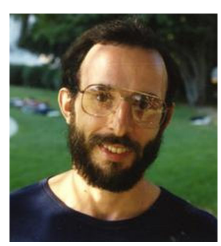
Beauville-Donagi ($\sim$ 1985) The Abel-Jacobi map

$$\alpha = q_* p^* : H^6(F, \mathbb{Z}) \longrightarrow H^4(X, \mathbb{Z})$$

is an isomorphism.



A. Beauville



R. Donagi

$\rightsquigarrow$ any integral Hodge class $\in H^{2,2}(X, \mathbb{Z})$ is of the form

$$q_* p^*(\Gamma)$$

Γ is homologous to a sum of rational curves

for some class $\Gamma \in H^{3,3}(F, \mathbb{Z})$

$\rightsquigarrow$ it is algebraic

Rmk If Γ is a rational curve on F, then $p^*\Gamma \subset P$ is a $\mathbb{P}^1$-bundle over Γ

$\rightsquigarrow$ $q_* p^* \Gamma$ is a rational surface on X.

Varieties of $K3^{[n]}$-type

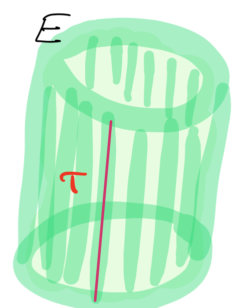
S K3 surface

$X = S^{[n]}$ = Hilbert scheme of n points on S

The Hilbert-Chow morphism

$$HC : S^{[n]} \longrightarrow S^{(n)}$$

induces decompositions



$$H^2(X, \mathbb{Z}) = H^2(S, \mathbb{Z}) \oplus \mathbb{Z} B$$

$$H_2(X, \mathbb{Z}) = H_2(S, \mathbb{Z}) \oplus \mathbb{Z} \tau$$

$B = \frac{1}{2}[E]$, E = exc. divisor of HC

τ = rational curve in a fiber of $HC|_E$

$q : H^2(X, \mathbb{Z}) \longrightarrow \mathbb{Z}$ Beauville-Bogomolov-Fujiki form

$\rightsquigarrow$ the decompositions above are orthogonal wrt q.

Rmk This shows that the IHC holds for 1-cycles for IHS varieties of the form $S^{[n]}$.

Deforming rational curves

X = IHS variety of dim $2n$

$f : \mathbb{P}^1 \longrightarrow X$ non-constant

$R = f(\mathbb{P}^1) \subset X$

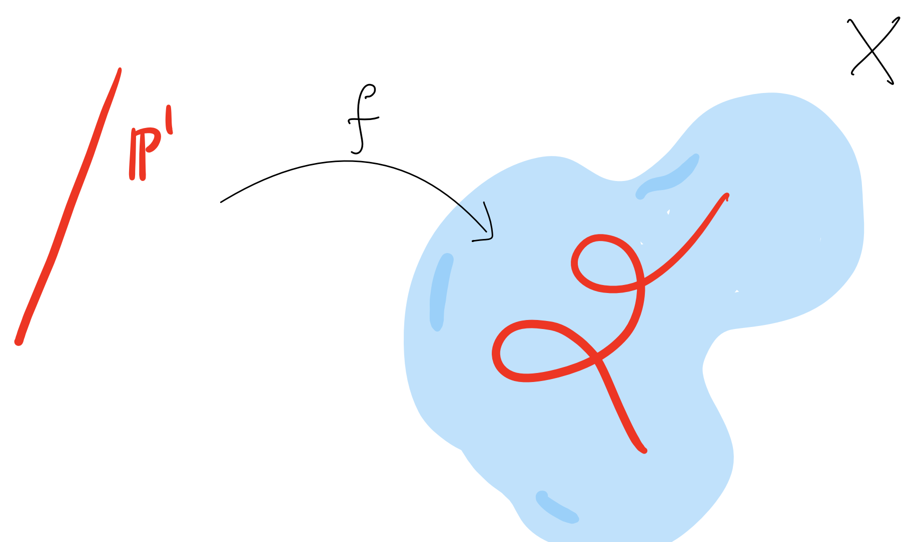
$$\mathrm{Def}(X, [R]) \subset \mathrm{Def}(X) = \text{deformation space of } X$$

$\parallel$ **Hodge locus**

$\{$deformations for which $[R]$ stays Hodge type $(n-1, n-1)\}$

$\overline{M}_0(X, \beta)$ = Kontsevich moduli space of stable maps $f : \mathbb{P}^1 \to X$

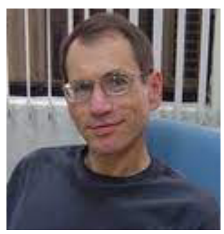
with $f_* [\mathbb{P}^1] = \beta$



Prop (Z. Ran)

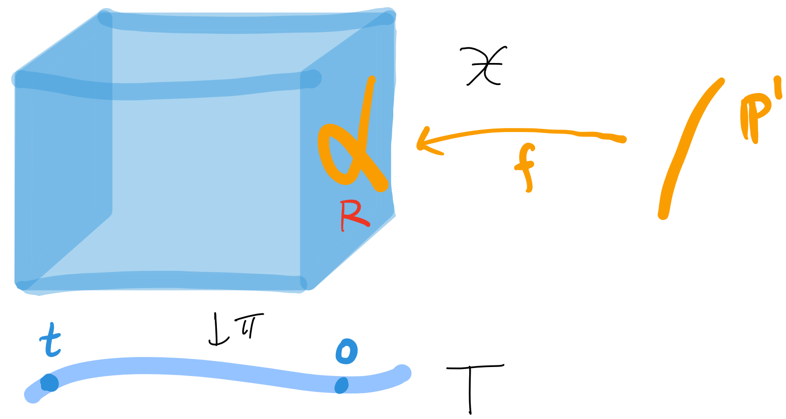
(1) Every component $M \subseteq \overline{M}_0(X, [R])$ **(RR)**
containing $[f]$ has dimension $\geq 2n-2$

(2) If there is a component M of $\dim = 2n-2$
then the curve R deforms in the
Hodge locus $\mathrm{Def}(X, [R])$



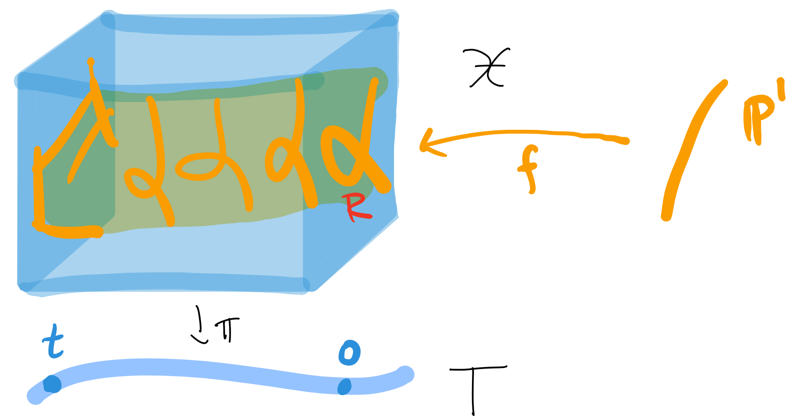
Z. Ran

If $\exists$ component of $\overline{M}_0(X, \beta)$ of dim $2n-2$



$\Downarrow$

The curve R deforms in the fibers of π where [R] stays Hodge.

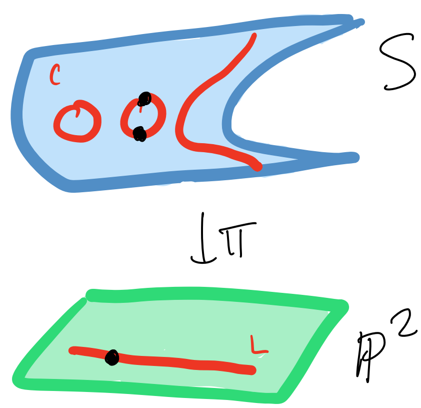


Constructing rational curves on $S^{[n]}$

Want: rational curves $C \subset S^{[n]}$ deforming in a family of dimension exactly $2n-2$. $\hookrightarrow$ then deform

ex (S, H) degree 2 K3 surface

$\pi : S \to \mathbb{P}^2$ double cover



$C \in |H|$ is a genus 2 curve $\rightsquigarrow$ admits a g^1_2

$\rightsquigarrow$ rational curve $R_C \subset S^{[2]}$

via the incidence correspondence

$$I = \{ (p, [Z]) \in S \times S^{[2]} \mid p \in \operatorname{Supp} Z \}$$

$$\begin{tikzcd} & I \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ S & & S^{[2]} \end{tikzcd}$$

The class is given by

$$R_C = H - bT \qquad b \in \mathbb{Z}$$

$\rightsquigarrow$ it is primitive in $H_2(X, \mathbb{Z})$.

$[R_C] = aH - bB$

$[R_C] \cdot H = aH^2 = 2a$

$\shortparallel$

2 = # subschemes in the g^1_2 incident to another H'

R_C moves in a family of dimension exactly 2:

π gives a plane $P \subset S^{[2]}$ containing all of the curves R_C.

P can be contracted by a birational map

$\Longrightarrow$ R_C deforms in a family of dimension exactly $= 2$. $= 2n-2$

$\therefore$ For any IHS (X, D) deformation equivalent to $(S^{[2]}, H)$, the IHC holds.

To get classes like these on any polarized K3 (S,H), we need to consider singular curves on K3s.

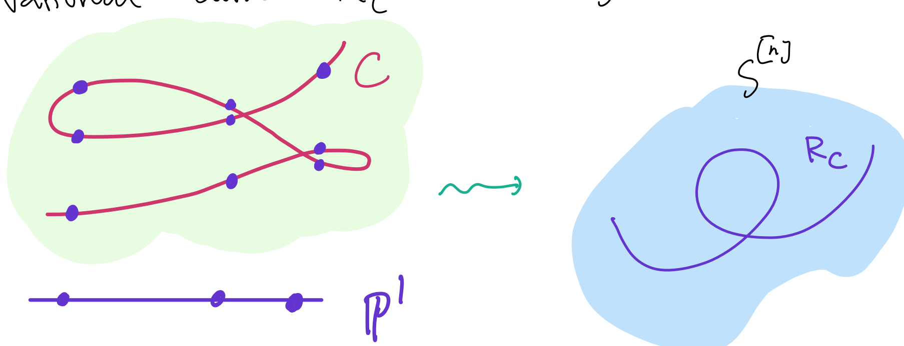
Set-Up

(S,H) K3 of degree $2p-2$

$C \in |H|$ a curve with δ nodes

Given a g^1_n on $\widetilde{C}$ (= normalization of C)

$\rightsquigarrow$ rational curve R_C on $S^{[n]}$



If things are sufficiently generic, we have

$$[R_C] = H - (p - \delta + n - 1)\,T$$

$\rightsquigarrow$ $[R_C]$ is primitive in $H_2(X,\mathbb{Z})$.

A result of Ciliberto–Knutsen

$V^n_{|H|,\delta}$ = Severi variety of curves in $|H|$

$$= \left\{ C \in |H| \;\middle|\; \begin{array}{l} C \text{ has } \delta \text{ nodes} \\ + \text{ the normalization } \tilde{C} \\ \text{admits a } g^1_n \end{array} \right\}$$

Theorem 2.1. *[CK, Thm 0.1] Let (S, H) be a very general primitively polarized $K3$ of genus $p := p_a(H) \geq 2$. Let δ and n be integers satisfying $0 \leq \delta \leq p$ and $n \geq 2$. Then the following statements hold:*

(i) $V^n_{|H|,\delta}$ *is non-empty if and only if*

automatic if $p-\delta \leq 2n-2$

$$\delta \geq \alpha\Big(p - \delta - (n-1)(\alpha+1)\Big), \quad \textit{where } \alpha = \left\lfloor \frac{p-\delta}{2n-2} \right\rfloor. \tag{6}$$

(ii) Whenever non-empty, $V^n_{|H|,\delta}$ *is equidimensional of the expected dimension* $\min\{2n-2, p-\delta\}$*, and a general point on each component corresponds to an irreducible curve with normalization* $\widetilde{C}$ *of genus* $g = p - \delta$*, such that the set of* $\mathfrak{g}^1_n$*'s on* $\widetilde{C}$ *is of dimension* $\max\{0, 2n-2-g\}$.

C. Ciliberto

A. L. Knutsen

Consequences

If $p - \delta \geq 2n-2 \Rightarrow$ $V^n_{|H|,\delta}$ has dim $2n-2$ and the set of g^1_n's on a general curve is 0-dimensional

If $p - \delta \leq 2n-2 \Rightarrow$ $V^n_{|H|,\delta}$ has dim $p-\delta$ and the set of g^1_n's have dim $(2n-2-p+\delta)$

$\therefore$ In both cases the family of rational curves R_C is exactly $(2n-2)$-dimensional

One then shows that for any IHS variety X of $K3^{[n]}$-type, $H_{1,1}(X,\mathbb{Z})$ is generated by classes $[C]$, where C comes from a deformation $(S^{[n]}, H)$ of X.

Open problems

1) Rationally connected varieties

Q Let X be a smooth rationally connected variety $/\mathbb{C}$.
Is $H_2(X, \mathbb{Z})$ generated by classes of curves?
rational curves?

Schoen: Yes, if the Tate conjecture holds for divisors on all sm. proj. surfaces

2) Abelian varieties

Let (A, Θ) be an abelian g-fold
Θ = primitive polarization.

Q Is the minimal cohomology class

$$\frac{\Theta^{g-1}}{(g-1)!} \in H_2(A, \mathbb{Z})$$

algebraic?

relates to rationality problems

Grabowski: If yes, then the IHC holds for 1-cycles on abelian varieties.

the proof uses the Fourier-Mukai transform

3) A question of Diaz

Is there a simply connected 3-fold X so that X admits a non-algebraic torsion class?

4) Griffiths – Harris conjecture

Does the IHC hold for very general hypersurfaces $X \subseteq \mathbb{P}^4$ of degree $d \geq 6$?