Tropical Degenerations and Stable Rationality

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(joint work with J. Nicaise)

The paper [NO] gives a general technique for studying rationality problems of hypersurfaces and complete intersections in toric varieties, based on the motivic volume formula of Nicaise–Shinder [NS19]. Here are two of the main applications:

Theorem 1. *A very general complex quartic fivefold is not stably rational.*

Theorem 2. *A very general complex complete intersection of a quadric and a cubic in $\mathbb{P}^6$ is not stably irrational.*

These theorems fit into a long list of results on the rationality problem for hypersurfaces and complete intersections, going back to the works of Clemens–Griffiths, Iskovskih–Manin in the 1970s, and more recently, Colliot-Thelene–Pirutka, Kollár, Schreieder, Totaro, Voisin. In these works, the approaches typically combine specialization arguments with various types of birational invariants (Brauer groups, differential forms, decomposition of the diagonal, etc). The method of [NO] however, seems to be the first instance where one specializes to a union of several components, and deduces irrationality from that of lower-dimensional varieties.

0.1. **Background.** Two varieties X and Y are said to be *stably birational* if $X \times \mathbb{P}^m$ and $Y \times \mathbb{P}^n$ are birational for some positive integers m, n. A variety X is *stably rational* if it is stably birational to a projective space.

For a field F, we let SB_F denote the set of stable birational equivalence classes of integral F-varieties. For a variety X, we write $[X]_{\mathrm{sb}}$ for its equivalence class.

We will work in $\mathbb{Z}[\mathrm{SB}_F]$, the free abelian group on the set SB_F. Elements of $\mathbb{Z}[\mathrm{SB}_F]$ are formal sums of the form

$$a_1[X_1]_{\mathrm{sb}} + \ldots + a_r[X_r]_{\mathrm{sb}}$$

for integers $a_1, \ldots, a_r$. In fact, $\mathbb{Z}[\mathrm{SB}_F]$ is a *ring*, with multiplication defined via the fiber product, i.e., $[X]_{\mathrm{sb}} \cdot [Y]_{\mathrm{sb}} = [X \times_F Y]_{\mathrm{sb}}$.

The motivic volume. We work over the field of Puiseux series

$$K = \mathbb{C}\{\{t\}\} = \bigcup_{m>0} \mathbb{C}((t^{1/m}))$$

and its valuation ring

$$R = \bigcup_{m>0} \mathbb{C}[[t^{1/m}]].$$

In short, we consider families $\mathcal{X} \to \operatorname{Spec} R$, and want to compare the rationality properties of the generic fiber $\mathcal{X}_K$, to that of the special fiber $\mathcal{X}_{\mathbb{C}}$. Note however that $\mathcal{X}_{\mathbb{C}}$ may have several irreducible components, so it makes most sense to perform this comparison in $\mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$. Indeed, the *motivic volume* will be a map $\mathbb{Z}[\mathrm{SB}_K] \to \mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$.

It suffices to define the motivic volume on proper R-schemes $\mathcal{X}$ which are *strictly semi-stable*, i.e., $\mathcal{X}_{\mathbb{C}}$ is a reduced simple normal crossing divisor on $\mathcal{X}$. In the formula (2) below, $\mathcal{X}$ will be a proper strictly semi-stable R-scheme, and we decompose special fiber into irreducible components

$$\mathcal{X}_{\mathbb{C}} = \sum_{i \in I} X_i. \tag{1}$$

Theorem (Nicaise–Shinder). *There exists a unique ring homomorphism*

$$\mathrm{Vol} : \mathbb{Z}[\mathrm{SB}_K] \to \mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$$

such that, for any $\mathcal{X}$ as above,

$$\mathrm{Vol}([\mathcal{X}_K]_{\mathrm{sb}}) = \sum_{\emptyset \neq J \subseteq I} (-1)^{|J|-1} [X_J]_{\mathrm{sb}} \tag{2}$$

where $X_J = X_{j_1} \cap \ldots \cap X_{j_r}$.

Note that Vol sends $[\mathrm{Spec}\, K]_{\mathrm{sb}}$ to $[\mathrm{Spec}\, \mathbb{C}]_{\mathrm{sb}}$. This simple observation gives an obstruction to stable rationality: *If $\mathcal{X}/R$ is a family such that the alternating sum* (2) *does not cancel out to* $[\mathrm{Spec}\, \mathbb{C}]$ *in* $\mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$*, then the generic fiber $\mathcal{X}_K$ is not stably rational.*

Example 3. Suppose the special fiber $\mathcal{X}_{\mathbb{C}}$ consists of two components, X_0 and X_1, intersecting along X_{01}. The motivic volume takes the form

$$\mathrm{Vol}(\mathcal{X}_K) = [X_0]_{\mathrm{sb}} + [X_1]_{\mathrm{sb}} - [X_{01}]_{\mathrm{sb}}. \tag{3}$$

From this, we deduce that either of the following conditions guarantee that the generic fiber $\mathcal{X}_K$ is not stably rational:

i) Exactly one of X_0, X_1, X_{01} is stably irrational.
ii) X_0 and X_1 are both stably irrational.
iii) X_0 and X_{01} are stably irrational, but they are not stably birational to each other.
iv) X_0, X_1, X_{01} are all stably irrational.

Remark 4. The condition of strict semi-stability is quite restrictive, and producing a semi-stable model often leads to many blow-ups which which are hard to analyze. An important point is that the formula (2) also applies when $\mathcal{X}$ is *strictly toroidal* (see [NO21]). This condition is much more flexible, and reduces the computations substantially.

0.2. **Quartic fivefolds.** We are now in position to prove Theorem 1. Let $F \in \mathbb{C}[x_0, \ldots, x_6]$ be a very general homogeneous polynomial of degree 4. Consider the following R-scheme

$$\mathcal{X} = \mathrm{Proj}\, R[x_0, \ldots, x_6, y]/(x_5 x_6 - ty, y^2 - F) \tag{4}$$

where the variable y has weight 2. Note that the generic fiber $\mathcal{X}_K$ is isomorphic to a smooth quartic hypersurface in $\mathbb{P}^6_K$ (inverting t allows us to eliminate y using the first equation). Moreover, $\mathcal{X}$ is strictly toroidal.

The special fiber has two components:

$$\begin{aligned} X_0 &= \mathrm{Proj}\, \mathbb{C}[x_0, \ldots, x_6, y]/(x_5, y^2 - F) \\ X_1 &= \mathrm{Proj}\, \mathbb{C}[x_0, \ldots, x_6, y]/(x_6, y^2 - F). \end{aligned}$$

Note that these are both very general quartic double fivefolds. We do not know whether these are stably rational or not. However, their intersection,

$$X_{01} = \operatorname{Proj} \mathbb{C}[x_0, \ldots, x_4, y]/(y^2 - F),$$

is a very general quartic double fourfold, and thus stably irrational by Hassett–Pirutka–Tschinkel [HPT19]. In any event, we are in either case i) or iv) in Example 3, so we conclude also that $\mathcal{X}_K$, and hence the also the very general quartic fivefold, is stably irrational.

0.3. **Further results.** The paper [NO] gives many other applications to rationality problems. For instance, we give logarithmic bounds for irrationality of complete intersections à la Schreieder, and study hypersurfaces in products of projective spaces.

While the main idea is simple, the challenge is now to write down suitable degenerations where one can apply the obstruction. For this, we utilize the theory of tropical degenerations. This has the advantage that degenerations can be found and studied using combinatorial methods, i.e., finding regular subdivisions of polytopes. For instance, the combinatorial picture for the 2-dimensional analogue of the above degeneration is shown in Figure 1.

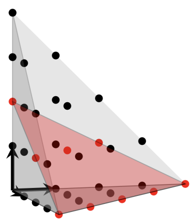

Figure 1. Degenerating a quartic surface into a union of two quartic double surfaces intersecting along a quartic double curve.

In fact, while the family (4) is completely explicit, most of the applications in [NO] require degenerations which are much more involved and harder to write down concretely. For instance, proving irrationality of a $(2, 3)$-divisor in $\mathbb{P}^1 \times \mathbb{P}^4$ involves degenerating $\mathbb{P}^1 \times \mathbb{P}^4$ into a union of 26 toric varieties.

References

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