STABLE IRRATIONALITY OF THE VERY GENERAL QUARTIC FIVEFOLD

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These are notes for a talk based on the paper [NO], joint with Johannes Nicaise. The paper gives a general technique for rationality problems of hypersurfaces and complete intersections in toric varieties, based on the motivic volume formula of Nicaise–Shinder [NS19]. One of the main applications is the following:

Theorem 0.1. *A very general complex quartic fivefold is not stably rational.*

The rationality problem for hypersurfaces is a classical topic in algebraic geometry, see e.g., [AM72, CG72, IM71] (in dimension 3), and more recently [CTP16, HPT19, Ko95, Sch19a, To16] (in higher dimensions). In these works, the approaches typically combine specialization arguments with various types of birational invariants (Brauer groups, unramified cohomology, differential forms, ..). The proof of Theorem 0.1, however, seems to be the first instance where one specializes a hypersurface to a union of several components, and deduces irrationality from that of lower-dimensional varieties. We will explain this proof in Section 2. In Section 3, we discuss the corresponding toric picture and further results of the paper [NO].

1. Background

Two varieties X and Y are said to be *stably birational* if $X \times \mathbb{P}^m$ and $Y \times \mathbb{P}^n$ are birational for some positive integers m, n. A variety X is *stably rational* if it is stably birational to a projective space.

Of course, we care about stable rationality mainly because we care about *rationality*. However, it turns out to be convenient to work with invariants that obstruct stable rationality rather than rationality. For instance, the main ingredient in the proof of Theorem 0.1 is the motivic volume formula of Nicaise and Shinder [NS19] which is naturally formulated in this setting.

For a field F, we let SB_F denote the set of stable birational equivalence classes of integral F-varieties. For a variety X, we write $[X]_{\mathrm{sb}}$ for the equivalence class of X.

We will work in $\mathbb{Z}[\mathrm{SB}_F]$, the free abelian group on the set SB_F. Elements of $\mathbb{Z}[\mathrm{SB}_F]$ are formal sums of the form

$$a_1[X_1]_{\mathrm{sb}} + \ldots + a_r[X_r]_{\mathrm{sb}}$$

for integers $a_1, \ldots, a_r$. In fact, $\mathbb{Z}[\mathrm{SB}_F]$ is a *ring*, with multiplication defined by the fiber product, $[X]_{\mathrm{sb}} \cdot [Y]_{\mathrm{sb}} = [X \times_F Y]_{\mathrm{sb}}$.

For any F-scheme X of finite type, we set

$$[X]_{\mathrm{sb}} = [X_1]_{\mathrm{sb}} + \ldots + [X_r]_{\mathrm{sb}}$$

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where $X_1, \dots, X_r$ are the irreducible components. In particular, $[X_{\mathrm{red}}]_{\mathrm{sb}} = [X]_{\mathrm{sb}}$ in this group.

1.1. **The motivic volume.** The aim of this section is to explain the *motivic volume* formula of Nicaise and Shinder [NS19]. We work over the field of Puiseux series

$$K = \mathbb{C}\{\{t\}\} = \bigcup_{m>0} \mathbb{C}((t^{1/m}))$$

and its valuation ring

$$R = \bigcup_{m>0} \mathbb{C}[[t^{1/m}]].$$

In short, we consider families $\mathcal{X} \to \operatorname{Spec} R$, and want to compare the rationality properties of the generic fiber $\mathcal{X}_K$, to that of the special fiber, $\mathcal{X}_{\mathbb{C}}$. Note however that $\mathcal{X}_{\mathbb{C}}$ may have several irreducible components, so it makes most sense to do this comparison in $\mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$. Indeed, the motivic volume will be a map $\mathbb{Z}[\mathrm{SB}_K] \to \mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$.

It suffices to define the motivic volume on proper R-schemes $\mathcal{X}$ which are *strictly semi-stable*, i.e., $\mathcal{X}_{\mathbb{C}}$ is a reduced simple normal crossing divisor on $\mathcal{X}$. In the formula (1.2) below, $\mathcal{X}$ will be a proper strictly semi-stable R-scheme, and we decompose special fiber into irreducible components

$$\mathcal{X}_{\mathbb{C}} = \sum_{i \in I} X_i. \tag{1.1}$$

Theorem 1.1 (Nicaise–Shinder)**.** *There exists a unique ring homomorphism*

$$\mathrm{Vol} : \mathbb{Z}[\mathrm{SB}_K] \to \mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$$

such that

$$\mathrm{Vol}([\mathcal{X}_K]_{\mathrm{sb}}) = \sum_{\emptyset \neq J \subseteq I} (-1)^{|J|-1} [X_J]_{\mathrm{sb}} \tag{1.2}$$

for any $\mathcal{X}$ as above, where $X_J = X_{j_1} \cap \ldots \cap X_{j_r}$.

Let us make the following observations:

- Vol sends $[\operatorname{Spec} K]_{\mathrm{sb}}$ to $[\operatorname{Spec} \mathbb{C}]_{\mathrm{sb}}$.
- If $\mathcal{X} \to \operatorname{Spec} R$ is *smooth and proper*, then $\mathrm{Vol}([\mathcal{X}_K]_{\mathrm{sb}}) = [\mathcal{X}_{\mathbb{C}}]_{\mathrm{sb}}$.

These two in conjunction have an important consequence, namely that if $\mathcal{X} \to \operatorname{Spec} R$ is smooth and proper, and the generic fiber $\mathcal{X}_K$ is geometrically stably rational, then so is the special fiber. In other words, *stable rationality specializes in smooth and proper families.* This was a long-standing open question, solved in [NS19] (and in [KT19] with 'stable rationality' replaced by 'rationality').

Remark 1.2. The condition of strict semi-stability is quite restrictive, and producing a semi-stable model often leads to many blow-ups which which are hard to analyze. An important point is that the formula (1.2) also applies when $\mathcal{X}$ is *strictly toroidal* (see [NO21]). This condition is much more flexible, and reduces the computations substantially.

1.2. **Obstructions to rationality.** In Theorem 0.1, the statement is that the very general quartic fivefold over the complex numbers is not stably rational. By the specialization properties of stable rationality due to Nicaise–Shinder, it suffices to exhibit a single example of a stably irrational smooth quartic fivefold over some algebraically closed field of characteristic 0. In our set-up, we of course use the generic fiber $\mathcal{X}_K$ over the field of Puiseux series.

A key idea in [NO], is to use Theorem 1.1 as an obstruction to stable rationality of $\mathcal{X}_K$. This is formulated in the following criterion:

Proposition 1.3. *Let $\mathcal{X}$ be a proper, strictly semi-stable (or strictly toroidal) R-scheme. Assume that for the strata $X_J = X_{j_1} \cap \ldots \cap X_{j_r}$ in the special fiber, we have*

$$\sum_{\emptyset \neq J \subseteq I} (-1)^{|J|-1} [X_J]_{\mathrm{sb}} \neq [\operatorname{Spec} \mathbb{C}]_{\mathrm{sb}} \tag{1.3}$$

in $\mathbb{Z}[\mathrm{SB}_F]$. Then $\mathcal{X}_K$ is not stably rational.

The strategy for Theorem 0.1 is therefore to look for suitable degenerations $\mathcal{X} \to \operatorname{Spec} R$ with $\mathcal{X}_K$ a quartic hypersurface in $\mathbb{P}^6_K$, with the property that the special fiber $\mathcal{X}_{\mathbb{C}}$ is a union of several components, and ensure that the corresponding strata do not cancel out to $[\operatorname{Spec} \mathbb{C}]_{\mathrm{sb}}$ in the alternating sum (1.3).

A point we would like to emphasise is that with this technique one typically encounters special fibers with several components, and deduces irrationality of $\mathcal{X}_K$ from that of varieties of lower dimension. In our main application, we deduce the irrationality of a quartic fivefold using a stably irrational fourfold.

Example 1.4. Suppose the special fiber $\mathcal{X}_{\mathbb{C}}$ consists of two components, X_0 and X_1, intersecting along X_{01}. The motivic volume takes the form

$$\mathrm{Vol}(\mathcal{X}_K) = [X_0]_{\mathrm{sb}} + [X_1]_{\mathrm{sb}} - [X_{01}]_{\mathrm{sb}} \tag{1.4}$$

From this, we deduce that either of the following conditions guarantee that the generic fiber $\mathcal{X}_K$ is not stably rational:

i) Exactly one of X_0, X_1, X_{01} is stably irrational.
ii) X_0 and X_1 are both stably irrational.
iii) X_0 and X_{01} are stably irrational, but they are not stably birational to each other.
iv) X_0, X_1, X_{01} are all stably irrational.

In the paper [NO21], there are examples illustrating each case i)-iv). While iii) seems like the hardest to check, it is remarkably effective when combined with "variation of birational type"-arguments à la Shinder [Sh19]

2. Quartic fivefolds

We are now in position to prove Theorem 0.1. Let $F \in \mathbb{C}[x_0, \ldots, x_6]$ be a very general homogeneous polynomial of degree 4. Consider the following R-scheme

$$\mathcal{X} = \operatorname{Proj} R[x_0, \ldots, x_6, y]/(x_5 x_6 - ty, y^2 - F) \tag{2.1}$$

where the variable y has weight 2. Note that the generic fiber $\mathcal{X}_K$ is isomorphic to a smooth quartic hypersurface in $\mathbb{P}^6_K$ (inverting t allows us to eliminate y using the first equation). Moreover, $\mathcal{X}$ is strictly toroidal.

The special fiber has two components:

$$\begin{aligned} X_0 &= \operatorname{Proj} \mathbb{C}[x_0, \dots, x_6, y]/(x_5, y^2 - F) \\ X_1 &= \operatorname{Proj} \mathbb{C}[x_0, \dots, x_6, y]/(x_6, y^2 - F). \end{aligned}$$

Note that these are both very general quartic double fivefolds. We do not know whether these are stably rational or not. However, their intersection,

$$X_{01} = \operatorname{Proj} \mathbb{C}[x_0, \dots, x_4, y]/(y^2 - F)$$

is a very general quartic double fourfold, and thus stably irrational by Hassett–Pirutka–Tschinkel [HPT19]. In any event we are in either case i) or iv) in Example 1.4, so we conclude also that $\mathcal{X}_K$, and hence the also the very general quartic fivefold, is stably irrational.

3. The paper [NO]

Proposition 1.3 gives a method to approach rationality problems of hypersurfaces and complete intersections in general. While the main idea is simple, the challenge is now to write down suitable degenerations where one can apply the obstruction (1.3). In fact, while the family (2.1) is completely explicit, most of the applications in [NO] require degenerations which are much more involved and harder to write down concretely.

These constructions use the theory of tropical (or toric) degenerations. This theory has the advantage that degenerations can be found and studied using combinatorial methods, i.e., finding regular subdivisions of polytopes. For instance, the combinatorial picture for the 2-dimensional analogue of the above family is shown in Figure 1.

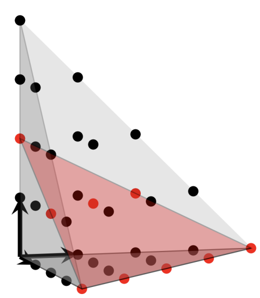

Figure 1. Degenerating a quartic surface into a union of two quartic double surfaces intersecting along a quartic double curve.

The figure shows the Newton polytope of a general quartic surface in $\mathbb{P}^3$, that is, the 3-simplex dilated by a factor of 4. It is subdivided into two smaller polytopes Δ_1, Δ_2 which intersect along the red polytope $\Delta_1 \cap \Delta_2$. Each polytope gives rise to a toric variety Y_i, and the subdivision gives rise to a degeneration of $\mathbb{P}^3$ into a union $Y_1 \cup Y_2$. Moreover, these components intersect according to the polytopes in the subdivision:

$Y_1 \cap Y_2$ is the toric variety corresponding to the red polytope $\Delta_1 \cap \Delta_2$ (i.e., $\mathbb{P}(1,1,2)$ polarized by $\mathcal{O}(4)$). From this, we obtain a degeneration of a quartic surface into a union of two surfaces $X_1 \cup X_2$ which intersect along a quartic double curve.

Other regular subdivisions of the polytope, give rise to different degenerations. To illustrate the flexibility of the method, let us try to construct a family $\mathcal{X} \to \operatorname{Spec} R$ with the property that *exactly one stratum X_J in the special fiber is stably irrational.* The subdivision we will use is shown below:

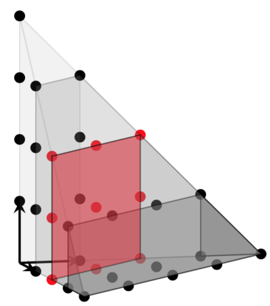

FIGURE 2. Degenerating a quartic fivefold into a union of four rational fivefolds, with one stably irrational stratum.

In this case, the dilated simplex is subdivided into four smaller polytopes $\Delta_1, \dots, \Delta_4$, and we obtain a degeneration of a quartic fivefold into a union of four components X_1, X_2, X_3, X_4. The red polytope corresponds to the stratum $X_{23} = X_2 \cap X_3$, which is a very general $(2,2)$-divisor in $\mathbb{P}^2 \times \mathbb{P}^3$ (which is also known to be stably irrational [HPT19]). All of the other strata are represented by polytopes of *lattice width one*; in suitable coordinates they are therefore given as hypersurfaces which are linear in one variable, hence they are rational. Thus we have the desired degeneration.

The motivic volume takes the following form

$$\begin{aligned} \operatorname{Vol}(\mathcal{X}_K) &= \sum_{1\le i\le 4} [X_i]_{\mathrm{sb}} - \sum_{1\le i<j\le 4} [X_{ij}]_{\mathrm{sb}} + \sum_{1\le i<j<k\le 4} [X_{ijk}]_{\mathrm{sb}} - [X_{1234}]_{\mathrm{sb}} \\ &= 4[\operatorname{Spec}\mathbb{C}]_{\mathrm{sb}} - 2[\operatorname{Spec}\mathbb{C}]_{\mathrm{sb}} - [X_{23}]_{\mathrm{sb}} \\ &\neq [\operatorname{Spec}\mathbb{C}]_{\mathrm{sb}} \end{aligned}$$

because $[X_{23}]_{\mathrm{sb}} \neq [\operatorname{Spec}\mathbb{C}]$. Hence the quartic fivefold $\mathcal{X}_K$ is not stably rational.

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