

On the Cox Ring of \mathbf{P}^2 Blown Up in Points on a Line

John Christian Ottem

Abstract

We show that the blow-up of \mathbf{P}^2 in n points on a line has finitely generated Cox ring. We give explicit generators for the ring and calculate its defining ideal of relations.

1 Introduction

In 2000, Hu and Keel [7] introduced the *Cox ring* of an algebraic variety, aiming to generalize the Cox construction for toric varieties. If X is a normal variety with freely finitely generated Picard group $\text{Pic}(X)$, this ring is essentially defined by

$$\text{Cox}(X) = \bigoplus_{D \in \text{Pic}(X)} H^0(X, \mathcal{O}_X(D)),$$

where the ring product is given by multiplication of sections as rational functions. Varieties whose Cox ring is finitely generated are called *Mori dream spaces*, and have interesting properties from the viewpoint of birational geometry.

Even though the definition of the Cox ring is quite explicit, calculating its presentation for a concrete variety can be a hard problem. For example, the Cox rings of Del Pezzo surfaces have been the subject of much recent literature in algebraic geometry (e.g., [2], [9] and [11]) and show that the behaviour of the Cox ring under blow-up is highly non-trivial. For general blow-ups of \mathbf{P}^2 , $\text{Cox}(X)$ may even fail to be finitely generated, since the surfaces may have infinitely many curves of negative self-intersection.

In this paper, we consider blow-ups of \mathbf{P}^2 in points p_1, \dots, p_n lying on a fixed line $Y \subset \mathbf{P}^2$. The blow-up $\pi : X \rightarrow \mathbf{P}^2$ in these points is a smooth projective surface with Picard group generated by the classes of the exceptional divisors E_1, \dots, E_n and L , the pullback of a general line in \mathbf{P}^2 . Our first result is that $\text{Cox}(X)$ is finitely generated for any number of points on a line. This result was first shown in [5] and correlates with recent results of Hausen and Süß in [8], since the surface X has a complexity one torus action.

Furthermore, we also find explicit generators for $\text{Cox}(X)$, i.e., generating sections x_1, \dots, x_r from respective vector spaces $H^0(X, \mathcal{O}_X(D_1)), \dots, H^0(X, \mathcal{O}_X(D_r))$, so that $\text{Cox}(X)$ may be regarded as a quotient

$$\text{Cox}(X) \simeq k[x_1, \dots, x_r]/I.$$

Here we consider a $\text{Pic}(X)$ -grading on $k[x_1, \dots, x_r]$ and I given by $\deg(x_i) = D_i$. In Section 4 we find explicit generators and a Gröbner basis for the ideal I . Our main result is the following theorem:

Theorem. Let X be the blow-up of \mathbf{P}^2 in $n \geq 3$ distinct points lying on a line. Then $\text{Cox}(X)$ is a complete intersection ring and its defining ideal is generated by quadric trinomials.

In particular, this means that the Cox ring is Gorenstein and a Koszul algebra.

Notation: We make the following standard shorthand notation for sheaf cohomology:

$$H^i(D) := H^i(X, \mathcal{O}_X(D)), \quad h^i(D) := \dim_k H^i(X, \mathcal{O}_X(D)), \quad i = 0, 1, 2.$$

2 Nef divisors and vanishing on X .

The surface X has good vanishing properties for nef divisors. For example, $H^2(D) = H^0(K - D) = 0$ is immediate by Serre duality, since K cannot be effective on X . It turns out that also $H^1(D) = 0$ for D nef, so all cohomology can be calculated from the Riemann-Roch theorem. To prove this, we first need some preparatory lemmas.

Lemma 1. *The monoid of effective divisor classes of X is finitely generated by the classes $L - E_1 - \dots - E_n, E_1, E_2, \dots, E_n$.*

Proof. It is clear that the classes above are all effective, so their semigroup span is in the effective monoid. Conversely, note that these divisor classes actually form a \mathbf{Z} -basis for $\text{Pic } X$. So given an *irreducible* effective divisor D , we let

$$m(L - E_1 - \dots - E_n) + a_1 E_1 + \dots + a_n E_n$$

represent the corresponding divisor class. If D is not one of the generators above we have $D \cdot E_i = m - a_i \geq 0$ and $D \cdot (L - E_1 - \dots - E_n) = m - \sum_{i=1}^n (m - a_i) \geq 0$. Together these inequalities imply that $m, a_i \geq 0$, and we are done. \square

Lemma 2. *The nef monoid is generated by the divisor classes $L, L - E_1, L - E_2, \dots, L - E_n$.*

Proof. Let $D = dL - \sum a_i E_i$ be a nef divisor class. Intersecting D with the classes in Lemma 1 gives the following set of inequalities:

$$d \geq a_1 + a_2 + \dots + a_n, \quad a_i \geq 0, \quad \forall i = 1, \dots, n$$

Now it is easy to see that we can decompose each D as a sum of the $L - E_i$'s by using a_i of $L - E_i$ and finally add $d - a_1 - a_2 - \dots - a_n \geq 0$ times L . \square

Note that since the classes above are effective, so is every nef divisor on X .

Lemma 3. *Let $D = dL - a_1 E_1 - \dots - a_n E_n$ be a divisor class on X , with $d + 1 \geq \sum_{i=1}^n a_i$ and $a_i \geq 0$. Then $h^1(D) = 0$.*

Proof. If $D = dL$, we have $h^1(D) = h^1(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(d)) = 0$ for $d \geq -1$. If say $a_1 > 0$, consider the divisor class $D' = D - (L - E_1)$. D' satisfies the conditions of the lemma, so by induction on d we have $h^1(D') = 0$. Let C be a smooth rational curve with class $L - E_1$, then $h^1(C, D|_C) = h^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(D \cdot C)) = 0$, since $D \cdot C \geq -1$. Now taking cohomology of the exact sequence

$$0 \rightarrow \mathcal{O}_X(D') \rightarrow \mathcal{O}_X(D) \rightarrow \mathcal{O}_C(D|_C) \rightarrow 0$$

gives $h^1(D) = 0$. \square

This gives us the multigraded Hilbert function of $\text{Cox}(X)$ in nef degrees:

Corollary 4. *For a nef divisor class D , we have $h^1(D) = 0$ and*

$$\dim_k \text{Cox}(X)_D = \chi(\mathcal{O}_X(D))$$

3 Generators for $\text{Cox}(X)$.

We are now in position to find explicit generators for $\text{Cox}(X)$ as a k -algebra. When $n = 1$, the surface $Bl_p\mathbf{P}^2$ is a toric variety, and by [3], its Cox ring coincides with the usual homogenous coordinate ring

$$\text{Cox}(Bl_p\mathbf{P}^2) = k[x, s_1, s_2, e]$$

where $\deg x = L$, $\deg s_i = L - E$ and $\deg e = E$. Therefore, we will in the following suppose that $n \geq 2$.

We first choose generators e_1, \dots, e_n for the 1-dimensional vector spaces $H^0(E_i)$ for $i = 1, \dots, n$ and a generator l of $H^0(L - E_1 - \dots - E_n)$. For the classes $L - E_i$, for which $H^0(L - E_i)$ is 2-dimensional, we need in addition to the section $le_1 \cdots e_{i-1}e_{i+1} \cdots e_n$, a new section s_i to form a basis. To specify these explicitly, we fix a point $q \in \mathbf{P}^2$ and for each i take a section corresponding to the strict transform of the line going through q and p_i . The projections of these sections to \mathbf{P}^2 are shown in Figure 1:

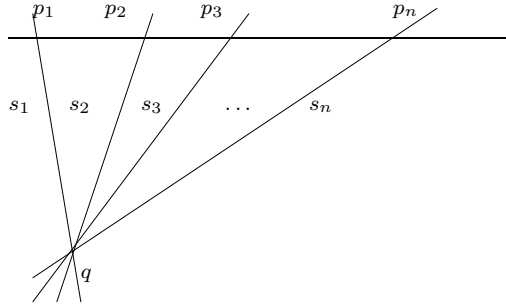


Figure 1: The choice of the sections s_1, s_2, \dots, s_n .

We will show that $\text{Cox}(X)$ is generated by the sections l, e_i, s_i for $i = 1, \dots, n$. The following lemma is a variant of Castelnuovo's base point free pencil trick ([4, Ex. 17.18]) and will be our main tool for proving this.

Lemma 5. *Let X be an algebraic variety over a field k , let \mathcal{F} be a locally free sheaf of \mathcal{O}_X -modules on X , let \mathcal{L} be an invertible sheaf on X and V a two-dimensional base-point free subspace of $H^0(X, \mathcal{L})$. If $H^1(\mathcal{L}^{-1} \otimes \mathcal{F}) = 0$, then the multiplication map*

$$V \otimes H^0(X, \mathcal{F}) \rightarrow H^0(X, \mathcal{L} \otimes \mathcal{F})$$

is surjective.

Proposition 6. *Let X be the blow-up of \mathbf{P}^2 in $n \geq 2$ distinct points on a line. Then there is a multigraded surjection*

$$k[l, e_1, \dots, e_n, s_1, \dots, s_n] \rightarrow \text{Cox}(X). \quad (1)$$

where $\deg(l) = L - E_1 - \dots - E_n$, $\deg e_i = E_i$ and $\deg s_i = L - E_i$.

Proof. Let D be an effective divisor on X . We need to show that $H^0(D)$ has a basis of sections which are polynomials in l, e_i, s_i .

We first show that we may take D to be nef. Indeed, suppose E is a curve such that $D \cdot E < 0$. Without loss of generality, we may suppose that E is one of the divisor classes generating the effective monoid. Let $x_E \in \{l, e_1, \dots, e_n\}$ be the corresponding section in $H^0(E)$. Then since E is a base component of the linear system $|D|$, multiplication by x_E induces an isomorphism $H^0(D - E) \rightarrow H^0(X, D)$. By induction on the number of fixed components, $H^0(D - E)$ is generated by monomials in the l, e_i, s_i and hence the same applies to $H^0(D)$.

Now suppose that D is a nef divisor class and write D (uniquely) in terms of the nef classes of Lemma 2:

$$D = aL + a_1(L - E_1) + a_2(L - E_2) + \dots + a_n(L - E_n)$$

where $a, a_i \geq 0$. If say, $a_1 \geq 2$, then $H^1(D - 2(L - E_1)) = 0$, since $D - 2(L - E_1)$ is nef, and so applying Lemma 5 with $V = H^0(L - E_1)$ and $\mathcal{F} = \mathcal{O}_X(D - (L - E_1))$, we get a surjection

$$H^0(D - (L - E_1)) \otimes H^0(L - E_1) \rightarrow H^0(D).$$

By induction on the number $D.L \geq 0$, $H^0(D - (L - E_1))$ is generated by monomials in l, e_i, s_i , and therefore so is $H^0(D)$.

If $a_i \leq 1$ for all i , and say, $a_1 = 1$, then $D - 2(L - E_1) = N + E_1$ for some divisor N satisfying the assumptions of Lemma 3 and $N.E_1 = 0$. In particular, $h^1(N) = 0$. Now also $h^1(N + E_1) = 0$, by the exact sequence

$$0 \rightarrow \mathcal{O}_X(N) \rightarrow \mathcal{O}_X(N + E_1) \rightarrow \mathcal{O}_{E_1}(-1) \rightarrow 0$$

and we proceed as above.

If $a_i = 0$ for all i , then $D = aL$ for some $a \geq 1$ and $H^0(X, \mathcal{O}_X(D)) \simeq H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(a))$. This implies that

$$H^0(X, (a - 1)L) \otimes H^0(X, L) \rightarrow H^0(X, aL).$$

is surjective. By induction on a , $H^0((a - 1)L)$ is generated by monomials in l, e_i, s_i , and therefore so is $H^0(D)$.

It remains to show that $H^0(L)$ has a basis of monomials in l, e_i, s_i . But $H^0(L) = \pi^*H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1))$, so it suffices to find three monomials of degree L that project to linearly independent sections in $\mathcal{O}_{\mathbf{P}^2}(1)$. By construction, this works for the three sections

$$\sigma_1 = s_1e_1, \quad \sigma_2 = s_2e_2, \quad \sigma_3 = le_1e_2e_3 \cdots e_n.$$

□

4 Relations

We now turn to the defining ideal I of relations of $\text{Cox}(X)$, i.e., the kernel of the map (1). Consider again the divisor class L : We have $h^0(L) = 3$, while there are $n + 1$ monomials of degree L in $k[l, e_i, s_i]$:

$$s_1e_1, \quad s_2e_2, \quad \dots \quad s_n e_n, \quad le_1e_2e_3 \cdots e_n$$

This means that there are $n - 2$ linear dependence relations between them. To see what they look like, consider again the projection of these sections in Figure 1. Of course any three of these lines through q satisfy a linear dependence relation, and these pull back via π to relations in $\text{Cox}(X)$ of the following form:

$$\begin{aligned} g_1 &= \underline{s_1 e_1} + a_1 s_{n-1} e_{n-1} + b_1 s_n e_n = 0 \\ g_2 &= \underline{s_2 e_2} + a_2 s_{n-1} e_{n-1} + b_2 s_n e_n = 0 \\ &\vdots \\ g_{n-2} &= \underline{s_{n-2} e_{n-2}} + a_{n-2} s_{n-1} e_{n-1} + b_{n-2} s_n e_n = 0 \end{aligned} \tag{2}$$

where each of the coefficients a_i, b_i are non-zero. We denote the ideal generated by these relations by J . The leftmost terms above are underlined since as the next lemma shows, they form an initial ideal for J .

Lemma 7. *The set $\{g_1, \dots, g_{n-2}\}$ is a Gröbner basis for J with respect to the graded lexicographical order, and $(s_1 e_1, \dots, s_{n-2} e_{n-2})$ is an initial ideal of J .*

Proof. It is well-known (e.g., see [1]) that a collection of polynomials with relatively prime leading terms is a Gröbner basis for the ideal they generate. \square

We will show that that the expressions (2) in fact generate all the relations, i.e., that $I = J$. For this, we will make use of the $\text{Pic}(X)$ -grading on $R = k[l, e_i, s_i]$ and I . The next lemma shows that it is sufficient to consider generators for the ideal of degrees corresponding to nef divisor classes.

Lemma 8. *The ideal I is generated by elements of degree D , where D is a nef divisor class.*

Proof. Suppose D is an effective divisor class and that there is a negative curve E such that $D \cdot E < 0$. Then this implies that E is a fixed component of $|D|$ and as above every monomial in $k[l, s_i, e_i]_D$ is divisible by x_E , the variable corresponding to E . This shows that any element of I_D can be written as a product of x_E and a relation in I_{D-E} . Now the claim follows by induction on the number of fixed components of D . \square

We will now prove our main theorem.

Theorem 9. *Let X be the blow-up of $n \geq 2$ points on a line. Then $\text{Cox}(X)$ is a complete intersection with $n - 2$ quadratic defining relations given in (2).*

Proof. By Lemma 8, it is sufficient to show that $I_D = J_D$ for all nef classes $D = dL - a_1 E_1 - a_2 E_2 - \dots - a_n E_n$, (here $d \geq a_1 + \dots + a_n$). Note that since $J \subseteq I$, we have in any case a surjective homomorphism

$$R/J \rightarrow \text{Cox}(X).$$

To show that this is an isomorphism in degree D , we calculate the multigraded Hilbert function of both sides. From Corollary 4 and Riemann-Roch, we have

$$\dim_k \text{Cox}(X)_D = h^0(D) = \binom{d+2}{2} - \binom{a_1+1}{2} - \dots - \binom{a_n+1}{2}. \tag{3}$$

To calculate $\dim_k(R/J)_D$, we use the Gröbner basis for J . Since the Hilbert function is preserved when going to initial ideals, we have

$$\dim_k(R/J)_D = \dim_k R/(s_1 e_1, \dots, s_{n-2} e_{n-2})_D$$

We use a counting argument to evaluate this number. Note that any monomial in $R/(s_1 e_1, \dots, s_{n-2} e_{n-2})_D$ corresponds to a way of writing D as a non-negative sum of divisor classes from

$$L - E_1 - \dots - E_n, \quad L - E_1, \quad \dots \quad L - E_n, \quad E_1, \quad \dots, \quad E_n$$

such that not both $L - E_i$ and E_i occur in the sum for $i = 1, \dots, n-2$. Abusing notation, we let the numbers s_i, e_i, l represents respectively the non-negative coefficients of $L - E_i, E_i, L - E_1 - \dots - E_n$ in this sum. Working in $\text{Pic}(X) \simeq \mathbb{Z}^{n+1}$, this translates the problem of finding $\dim_k(R/J)_D$ into following counting problem: finding the number of non-negative solutions of

$$\begin{aligned} s_1 + s_2 + \dots + s_n &= d - l \\ s_1 - e_1 &= a_1 - l \\ &\vdots \\ s_n - e_n &= a_n - l \end{aligned}$$

such that $s_i \cdot e_i = 0$ for $i = 1, \dots, n-2$.

Lemma 10. *For each $l \leq d$ we have $d - l \geq \sum_{k=1}^n \max(a_k - l, 0)$.*

Proof. For $l = 0$, this inequality reduces to the nef condition on D . Now, increasing l by one decreases the left hand side by one, and if there is some $a_i - l > 0$, then $\max(a_i - l, 0)$ is decreased by one, as well, if not, the right hand side is zero, so in any case the inequality is preserved. \square

For each fixed l , we count the number of non-negative solutions $S(l)$ to the system (4). Note that s_i is completely determined as $s_i = \max(a_i - l, 0)$ for $1 \leq i \leq n-2$ by the condition $s_i \cdot e_i = 0$. Hence by the first equation in (4), we are looking for non-negative solutions to

$$s_n + s_{n-1} = d - l - \sum_{k=1}^{n-2} \max(a_k - l, 0) \geq 0$$

such that $s_n \geq \max(a_n - l, 0)$ and $s_{n-1} \geq \max(a_{n-1} - l, 0)$, of which there are in total

$$d - l - \sum_{k=1}^{n-2} \max(a_{n-2} - l, 0) + 1 - \sum_{k=n-1, n} \max(a_k - l, 0)$$

Hence the total number of solutions to (4) is

$$\begin{aligned} \sum_{l=0}^d S(l) &= \sum_{l=0}^d \left(d + 1 - l - \sum_{k=1}^n \max(a_k - l, 0) \right) \\ &= \binom{d+2}{2} - \sum_{i=0}^{a_1} (a_1 - i) - \sum_{i=0}^{a_1} (a_1 - i) - \dots - \sum_{i=0}^{a_n} (a_n - i) \\ &= \binom{d+2}{2} - \binom{a_1+1}{2} - \binom{a_2+1}{2} - \dots - \binom{a_n+1}{2} = h^0(D). \end{aligned}$$

This finishes the proof that $I = J$. Now, from [2, Remark 1.4] we have $\dim \text{Cox}(X) = n + 3$, furthermore by Proposition 6 we have that $\text{codim} \text{Cox}(X) = (2n + 1) - (n + 3) = n - 2$, which is exactly the number of relations in I . \square

Remark. The above result can also be proved in another way, using the following lemma, proved by Stillman in [10]:

Lemma 11. *Let $J \subset k[x_1, x_2, \dots, x_n]$ be an ideal containing a polynomial $f = gx_1 + h$, with g, h not involving x_1 and g a non-zero divisor modulo J . Then, J is prime if and only if the elimination ideal $J \cap k[x_2, \dots, x_n]$ is prime.*

The above lemma can be used to prove that $J = (g_1, \dots, g_{n-2})$ is prime, using induction on n . For $n = 3$, this is clear. Next, note that the elimination ideal $J \cap k[s_2, \dots, s_n, e_2, \dots, e_n, l]$ is just (g_2, \dots, g_{n-2}) since $\{g_1, \dots, g_{n-2}\}$ is a Gröbner basis. By induction on n , (g_2, \dots, g_{n-2}) is the defining ideal of $\text{Cox}(X)$ for \mathbf{P}^2 blown up in the points p_2, \dots, p_n and hence is prime. Taking now $x_1 = e_1, g = s_1, h = a_1 s_{n-1} e_{n-1} + b_1 s_n e_n$ shows that J is prime. Then, since $J \subseteq I$ are two prime ideals with the same Krull dimension, they must be equal.

Acknowledgement. I wish to thank Kristian Ranestad, Jørgen Vold Rennemo and Mauricio Velasco for interesting discussions. I would also like to thank Brian Harbourne for pointing out the reference [6] and the anonymous referee for helpful comments.

References

- [1] W. Adams and P. Loustau. *An Introduction to Gröbner Bases*. Providence, RI: Amer. Math. Soc., 1994.
- [2] V. Batyrev and O. N. Popov. The Cox ring of a Del Pezzo surface. In *Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002)*, volume 226 of *Progr. Math.*, pages 85–103. Birkhäuser Boston, Boston, MA, 2004.
- [3] D. Cox, The homogeneous coordinate ring of a toric variety, *J. Algebraic Geom.* 4(1), pages 17-50, 1995
- [4] D. Eisenbud, *Commutative Algebra With a View Towards Algebraic Geometry*, Graduate Texts in Mathematics 150, Springer, 1995
- [5] J. Elizondo, K. Kurano, K. Watanabe: The total coordinate ring of a normal projective variety. *J. Algebra* 276, pages 625-637 (2004)
- [6] C. Galindo and F. Monserrat, The total coordinate ring of a smooth projective surface, *J. Algebra* 284 (2005), pp. 91–101
- [7] Y. Hu, S. Keel. Mori Dream Spaces and GIT. *Michigan Math. J.* Volume 48, Issue 1, pages 331-348, 2000.
- [8] J. Hausen, H. Süß, The Cox ring of an algebraic variety with torus action, arXiv:0903.4789, Mar 2009.

- [9] A Laface, and M Velasco. Picard-graded Betti Numbers and the Defining Ideals of Cox Rings. *Journal of Algebra* 322, Issue 2, pages 353-372. 2009.
- [10] M. Stillman: Tools for computing primary decompositions and applications to ideals associated to Bayesian networks, A. Dickenstein and I. Emiris (eds.): *Solving Polynomial Equations. Foundations, Algorithms and Applications*, Springer Verlag, Heidelberg, 2005, 203–239.
- [11] M Stillman, D Testa, and M Velasco. Groebner bases, monomial group actions, and the Cox rings of Del Pezzo surfaces. *Journal of Algebra* 316, Issue 2, Pages 777-801, 2007.