

Mori dream hypersurfaces in products of projective spaces

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In my talk I presented a few results about the birational structure of hypersurfaces in products of projective spaces. These hypersurfaces are in many respects simple as algebraic varieties, but it turns out that they can have surprisingly complicated behaviour from the viewpoint of birational geometry. For example, a hypersurface of tridegree $(2, 2, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$, has infinite birational automorphism group and the effective cone is rational non-polyhedral.

A natural question is when such hypersurfaces are so-called Mori dream spaces. These varieties were introduced by Hu and Keel in [2] as a class of varieties with good birational geometry properties. By definition, a variety is Mori dream if its Cox ring is finitely generated. This ring is essentially defined as

$$\mathcal{R}(X) = \bigoplus_{D \in \text{Pic}(X)} H^0(X, D).$$

Choosing a presentation for the Cox ring gives an embedding of X into a simplicial toric variety Y such that each small modification of X is induced from a modification of the ambient toric variety Y (see [2]). From this one shows that the Minimal Model Program can be carried out for any divisor and has a combinatorial structure as in the case of toric varieties.

Being a Mori dream space is a relatively strong condition and there are classical examples of varieties that are not. Perhaps the most famous of these is Nagata's counterexample to Hilbert's 14th problem, in which he proves that the blow-up of \mathbb{P}^2 along the base-locus of a general cubic pencil has infinitely many (-1) -curves. This blow-up is clearly not a Mori dream space since each of the (-1) -curves would require a generator of the Cox ring.

When X is a hypersurface in $\mathbb{P}^m \times \mathbb{P}^n$, the Picard number is 2, so this phenomenon can not occur. However, there are other obstructions to having finitely generated Cox ring. Here the main interesting case occurs when $m = 1$. In the case $m, n \geq 2$, it is straightforward to show that the Cox ring of X is quotient of that of $\mathbb{P}^m \times \mathbb{P}^n$ by the defining polynomial, and so X is clearly a Mori dream space. For hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^n$ we have the following:

Theorem. *Let X be a very general hypersurface of bidegree (d, e) in $\mathbb{P}^1 \times \mathbb{P}^n$ and let $H_i = \text{pr}_i^* \mathcal{O}(1)$. Then X is a Mori dream space if and only if it belongs to the following cases:*

- $d \leq n$ in which case the Cox ring has the following presentation

$$\mathcal{R}(X) = k[x_0, x_1, y_0, \dots, y_n, z_1, \dots, z_d]/I$$

where I is generated by $d + 1$ forms of bidegree dH_2 .

- $e = 1$, in which case X is a projective bundle over \mathbb{P}^1 .

In all other cases, we have

$$\overline{\text{Eff}}(X) = \overline{\text{Mov}}(X) = \text{Nef}(X) = \mathbb{R}_{\geq 0}H_1 + \mathbb{R}_{\geq 0}(neH_2 - dH_1),$$

but $\text{Eff}(X)$ is not closed. Hence X is not a Mori dream space.

In each case one can describe the birational structure of X fairly explicitly using the defining polynomial of X . To show that some of the hypersurfaces are not Mori dream spaces, a degeneration argument is used.

As a pleasant by-product we get simple analogues of many classical pathologies in birational geometry:

(i) A surface with $\text{Nef}(X)$, $\text{Eff}(X)$, $\text{Mov}(X)$ all rational polyhedral, but $\text{Eff}(X)$ not closed. This is true for surfaces of large bidegrees in $\mathbb{P}^1 \times \mathbb{P}^2$.

(ii) A rational surface with infinitely many (-1) -curves. The blow-up of \mathbb{P}^2 along the base-locus of two degree e curves embeds as a $(1, e)$ -hypersurface in $\mathbb{P}^1 \times \mathbb{P}^2$. For example, Nagata's example is a $(1, 3)$ -hypersurface.

(iii) A non-ample line bundle with positive intersection numbers with any curve. If X is a hypersurface of bidegree $(3, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$, then the line bundle $L = \mathcal{O}(2H_2 - H_1)$ satisfies $L^2 = 0$ and $L \cdot C > 0$ for every curve $C \subset X$. Such examples were first constructed by Mumford using projective bundles over curves of genus ≥ 2 .

(iv) A Calabi-Yau threefold with infinitely many $(-1, -1)$ -curves. If X has tridegree $(2, 2, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$, then one can consider the two projections $p_{13}, p_{23} : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^2$. These projections are generically $2 : 1$, and so X possesses two pseudoautomorphisms σ_1, σ_2 given by interchanging the two sheets of the double cover. It can be shown that σ_1, σ_2 generate an infinite subgroup of $\text{Bir}(X)$.

(v) A rationally connected variety which is not birational to a log Fano. Finally, using these results, I constructed a counterexample to a question of Cascini and Gongyo [1], which asks whether every rationally connected variety is birational to a log Fano variety. It turns out that many Fano fibrations over \mathbb{P}^1 are not birational to a log Fano, since they are so-called *birationally superrigid*, meaning that they have essentially only one Mori fiber structure.

REFERENCES

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- [3] J. C. Ottem. Birational geometry of hypersurfaces in products of projective spaces. (2013) arXiv:1305.0537

Extremal rays on hyperkähler manifolds and relations to Brill-Noether theory

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(joint work with C. Ciliberto, and with M. Lelli-Chiesa and G. Mongardi)

A hyperkähler manifold is a simply-connected compact Kähler manifold X carrying an everywhere non-degenerate holomorphic 2-form, unique up to scale. In particular, it has even dimension. By the Beauville–Bogomolov decomposition theorem, hyperkähler manifolds form one of the three basic building blocks —