

- [LM09] Lazarsfeld, R., Mustață, M.; *Convex bodies associated to linear series*, Ann. Sci. Ecole Norm. Sup. (4) **42** (2009) 783–835.
- [Mac17] Maclean, C. *Approximable algebras and a question of Chen’s*. Arxiv preprint 1703.01801, 2017.
- [Oko03] Okounkov, A.; *Why would multiplicities be log-concave?* in: The Orbit Method in Geometry and Physics, in: Progr. Math., vol. 213, 2003, pp. 329–347.

Positivity of the diagonal

JOHN CHRISTIAN OTTEM

(joint work with Brian Lehmann)

In algebraic geometry, projective varieties are studied and classified in terms of the positivity of their tangent bundle. The paper [LO17] proposes a parallel viewpoint to this, by studying a projective variety X in terms of the positivity of the diagonal Δ (as a higher codimension cycle) on the self-product $X \times X$. This is motivated by the fact that the normal bundle of Δ is the tangent bundle T_X of X , and one expects there to be interplay between the ampleness properties of T_X (as a vector bundle) and the cycle-type positivity of Δ . This perspective is quite vivid already for curves:

Example 1 (Curves). Let C be a curve and let $\Delta \subset C \times C$ be the diagonal. We have the following table:

| g | type | K_X | Δ |
|----------|----------------|-----------|---|
| 0 | \mathbb{P}^1 | $K_X < 0$ | Δ is ample (it is a (1,1)-divisor on $\mathbb{P}^1 \times \mathbb{P}^1$.) |
| 1 | elliptic | $K_X = 0$ | Δ nef but not big (any effective divisor is nef on an abelian surface, and $\Delta^2 = 0$) |
| ≥ 2 | general type | $K_X > 0$ | Δ negative, can be contracted by the subtraction map $C \times C \rightarrow C - C \subset J(C)$. |

In general, given a variety X , we have a canonical cycle class $[\Delta] \in N_n(X \times X)$, and there are several ways in which this can be ‘positive’ (nef, big, movable, ample,...). Our motivating question is:

How do the positivity properties of the cycle $[\Delta]$ reflect the geometric properties of X ?

Here geometric properties will refer to algebro–geometric properties (e.g., rational curves) as well as topological invariants (e.g., $\pi_1(X)$, $H^k(X, \mathbb{Z}), \dots$). Intuitively, one can expect varieties with Δ positive to be more similar to projective spaces.

1.1. Varieties with big diagonal. Most of the notions of positivity for divisors have analogues for cycles of higher codimension. We define the effective cone $\text{Eff}_k(X)$ as the cone spanned by effective k -cycles, and let $\text{Nef}^k(X)$ be its dual cone, the nef cone. For most of the varieties in this note, numerical and cohomological equivalence coincide, so we will for simplicity think of these cones in $H_{2k}(X, \mathbb{R})$ and $H^{2k}(X, \mathbb{R})$ respectively. We say that a class is *big* if it lies in the interior of $\text{Eff}_k(X)$, and *nef* if it lies in $\text{Nef}^k(X)$. So the question becomes: For which varieties have is $[\Delta]$ big or nef cycle on $X \times X$? Here is a sample theorem:

Theorem 2. *The only surfaces with nef and big diagonal are the projective planes, and the fake projective planes.*

It is interesting to compare this result to Mori’s theorem, which states that the only smooth variety with ample tangent bundle is \mathbb{P}^n . In light of this, the above statement is somewhat surprising, given that these varieties really fall on the opposite sides of the spectrum in the classification; fake projective planes have *anti-ample* tangent bundles and are consequently of general type. So by switching to the perspective of numerical positivity of Δ , we also include varieties with the same cohomological behavior as projective space.

Example 3. Let us verify that Δ_X is nef and big for a (fake) projective space. By the Künneth formula we have $H^{2k}(X \times X) = \bigoplus_{p+q=k} \mathbb{R}(\pi_1^* h^p \cdot \pi_2^* h^q)$ where h is an ample divisor on X . Then the diagonal can be written as a sum

$$\Delta = \sum_{p+q=n} c_{pq} (\pi_1^* h^p \cdot \pi_2^* h^q)$$

with $c_{pq} > 0$. This class is obviously nef and big.

Some examples of fake projective planes are: (i) Odd dimensional quadrics; (ii) the 100 fake projective planes of [CS10]; (iii) the Del Pezzo quintic threefolds V_5 ; and (iv) the Fano threefolds V_{22} . Our results imply that these are in fact all the examples in dimension at most 3: that is, every variety of dimension ≤ 3 with nef and big diagonal is either \mathbb{P}^2 , \mathbb{P}^3 or one of the varieties (i)–(iv).

So having big diagonal should be a quite restrictive condition. How would one prove that a given variety does not have big diagonal? This is equivalent to finding a nef class β having intersection product 0 with Δ ; finding explicit classes like this can be non-trivial, especially given the examples of Debarre–Ein–Lazarsfeld–Voisin [DELV11]. One way of constructing β is via products of divisors. The following lemma is elementary, but turns out to be remarkably effective:

Lemma 4. *Let X be an n -dimensional smooth projective variety admitting a nef class $D \in N^1(X, \mathbb{R})$ such that $D^n = 0$. Then Δ is not big.*

The proof is straightforward: If D is the above divisor, a suitable product of the form $\beta = \pi_1^* D^k \cdot \pi_2^* D \cdot H^{n-k-1}$ (where $k > 0$ and H is ample), is a nef class which dots Δ to zero, and so Δ is not big. (Using a similar argument, one can show that the same statement holds also when $D \in H^{1,1}(X, \mathbb{R})$ is a (possibly non-algebraic) nef cohomology class.)

This lemma already puts strong restrictions on the possible varieties with big diagonal. For instance, varieties with big diagonal admit no maps to lower dimensional varieties. This implies in particular that the only smooth uniruled surface with big diagonal is \mathbb{P}^2 . The main geometric implication of the diagonal being big is the following:

Theorem 5. *Let X be a smooth projective variety with big diagonal. Then $h^{k,0}(X) = 0$ for all $k > 0$.*

The proof is inspired by a proof of Voisin using the Hodge–Riemann relations to bound the effective cone. To give some details, we fix a Kahler form ω on $X \times X$ and assume that we have a non-zero closed $(k, 0)$ -form α on X . Now, the class

$$\beta = (-1)^{\frac{k(k-1)}{2}} i^k (\pi_1^* \alpha - \pi_2^* \alpha) \cup (\pi_1^* \bar{\alpha} - \pi_2^* \bar{\alpha}) \cup \omega^{n-k}$$

is represented by a non-zero (n, n) -form on $X \times X$, which by construction restricts to 0 on the diagonal. Moreover, since $(k, 0)$ -classes are automatically primitive, the Hodge–Riemann relations imply that the class of β is nef, which contradicts the bigness of Δ .

1.2. Nefness of Δ . Also nefness of Δ imposes strong restrictions on the geometry of X . The primary examples here are the varieties with nef tangent bundle (which automatically have nef diagonals). Such varieties are expected to have very special properties (e.g., Campana–Peternell conjecture that they should be homogeneous space-fibrations over abelian varieties). As for bigness, to prove that Δ is not nef, one has to produce subvarieties of $X \times X$ that intersect Δ non-transversely. Some ways of producing such subvarieties include: (i) products of divisors; (ii) subvarieties linked to the diagonal via such products; (iii) graphs of (birational) automorphisms; and (iv) Δ itself. The latter condition implies in particular that a variety with nef diagonal must have non-negative Euler characteristic. One particularly useful criterion is the following:

Proposition 6. *Let X be a smooth variety. If Δ_X is nef, then every pseudoeffective class on X is nef.*

For instance, if S is a smooth surface with Δ then S is minimal (since (-1) -curves are not nef). By combining this result with Theorem 5, we obtain:

Corollary 7. *The only smooth projective surfaces with big and nef diagonal are the projective plane and fake projective planes.*

Indeed, if Δ is big and nef, then $h^{1,0}(X) = h^{2,0}(X) = 0$, and $\rho = h^{1,1}(X)$. If $\rho > 1$, X either has a non-nef pseudoeffective divisor (which contradicts Proposition 6), or there is a nef divisor with self-intersection 0 (contradicting Lemma 4). Hence $\rho = 1$, and X is a (fake) projective plane.

A similar, but more involved analysis shows that also a threefold with nef and big diagonal has to be a fake projective space. Here one needs the following additional ingredients: (i) $\chi(X) \geq 0$; (ii) $\chi(\emptyset_X) = \frac{1}{24} c_1 c_2 = 1$; (iii) X is minimal; (iv) the Miyaoka inequality; and (v) The classification of Fano 3-folds. This indicates that a classification of varieties with nef and big diagonal in higher dimension will be more difficult; we pose the question whether a fourfold with nef and big diagonal must be a fake projective space.

1.3. Examples. The paper [LO17] studies how varieties with big or nef diagonal fit into the classification of varieties of low dimension. We conclude with a few examples illustrating these results:

Example 8.

- (1) Quadrics have big diagonal if and only if the dimension is even.
- (2) A Grassmannian has a big diagonal if and only if it is a projective space.
- (3) Let X be a toric variety. Then Δ_X is nef if and only if X has nef tangent bundle if and only if X is a product of projective spaces. There are also toric threefolds other than \mathbb{P}^3 with big diagonal.

Example 9 (Surfaces). Let us say a few words about how varieties with nef diagonal fit into the classification of surfaces. First of all, if S is a surface with nef diagonal, it must be minimal (otherwise there is a (-1) -curve, contradicting Lemma 6). We next consider the surfaces according to their Kodaira dimension κ . If $\kappa = -\infty$, S must either be \mathbb{P}^2 , $\mathbb{P}^1 \times \mathbb{P}^1$, or the ruled surface over an elliptic curve defined by a semistable rank 2 bundle. Each of these have nef tangent bundles and thus nef diagonal.

For $\kappa = 0$, abelian surfaces and hyperelliptic surfaces both have nef tangent bundles, and thus nef diagonal. Any Enriques surface admits an involution $i : S \rightarrow S$ exchanging the two sheets of a double cover $S \rightarrow T$. Intersecting Δ with the graph of the involution gives a negative number, so Δ is not nef. For K3 surfaces, we prove, using results of Bayer–Macrì [BM14], that the diagonal is not nef. In fact, we show that the diagonal is negative in a very strong sense: any effective cycle on $S \times S$ with class proportional to Δ , must itself be a multiple of Δ .

For $\kappa = 1$, we can consider the canonical map $\pi : S \rightarrow C$. By intersecting the diagonal with a cycle in from $\pi^{-1}(\Delta_C)$, one sees right away that the base C of the canonical map must have genus either 0 or 1. In the latter case, Δ has negative intersection with $\sigma \times \sigma$, where σ is a section of π . Furthermore, in [LO17] we give an example showing that it is in fact possible for Δ to be nef if π admits no sections.

Example 10 (Hypersurfaces). Let X be a smooth hypersurface of degree ≥ 3 and dimension ≥ 2 . An Euler characteristic computation shows that the diagonal of X is not nef. Bigness of the diagonal turns out to be more subtle. We show:

Theorem 11. *For a smooth Fano hypersurface of degree ≥ 3 and dimension ≤ 5 , the diagonal is not big.*

The strategy here is to use the rational map $\mathbb{P}^{n+1} \times \mathbb{P}^{n+1} \dashrightarrow Gr(2, n+2)$, which is resolved by blowing up Δ . Using this map, we can pull back Schubert cycles from the Grassmannian, and intersect Δ with these to argue that the diagonal is not big.

REFERENCES

- [BM14] Arend Bayer and Emanuele Macrì. MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations. *Invent. Math.*, 198(3):505–590, 2014.
- [CS10] Donald I. Cartwright and Tim Steger. Enumeration of the 50 fake projective planes. *C. R. Math. Acad. Sci. Paris*, 348(1-2):11–13, 2010.