STABLE RATIONALITY OF SOME COMPLETE INTERSECTIONS IN THE DEGREE FIVE GRASSMANNIAN

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We work over the complex numbers and consider the Grassmannian $\mathrm{Gr}(2,5)$ embedded in $\mathbb{P}^9$ by the Plücker embedding. In this note, we will be interested in the following instance of the *rationality problem*:

Theorem 0.1. *The intersection of* $\mathrm{Gr}(2,5)$ *with a very general hypersurface of degree at least 3 is not stably rational.*

The statement is optimal in the sense that hypersurfaces of degree 1 and 2 in $\mathrm{Gr}(2,5)$ are rational. The cases of degree 3 and 4 gives new classes of Fano 5-folds where irrationality was not previously known.

For complete intersections, we have

Proposition 0.2. *A very general complete intersection of two quadrics in* $\mathrm{Gr}(2,5)$ *is stably irrational.*

A very general complete intersection of a cubic hypersurface and a hyperplane in $\mathrm{Gr}(2,5)$ *is stably irrational.*

While these complete intersections are perhaps not the most studied among Fano varieties, the interest in Theorem 0.1 lies in the technique rather than the statement itself. Many important classes in the rationality problem concerns hypersurfaces of varieties different from projective space. In fact, the motivation for Theorem 0.1 comes from studying Gushel–Mukai varieties, which are quadric hypersurfaces in a linear section of the Grassmannian $\mathrm{Gr}(2,5)$.

The paper [5] introduced a technique for proving that hypersurfaces in toric varieties are irrational, based on the motivic volume formula of Nicaise–Shinder [7]. The main idea is that given a degeneration $\mathcal{X} \to B$, one can deduce stable irrationality of the geometric generic fiber $\mathcal{X}_{\overline{k(B)}}$ from the components in the special fiber $\mathcal{X}_k$ and their intersections. We recall this method in Section 1.

The proof of Theorem 0.1 roughly proceeds in three steps:

(1) Write down a toric degeneration $\mathscr{G} \to \mathbb{A}^1$ with generic fiber being the Grassmannian $\mathrm{Gr}(2,5)$ in $\mathbb{P}^9$, and a general Cartier divisor $\mathcal{X} \subset \mathscr{G}$, flat over $\mathbb{A}^1$.
(2) Blow-up $\mathscr{G}$ in the special fiber so that, $\widetilde{\mathscr{G}}$, or rather, the strict transform $\widetilde{\mathcal{X}}$, gives a family $\widetilde{X} \to \mathbb{A}^1$ which is semistable.
(3) Analyze the special fiber $\widetilde{X}_0$, and compute the motivic volume. Finally deduce that the geometric fiber is not stably rational.

While this is a rather general method, which applies to other ambient varieties, it is limited by the fact that the steps (2) and (3) usually require many computations (which are best performed using a computer).

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1. The motivic volume of Nicaise–Shinder

Here we recall the main birational obstruction of [5], based on the motivic volume formula of Nicaise–Shinder [NS].

For a field F, we let SB_F denote the set of stable birational equivalence classes of integral F-varieties. For a variety X, we write $[X]_{\mathrm{sb}}$ for its equivalence class. We will work in $\mathbb{Z}[\mathrm{SB}_F]$, the free abelian group on the set SB_F. Elements of $\mathbb{Z}[\mathrm{SB}_F]$ are formal sums of the form

$$a_1[X_1]_{\mathrm{sb}} + \ldots + a_r[X_r]_{\mathrm{sb}}$$

for integers $a_1, \ldots, a_r$.

We will work over the field of Puiseux series

$$K = \mathbb{C}\{\{t\}\} = \bigcup_{m>0} \mathbb{C}((t^{1/m}))$$

and its valuation ring

$$R = \bigcup_{m>0} \mathbb{C}[[t^{1/m}]].$$

We consider families $\mathcal{X} \to \operatorname{Spec} R$, and want to compare the rationality properties of the generic fiber $\mathcal{X}_K$, to that of the special fiber $\mathcal{X}_{\mathbb{C}}$. Note that $\mathcal{X}_{\mathbb{C}}$ may have several irreducible components, so it makes most sense to perform this comparison in $\mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$. Indeed, the *motivic volume* is a map $\mathrm{Vol} : \mathbb{Z}[\mathrm{SB}_K] \to \mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$.

It suffices to define the motivic volume on proper R-schemes $\mathcal{X}$ which are *strictly semi-stable*, i.e., $\mathcal{X}_{\mathbb{C}}$ is a reduced simple normal crossing divisor on $\mathcal{X}$. In the formula (1.2) below, $\mathcal{X}$ will be a proper strictly semi-stable R-scheme, and we decompose special fiber into irreducible components

$$\mathcal{X}_{\mathbb{C}} = \sum_{i \in I} X_i. \tag{1.1}$$

Theorem (Nicaise–Shinder)**.** *There exists a unique ring homomorphism*

$$\mathrm{Vol} : \mathbb{Z}[\mathrm{SB}_K] \to \mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$$

such that, for any $\mathcal{X}$ as above,

$$\mathrm{Vol}([\mathcal{X}_K]_{\mathrm{sb}}) = \sum_{\emptyset \neq J \subseteq I} (-1)^{|J|-1} [X_J]_{\mathrm{sb}} \tag{1.2}$$

where $X_J = X_{j_1} \cap \ldots \cap X_{j_r}$.

Note that Vol sends $[\operatorname{Spec} K]_{\mathrm{sb}}$ to $[\operatorname{Spec} \mathbb{C}]_{\mathrm{sb}}$. This simple observation gives an obstruction to stable rationality:

Obstruction. *If $\mathcal{X}/R$ is a family such that the alternating sum* (1.2) *does not cancel out to* $[\operatorname{Spec} \mathbb{C}]$ *in* $\mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$*, then the generic fiber $\mathcal{X}_K$ is not stably rational.*

The power of the method comes from the fact that the irrationality of $\mathcal{X}_K$ can often be deduced from that of the strata X_J, which are typically of smaller dimension. Thus the technique is a way to get more milage out of known irrationality results.

To use this obstruction in practice, one therefore needs two main inputs: (i) a source of interesting degenerations $\mathcal{X} \to R$ and (ii) irrationality of known lower-dimensional varieties.

Example 1.1. Suppose the special fiber $\mathcal{X}_{\mathbb{C}}$ consists of two components, X_0 and X_1, intersecting along X_{01}. The motivic volume takes the form

$$\mathrm{Vol}([\mathcal{X}_K]_{\mathrm{sb}}) = [X_0]_{\mathrm{sb}} + [X_1]_{\mathrm{sb}} - [X_{01}]_{\mathrm{sb}}. \tag{1.3}$$

From this, we deduce that either of the following conditions guarantee that the generic fiber $\mathcal{X}_K$ is not stably rational:

i) Exactly one of X_0, X_1, X_{01} is stably irrational.
ii) X_0 and X_1 are both stably irrational.
iii) X_0 and X_{01} are stably irrational, but they are not stably birational to each other.
iv) X_0, X_1, X_{01} are all stably irrational.

Proposition 1.2. *Let X be a smooth complex projective variety and let M be a base-point free divisor on X. Suppose that $M \sim D + D'$ where D, D' are two smooth divisors on X so that also $D \cap D'$ is smooth. Then if either*

i) Exactly one of D, D' and $D \cap D'$ is stably irrational.
ii) D and D' are both stably irrational
iii) D and $D \cap D'$ are stably irrational, but not stably birational to each other
iv) D, D' and $D \cap D'$ are all stably irrational.

Then a general divisor in $|M|$ is not stably rational.

Proof. Let $x \in H^0(X, \mathcal{O}_X(D))$, $y \in H^0(X, \mathcal{O}_X(D'))$ define the two divisors and let $z \in H^0(X, \mathcal{O}_X(M))$ be a general section. Consider the scheme

$$\mathcal{X} = V(tz - xy) \subset X \times_{\mathbb{C}} \operatorname{Spec} R.$$

As an R-scheme, $\mathcal{X} \to \operatorname{Spec} R$ is proper, but not strictly semi-stable. However, one checks using local charts that the blow-up $\widetilde{\mathcal{X}}$ along the smooth codimension 4 subscheme $Z = V(t, x, y, z)$ is semistable. In $\widetilde{\mathcal{X}}$, the special fiber $\widetilde{\mathcal{X}}_{\mathbb{C}}$ consists of three components E_0, E_1, E_2, so that

- E_0 (resp. E_1; E_2) is birational to D (resp. D'; $Z \times \mathbb{P}^2$).
- $E_0 \cap E_1$ (resp. $E_0 \cap E_2$; $E_1 \cap E_2$) is isomorphic to $D \cap D'$ (resp. $Z \times \mathbb{P}^1$; $Z \times \mathbb{P}^1$)
- $E_0 \cap E_1 \cap E_2$ is isomorphic to Z.

Thus the motivic volume takes the form

$$\begin{aligned}
\mathrm{Vol}(\mathcal{X}_K) &= [E_0]_{\mathrm{sb}} + [E_1]_{\mathrm{sb}} + [E_2]_{\mathrm{sb}} - [E_{01}]_{\mathrm{sb}} - [E_{02}]_{\mathrm{sb}} - [E_{12}]_{\mathrm{sb}} + [E_{012}]_{\mathrm{sb}} \\
&= [D]_{\mathrm{sb}} + [D']_{\mathrm{sb}} + [Z]_{\mathrm{sb}} - [D \cap D']_{\mathrm{sb}} - [Z]_{\mathrm{sb}} - [Z]_{\mathrm{sb}} + [Z]_{\mathrm{sb}} \\
&= [D]_{\mathrm{sb}} + [D']_{\mathrm{sb}} - [D \cap D']_{\mathrm{sb}}.
\end{aligned}$$

If we are in any of the cases i)–iv), this expression is not equal to $[\operatorname{Spec} \mathbb{C}]$ in $\mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$. □

Proof of Proposition 0.2. For the intersection of two quadrics, we fix very general polynomials L_1, L_2, Q_1, Q_2 of degrees 1,1,2,2 respectively. Now apply the previous proposition to $X = \mathrm{Gr}(2,5) \cap V(Q_1)$, $D = V(L_1)$, $D' = V(L_2)$. Then D and D' are Gushel-Mukai 4-folds, while $D \cap D'$ is a Gushel-Mukai 3-fold. The latter is known to be stably irrational by [2]. It follows that we are either in case *i)* or *iv)* in Proposition 1, and we conclude.

In the second case, we fix very general polynomials H, L, Q of degrees 1,1,2 respectively. We apply the proposition to $X = V(H)$, $D = V(L)$, $D' = V(Q)$. As above, the intersection $D \cap D'$ is a Gushel-Mukai 3-fold (stably irrational); $D = V(H, L)$ is rational; and $D' = V(H,Q)$ is a Gushel-Mukai 4-fold. If D' is stably rational, we are done by the previous proposition. If D' is stably irrational, we need to show that it is not stably birational to $D \cap D'$. But this follows because we may vary the quadric Q and the linear form L, keeping $V(Q,L)$ fixed, so that D' becomes rational. Indeed, we may for instance choose X_0 to contain a 2-plane of the form $\mathbb{P}(V_1 \wedge V_4) \cap \mathrm{Gr}(2,5)$ (this is a codimension 2 condition in the moduli space of Gushel-Mukai 4-folds), and in that case it is rational. It follows that there is variation among the stable birational types of the D''s that contain the (fixed) $D \cap D'$ containing it, and hence D' is not stably birational to $D \cap D'$ for generic choices of the coefficients. □

Remark 1.3. In general, producing a semi-stable model often leads to many blow-ups which which are hard to analyze. An important point is that the formula (1.2) also applies when $\mathcal{X}$ is *strictly toroidal* (see [6]). This condition is much more flexible, and reduces the computations substantially.

2. A toric degeneration of the degree 5 Grassmannian

Consider the Grassmannian $\mathrm{Gr}(2,5)$ embedded in $\mathbb{P}^9$ by the Plücker embedding. We choose homogeneous coordinates $x_0, \dots, x_9$ on $\mathbb{P}^9$ and consider the $\mathbb{C}^*$-action given by scaling the x_0 and the x_9 coordinate. This produces a family $\mathscr{G} \to \mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[t]$ with generic fiber isomorphic to $\mathrm{Gr}(2,5)$ over $\mathbb{C}(t)$. Explicitly, $\mathscr{G}$ is defined by the five 4×4-minors of the matrix

$$M = \begin{pmatrix} 0 & x_0\,t & x_1 & x_2 & x_3 \\ -x_0\,t & 0 & x_4 & x_5 & x_6 \\ -x_1 & -x_4 & 0 & x_7 & x_8 \\ -x_2 & -x_5 & -x_7 & 0 & x_9\,t \\ -x_3 & -x_6 & -x_8 & -x_9\,t & 0 \end{pmatrix}$$

The special fiber over $0 \in \mathbb{A}^1$ is defined by the Pfaffians defined by setting $t = 0$ in the matrix M. Explicitly, the fiber is defined by the ideal

$$(x_6x_7 - x_5x_8, x_3x_7 - x_2x_8, x_3x_5 - x_2x_6, x_3x_4 - x_1x_6, x_2x_4 - x_1x_5)$$

The fiber $\mathscr{G}_0$ is irreducible, and toric, as its defining ideal is generated by binomial equations. Explicitly, $\mathscr{G}_0$ can be realized as the Zariski closure of the image of the map

$$\begin{array}{rcl} \phi : (\mathbb{C}^*)^6 & \to & \mathbb{P}^9 \\ (w_1, \dots, w_6) & \mapsto & (w_0, w_1, w_1w_4, 1, w_4, w_5, w_6, w_4w_6, w_5w_6, w_9) \end{array}$$

Looking at the monomials defining ϕ, we see that $\mathscr{G}_0$ is defined as a projective toric variety by a polytope $P \subset \mathbb{R}^6$ given by the convex hull of the colums of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

This means that a general hypersurface of degree n in $\mathbb{P}^9$ pulls back to a polynomial p in the $w_0, \dots, w_6$ of degree $2n$ with Newton polytope nP.

As in [5], we define a polytope $\Delta \subset \mathbb{R}^n$ to be *stably rational* if a very general polynomial with Newton polytope Δ defines a stably rational hypersurface of the torus $(\mathbb{C}^*)^n$.

Lemma 2.1. *For $n \geq 3$, the polytope nP is not stably rational.*

Proof. By [5, Theorem 3.14] it suffices to produce a subdivison of nP into smaller polytopes P_i such that (i) the subdivision is regular, i.e., the subdivision associated to of a piecewise linear function and (ii) one polytope P_0 is stably irrational, and not contained in the boundary ∂P, while all other polytopes are stably rational.

To find a stably irrational subpolytope, we use the *Hassett–Pirutka–Tschinkel quartic*

$$F = xyu^2 + xv^2 + yw^2 + (x^2 + y^2 + 1 - 2(xy + x + y))$$

The Newton polytope of F is given by the convex hull of the column of the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix}$$

and this is stably irrational by [1]. If we define the linear map $\iota : \mathbb{R}^5 \to \mathbb{R}^6$ by $(t_1, t_2, t_3, t_4, t_5) \mapsto (t_5, t_4, t_1, t_5, t_2, t_3)$, one checks that $\Delta_{\mathrm{HPT}} := \iota(\Delta_F)$ is contained in $3P$ (and hence all nP for $n \geq 3$). The subpolytope $\Delta_{\mathrm{HPT}} \subset nP$ is not contained in the boundary of $3P$.

To produce the desired subdivision of $3P$, the easiest choice is to take the regular subdivision associated to the convex function

$$\psi(z) = \max_{v \in \Delta_{\mathrm{HPT}}} \|z - v\|^2.$$

(Compare this with Theorem 6.3 in [5]). Using Macaulay2, one checks that the resulting subdivision contains 14 maximal polytopes, and all polytopes in $\mathscr{P}$ except Δ_{HPT} have lattice width 1, and hence are rational. ☐

Remark 2.2. The last step of the proof can be bypassed using the results of [3]. Indeed, [3, Theorem 3.16] implies that any polytope of dimension 5 which contains Δ_{HPT} is stably irrational (one needs that Δ_{HPT} satisfies condition (M) in that paper, but this follows from [3, Example 3.21]).

If the family $\mathscr{G} \to \mathbb{A}^1$ were strictly semistable (or strictly toroidal), we would have been in position to conclude the proof of Theorem 0.1 here. Indeed, $\mathscr{G} \to \mathbb{A}^1$ induces a family $\mathcal{X} \to \mathbb{A}^1$ where the generic fiber is a hypersurface in $\mathrm{Gr}(2,5)$, and where the central fiber $\mathcal{X}_0$ is a hypersurface in the toric variety $\mathscr{G}_0$. But this hypersurface is stably irrational, by Lemma 2.1. Thus by [7, Section 4] the generic fiber $\mathcal{X}_K$ and hence the very general hypersurface in $\mathrm{Gr}(2,5)$ is stably irrational as well.

Unfortunately, the family $\mathscr{G} \to \mathbb{A}^1$ is not strictly semistable, so a more detailed analysis of the singularities is required.

2.1. **Singularities.** $\mathscr{G}$ has a singular locus consisting of three strata

$$\begin{aligned} S_1 &= V(t, x_2, x_3, x_5, x_6, x_7, x_8, x_9), \\ S_2 &= V(t, x_0, x_1, x_2, x_3, x_4, x_5, x_6), \\ S_3 &= V(t, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \end{aligned}$$

Geometrically, S_1 and S_2 are two disjoint planes in $\mathbb{P}^9$ and S_3 is a line intersecting each of them in a point. Blowing up S_3, and then the strict transforms $\widetilde{S}_1$ and $\widetilde{S}_2$, one obtains a new family $\widetilde{\mathscr{G}} \to \mathbb{A}^1$. Now the special fiber consists of four components:

(1) The strict transform $\widetilde{\mathscr{G}_0}$.
(2) Exceptional divisors E_1, E_2, E_3 over the centers S_1, S_2, S_3 respectively.

One checks by explicit equations (e.g., by Macaulay2) that:

- Each of the E_i are birational to a product $S_i \times \mathbb{P}^{5-\dim S_i}$.
- The total space $\widetilde{\mathscr{G}}$ has a singular locus which map to the intersections $S_1 \cap S_3$ and $S_2 \cap S_3$.

This means that the family $\widetilde{\mathcal{X}} \subset \widetilde{\mathscr{G}}$ is strictly semistable. Moreover, the restrictions $e_i = \widetilde{\mathcal{X}} \cap E_i$ are in fact disjoint.

Consider the base change $\mathcal{Y} \to \operatorname{Spec} R$ of $\widetilde{\mathcal{X}}$ to the valuation ring R, with fraction field K and residue field $\mathbb{C}$. The motivic volume formula (1.2) is an alternating sum of terms $[\widetilde{\mathcal{X}_0}]$, $[\widetilde{e_i}]$, $[\widetilde{\mathcal{X}_0} \cap \widetilde{e_i}]$ (the other intersections do not appear, because the e_i are disjoint). Here there are a few cancellations as for each $i = 1, 2, 3$, e_1 and $\widetilde{\mathcal{X}}_0 \cap e_i$ are stably birational (they are both stably birational to $\mathcal{X} \cap S_i$). Thus, in light of these cancellations, we obtain

$$\operatorname{Vol}(\mathcal{Y}_K) = [\widetilde{\mathcal{X}_0}]_{\mathrm{sb}}$$

This is different from $[\operatorname{Spec} \mathbb{C}]_{\mathrm{sb}}$ by Lemma 2.1, and hence $\mathcal{X}_K$ is geometrically stably irrational as well. This implies that a very general cubic hypersurface in $\mathrm{Gr}(2, 5)$ is stably irrational.

Remark 2.3. As explained in the introduction, the motivation for proving Theorem 0.1. A hypersurface of degree 3 in $\mathrm{Gr}(2, 5)$ degenerates to two components, one linear section of $\mathrm{Gr}(2, 5)$ and one quadric section of $\mathrm{Gr}(2, 5)$. Both of these components are rational. Their intersection however, is a Gushel-Mukai 4-fold. Thus the expected irrationality of Gushel-Mukai 4-folds would imply that cubics in $\mathrm{Gr}(2, 5)$ are irrational.

In fact, the same type of argument gives a quick proof that quartic sections of $\mathrm{Gr}(2, 5)$ are stably irrational: degenerate the quartic to a union of two quadrics. Then the two components X_0 and X_1 are Gushel-Mukai 5-folds (hence rational), while $X_0 \cap X_1$ is stably irrational by Proposition 0.2.

Appendix A. A Macaulay2 session

The following code defines the total space of the degeneration.

```
kk=QQ
R=kk[x0,x1,x2,x3,x4,x5,x6,x7,x8,x9,t,u0,u1,u2,u3,u4,u5,u6,u7,u8]
B=ideal(x0,x1,x2,x3,x4,x5,x6,x7,x8,x9)
U=ideal(u0,u1,u2,u3,u4,u5,u6,u7,u8)
G=Grassmannian(1,4,R)
e={1, 0, 0, 0, 0, 0, 0, 0, 0, 1}

-- the total space of the C^*-action
scale=map(R,R,{t^(e_0)*x0,t^(e_1)*x1,t^(e_2)*x2,t^(e_3)*x3,t^(e_4)*x4,t^(e_5)*x5,...)
I=scale(G)
I=trim saturate(I,t) -- remove components supported along t=0
lim=trim sub(I,t=>0) -- this defines is the flat limit, the special fiber G_0

ss=trim ideal singularLocus I; -- the singular locus of the total space
ss=saturate(ss,B);
ss=radical ss
S=decompose ss -- this has 3 components
```

The singular locus of $V(I)$ consists of the three components $S_0 = V(t, x_0, x_1, x_2, x_3, x_4, x_6, x_7)$ and $S_1 = V(t, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ and $S_2 = V(t, x_3, x_4, x_5, x_6, x_7, x_8, x_9)$

Our aim is to show that the blow-up is generically smooth along the exceptional divisors E_0, E_1, E_2. In other words, we want to show that the image of the singular locus of the blow-up is contained in V of the following ideal:

```
intersections=intersect(S_0+S_1,S_0+S_2,S_1+S_2)

X=(random(3,R) % t)+t*(random(3,R) % t) + t*t*(random(3,R) % t); -- a random cubic in P9 x A1
X=X % U;
dim saturate(X+intersections,B) < 0
```

As the last few lines show, a generic cubic will then avoid the image of the singular locus of the blow-up and thus the induced total space $\widetilde{X}$ will be smooth.

```
mm=matrix{{t, x7, x6, x4, x3, x2, x1, x0},{u0,u1,u2,u3,u4,u5,u6,u7}}
U=ideal(u0,u1,u2,u3,u4,u5,u6,u7);
J=I+minors(2,mm)
decompose oo
J=oo_1 -- J defines the blow-up in P9xP7
```

The next bit checks that the blow-up is has singularities that map into the above intersection. (As we are blowing up a linear space, some of the variables can be eliminated to speed up the computation.)

```
JJ=trim sub(J,u0=>1)
JJJ=eliminate({x7, x6, x4, x3, x2, x1, x0},JJ)
ss=trim ideal singularLocus JJJ;
radical trim(JJ+ss+S_0)
imageOfSings=eliminate({u0,u1,u2,u3,u4,u5,u6,u7},oo)
imageOfSings==S_0
```

The above code is carried out in all the affine charts, and the conclusion is the same in each case: the blow-up is generically smooth along the exceptional divisor E_0.

We finally check that the exceptional divisor E_0 itself is rational, and $E_0 \to S_0$ is birational to a product.

```
E_0=J+S_0
E_0=saturate(E_0,B) -- remove support in V(x0..x9)
E_0=saturate(E_0,U) -- remove support in V(u0..u7)
isPrime E_0 --- So: the exceptional divisor is irreducible
eliminate(u0,E_0)
eliminate(u1,oo) -
```

The output is $V(t, x_7, x_6, x_4, x_3, x_2, x_1, x_0, x_5u_2 - x_8u_4)$ which is clearly a rational variety.

The same procedure is carried out for the blow-up of the other two components.

The Macaulay2 code for the example can be found here:

`https://www.mn.uio.no/math/personer/vit/johnco/papers/Gr25.m2`

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