

# On $(2,3)$ -fourfolds

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Joint work with with Johannes Nicaise.

$X$  is *rational* if it is birational to  $\mathbb{P}^n$ .

$X$  is *stably rational* if  $X \times \mathbb{P}^m$  is rational for some  $m \geq 0$ .

$X$  and  $Y$  are *stably birational* if  $X \times \mathbb{P}^m \sim_{bir} Y \times \mathbb{P}^l$  for some  $m, l \geq 0$ .

The paper [NO19] gives a quite general method for the (stable) rationality problem for complete intersections in toric varieties.

Everything is in characteristic 0.

# Hypersurfaces in $\mathbb{P}^n$

## Theorem

*A very general quartic fivefold  $X \subset \mathbb{P}^6$  is stably irrational.*

Also:

- New proofs of hypersurfaces of higher degree or lower dimension (eg quartic fourfolds, quintic fivefolds, ..)
- Improvements of Schreieder's bound  $d \geq \log_2 n + 2$

Hypersurfaces in  $\mathbb{P}^n$ [illegible]

# Hypersurfaces in $\mathbb{P}^n$

| d | curves | surfaces | 3-folds                 | 4-folds | 5-folds    | 6-folds | 7-folds | 8-folds | 9-folds | 10-folds |
|---|--------|----------|-------------------------|---------|------------|---------|---------|---------|---------|----------|
| 2 |        |          |                         |         | Rational   |         |         |         |         |          |
| 3 |        |          | Clemens-Griffiths       |         |            |         |         |         |         |          |
| 4 |        |          | Colliot-Thélène-Pirutka | Totaro  |            |         |         |         |         |          |
| 5 |        |          |                         | Kollár  | Schreieder |         |         |         |         |          |
| 6 |        |          |                         |         | Kollár     | Kollár  | Totaro  |         |         |          |
| 7 |        |          | Stably irrational       |         |            | Kollár  | Totaro  |         |         |          |
| 8 |        |          |                         |         |            |         | Kollár  | Kollár  | Kollár  | Totaro   |
| 9 |        |          |                         |         |            |         |         | Kollár  | Kollár  | Totaro   |

Hypersurfaces in  $\mathbb{P}^n$ [illegible]

## (2,3)-complete intersections

### Theorem

*Very general complete intersections of a quadric and a cubic in  $\mathbb{P}^n$  are stably irrational for  $n \leq 6$ .*

Relation to the *Lüroth problem*:

- Fano (1908): (Incorrect) proof of irrationality for  $n = 5$
- Enriques (1912): Proof of unirationality for  $n = 5$
- Hassett–Tschinkel (2018): Stable irrationality for  $n = 5$ .
- Morin (1955), Conte–Murre (1998): Unirationality for  $n = 6$ .

The above result settles the rationality problem for all complete intersections of dimension  $\leq 4$  - except cubic fourfolds.

## More complete intersections in $\mathbb{P}^n$

We consider generic complete intersections

$$X = H_1 \cap \cdots \cap H_r \subset \mathbb{P}^n$$

where  $\deg H_i = d_i \geq 2$ .

- Logarithmic bounds à la Schreieder for stable irrationality.
- Complete intersections of  $r$  quadrics in  $\mathbb{P}^n$  are stably irrational if  $r \geq 3$  and  $2r \geq n - 1$ .
- Many new cases also in low dimension.

In dimension 5:

$$(\mathbf{4}), (5), (6), (\mathbf{2}, \mathbf{4}), (2, 5), (\mathbf{3}, \mathbf{3}), (3, 4), (\mathbf{2}, \mathbf{2}, \mathbf{3}), (2, 2, 4), (2, 3, 3), \\ (\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}), (2, 2, 2, 3), (\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}).$$

# Products of projective spaces

Many new cases for hypersurfaces in  $\mathbb{P}^\ell \times \mathbb{P}^m$ .

## Theorem

*A very general hypersurface  $X$  of bidegree  $(a, b)$  in  $\mathbb{P}^1 \times \mathbb{P}^n$  ( $n \leq 4$ ) is stably rational if and only if*

- $a = 1$ ; or
- $b \leq 2$

This will be a consequence of the special case:

## Proposition ((2, 3)-divisors in $\mathbb{P}^1 \times \mathbb{P}^4$ )

Very general (2, 3)-hypersurfaces in  $\mathbb{P}^1 \times \mathbb{P}^n$  are stably irrational, for  $n \leq 4$ .

# Ingredients

The proof uses

- Specialization of birational types (Nicaise–Shinder, Kontsevich–Tschinkel)
- Tropical geometry, toric degenerations
- Stable irrationality of known lower-dimensional varieties

## The limits of rationality

$\mathrm{SB}_F$  = set of stable birational equivalence classes of integral  $F$ -varieties

$[X]_{\mathrm{sb}}$  = equivalence class of  $X$ .

We consider  $\mathbb{Z}[\mathrm{SB}_F]$ .

For any  $F$ -scheme  $X$  of finite type, we set

$$[X]_{\mathrm{sb}} = [X_1]_{\mathrm{sb}} + \dots + [X_r]_{\mathrm{sb}} \quad \text{in } \mathbb{Z}[\mathrm{SB}_F]$$

where  $X_1, \dots, X_r$  are the irreducible components.

Ring product:  $[X]_{\mathrm{sb}} \cdot [Y]_{\mathrm{sb}} = [X \times_F Y]_{\mathrm{sb}}$ .

**Larsen–Lunts (2003):** There is a natural isomorphism

$$\mathbf{K}(\mathrm{Var}/F)/(\mathbb{L}) \simeq \mathbb{Z}[\mathrm{SB}_F].$$

induced by  $[X] \mapsto [X]_{\mathrm{sb}}$  for  $X$  smooth and proper.

## Some notation

Field of Puiseux series:

$$K = \mathbb{C}\{\{t\}\} = \bigcup_{m>0} \mathbb{C}((t^{1/m}))$$

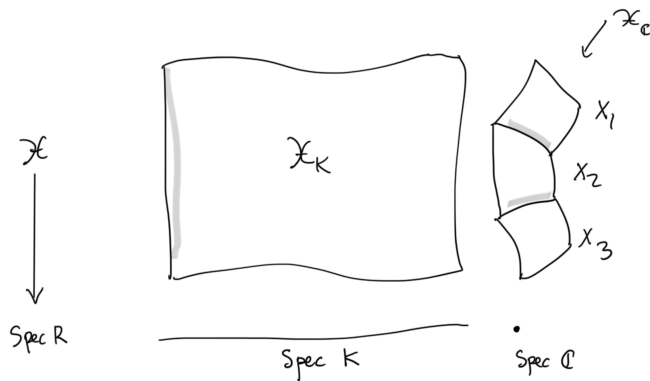
Valuation ring:

$$R = \bigcup_{m>0} \mathbb{C}[[t^{1/m}]]$$

An  $R$ -scheme is *strictly semi-stable* if, Zariski locally, it admits an étale morphism to a scheme of the form

$$\mathrm{Spec} R[z_1, \dots, z_s]/(z_1 \cdots z_r - t^q)$$

where  $s \geq r \geq 0$  and  $q$  is a positive rational number.



$\mathcal{X}$  = a proper semi-stable model over  $R$ , with special fiber

$$\mathcal{X}_{\mathbb{C}} = \sum_{i \in I} X_i$$

Theorem (Nicaise–Shinder 2019)

*There exists a unique ring homomorphism*

$$\mathrm{Vol} : \mathbb{Z}[\mathrm{SB}_K] \rightarrow \mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$$

*such that*

$$\mathrm{Vol}([\mathcal{X}_K]_{\mathrm{sb}}) = \sum_{\emptyset \neq J \subseteq I} (-1)^{|J|-1} [X_J]_{\mathrm{sb}}$$

*for any  $\mathcal{X}$  as above, where  $X_J = X_{j_1} \cap \dots \cap X_{j_r}$*

## Basic consequences

- $\text{Vol}$  sends  $[\text{Spec } K]_{\text{sb}}$  to  $[\text{Spec } \mathbb{C}]_{\text{sb}}$   
 $\rightsquigarrow$  obstruction to stable rationality of  $\mathcal{X}_K$ :

If

$$\sum_{\emptyset \neq J \subseteq I} (-1)^{|J|-1} [X_J]_{\text{sb}} \neq [\text{Spec } \mathbb{C}] \quad \text{in } \mathbb{Z}[\text{SB}_{\mathbb{C}}]$$

then  $\mathcal{X}_K$  is stably irrational.

- If  $\mathcal{X}$  is a smooth and proper  $R$ -scheme, then the formula simplifies to

$$\text{Vol}([X_K]_{\text{sb}}) = [\mathcal{X}_{\mathbb{C}}]_{\text{sb}}$$

$\therefore$  Geometric stable rationality specializes in smooth and proper families in characteristic 0.

### Example (Voisin)

A very general double quartic threefold is irrational.

Follows by degenerating to the Artin-Mumford example.

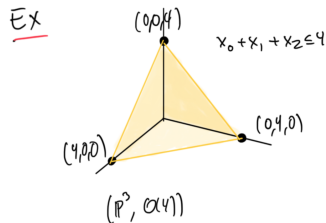
We get better results using degenerations with many components, where *irrational strata of low dimension may be shown to not cancel out in the alternating sum*

$$\mathrm{Vol}([\mathcal{X}_K]_{\mathrm{sb}}) = \sum_{\emptyset \neq J \subseteq I} (-1)^{|J|-1} [X_J]_{\mathrm{sb}}.$$

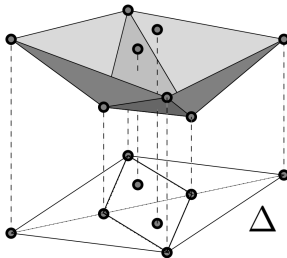
To carry this out we need a more powerful way of constructing degenerations.

# Tropical degenerations

Consider a lattice polytope  $\Delta \subset \mathbb{R}^n$  corresponding to a toric variety  $Y$ .



Let  $\mathcal{P}$  be a *regular subdivision* of  $\Delta$  into lattice polytopes.



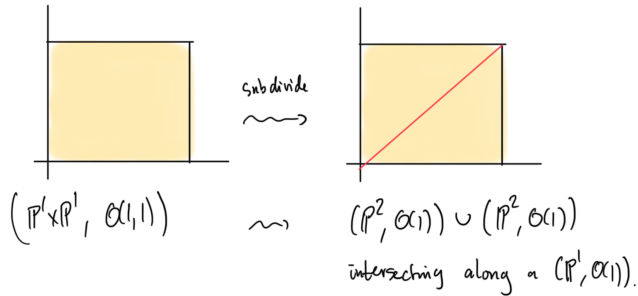
$\mathcal{P}$  induces a degeneration of  $Y$  into a union of toric varieties

$$\mathcal{Y}_0 = \bigcup_{P \in \mathcal{P}} Y_P$$

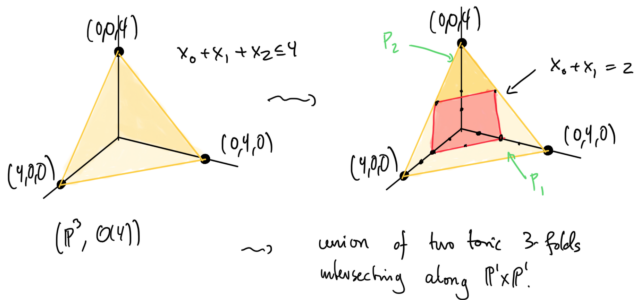
If  $P_1, P_2 \in \mathcal{P}$  intersect along a common face  $Q$ , then

$$Y_{P_1} \cap Y_{P_2} = Y_Q$$

Ex



Ex



Let  $f$  be a general Laurent polynomial with Newton polytope  $\Delta \subset \mathbb{R}^{n+1}$ .

For every face  $\delta$  of  $\mathcal{P}$ , set

$$f_\delta = \sum_{\mathbb{Z}^{n+1} \cap \delta} c_m x^m$$

Non-degeneracy: We assume that  $Z(f_\delta)$  is smooth for all  $\delta$ .

### Theorem

*If*

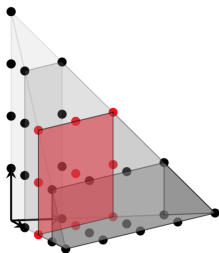
$$\sum_{\delta \subsetneq \partial \Delta} (-1)^{\dim \delta} [Z(f_\delta)]_{\text{sb}} \neq (-1)^n [\text{Spec } \mathbb{C}]_{\text{sb}}$$

*in  $\mathbb{Z}[\text{SB}_{\mathbb{C}}]$ , then a very general hypersurface in  $(\mathbb{C}^*)^{n+1}$  with Newton polytope  $\Delta$  is not stably rational.*

## The Quartic fivefold is stably irrational

Newton polytope:  $\Delta = \{(x_1, \dots, x_6) \in \mathbb{R}_{\geq 0}^6 \mid \sum_i x_i \leq 4\}$

Subdivision below  $\rightsquigarrow$  degeneration with special fiber  $X_1 \cup X_2 \cup X_3 \cup X_4$ .



**Red polytope** =  $(2, 2)$ -divisor in  $\mathbb{P}^2 \times \mathbb{P}^3$

$\rightsquigarrow$  stably irrational by [Hassett–Pirutka–Tschinkel 2016].

All other polytopes have *lattice width one*, hence rational.

Thus

$$\sum_{\delta \subsetneq \partial \Delta} (-1)^{\dim \delta} [Z(f_\delta)]_{\text{sb}} \neq (-1)^n [\text{Spec } \mathbb{C}]_{\text{sb}}$$

# Products of projective spaces

## Theorem

*Let  $X$  be a very general divisor of bidegree  $(a, b)$  in  $\mathbb{P}_k^1 \times_k \mathbb{P}_k^n$ , with  $n \leq 4$  and  $a, b \geq 1$ . Then  $X$  is rational if  $a = 1$  or  $b \leq 2$ , and stably irrational in all other cases.*

Subdivisions of the polytope  $a\Delta_1 \times b\Delta_n$  allows us to raise degree/dimension:

$(a, b)$  in  $\mathbb{P}^m \times \mathbb{P}^n$  stably irrational  $\implies (a, b+1)$  and  $(a+1, b)$  also stably irrational in  $\mathbb{P}^m \times \mathbb{P}^n$  and  $\mathbb{P}^m \times \mathbb{P}^{n+1}$ .

$\therefore$  It suffices to consider  $(2, 3)$ -fourfolds in  $\mathbb{P}^1 \times \mathbb{P}^4$ .

## The Hassett–Pirutka–Tschinkel quartic

$Y \subset \mathbb{P}^2 \times \mathbb{P}^3$ , bidegree  $(2, 2)$ , defined by

$$xyU^2 + xV^2 + yW^2 + (x^2 + y^2 + z^2 - 2(xy + xz + yz))T^2 = 0$$

**Hassett–Pirutka–Tschinkel/Schreieder:** Anything that specializes to  $Y$  does not admit a decomposition of the diagonal (hence is stably irrational).

## $(2, 3)$ -divisors in $\mathbb{P}^1 \times \mathbb{P}^4$

$P$  = the Newton polytope of the HPT quartic.

$$= \text{convex hull of column vectors of } \begin{pmatrix} 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Starting observation:  $P$  is contained in the Newton polytope of a general  $(2, 3)$ -divisor:

$$2\Delta_1 \times 3\Delta_4 = \{(u, v) \in \mathbb{R}_{\geq 0}^{1+4} \mid u \leq 2, v_1 + \dots + v_4 \leq 3\}.$$

In concrete terms, the following bidegree  $(2, 3)$  polynomial

$$\begin{aligned} & x_0^2 y_0^3 - 2x_0 x_1 y_0^3 + x_1^2 y_0^3 - 2x_0^2 y_0^2 y_1 - 2x_0 x_1 y_0^2 y_1 \\ & + x_0^2 y_0 y_1^2 + x_0 x_1 y_1 y_2^2 + x_0^2 y_1 y_3^2 + x_0 x_1 y_0 y_4^2 \end{aligned}$$

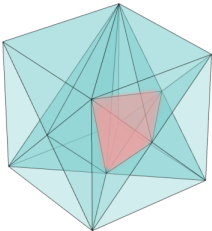
dehomogenizes to the HPT quartic.

Let  $\mathcal{P}$  denote the regular subdivision of the polytope  $2\Delta_1 \times 3\Delta_4$  induced by the convex function

$$f\colon \mathbb{R}^5 \rightarrow \mathbb{R}, \, x \mapsto \min_{z \in P} \|x - z\|^2$$

The cells in  $\mathcal{P}$ :

| dim $\delta$ | 0  | 1   | 2   | 3   | 4   | 5  |
|--------------|----|-----|-----|-----|-----|----|
| number       | 43 | 192 | 353 | 323 | 146 | 26 |



$\rightsquigarrow$  degeneration of  $\mathbb{P}^1 \times \mathbb{P}^4$  into a union of 26 toric varieties.

Going through the cells of dimension 2 and 4 reveals that any face  $\delta$  of even dimension either

- has lattice width one (hence rational)
- corresponds to a quadric bundle over  $\mathbb{P}_k^1$  (hence rational).
- defines a conic bundle over  $\mathbb{A}^3$  with a section (hence rational)

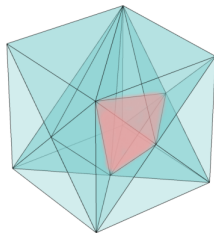
This gives

$$\mathrm{Vol}([\mathcal{X}]_{\mathrm{sb}}) = [HPT] + a[\mathrm{Spec} \mathbb{C}] \quad \text{for some } a \in \mathbb{Z}$$

As this is  $\neq [\mathrm{Spec} \mathbb{C}]$ , a very general  $X$  is stably irrational. □

## End remarks

**General strategy:** Construct subdivisions  $\mathcal{P}$  so that all but one lower-dimensional polytope is stably rational (or make sure that the various intersections do not cancel out).



If time permits:  $(2, 3)$ -complete intersections

Let  $\mathbb{P}^6 = \text{Proj } k[x_0, \dots, x_6]$  and let  $P = \{x_0 = \dots = x_3 = 0\} \simeq \mathbb{P}^2$ .

$$Y = \{q = c = 0\} \subset \mathbb{P}^6$$

for  $q$  and  $c$  very general of degree 2 and 3. We assume  $X$  contains  $P$ .

Blow-up:

$$\begin{array}{ccc} X \subset Bl_P \mathbb{P}^6 & \xrightarrow{\pi} & \mathbb{P}^6 \\ \downarrow p & & \\ \mathbb{P}^3 & & \end{array}$$

$X = Q \cap C$  where  $Q \in |2H - E|$  and  $C \in |3H - E|$ .

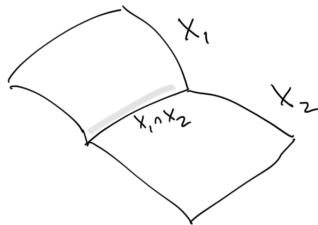
It suffices to show that generic intersections

$$X = Q \cap C \subset \text{Bl}_P \mathbb{P}^6$$

where  $Q \in |2H - E|$  and  $C \in |3H - E|$  are stably irrational.

Now degenerate  $Q$  to  $Q_0 + E$  where  $Q_0 \in |2H - 2E| = |2p^*h|$ .

This induces a degeneration of  $\mathcal{X} \rightarrow \mathbb{A}^1$  with special fiber  $\mathcal{X}_0 = X_1 \cup X_2$ :



There are three strata:

- $X_1 = Q_0 \cap C$
- $X_2 = E \cap C$
- $X_{12} = Q_0 \cap E \cap C$

**The stratum  $X_1 = Q_0 \cap C$ :**

$$\begin{array}{ccc}
 Q_0 = \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}^3 \oplus \mathcal{O}(1,1)) & \hookrightarrow & \mathbb{P}(\mathcal{O}^3 \oplus \mathcal{O}(1)) \xrightarrow{\pi} \mathbb{P}^6 \\
 \downarrow & & \downarrow p \\
 \mathbb{P}^1 \times \mathbb{P}^1 & \hookrightarrow & \mathbb{P}^3
 \end{array}$$

$C|_{Q_0}$  is a very general divisor in  $|\mathcal{O}(2) \otimes p^*\mathcal{O}(1,1)|$  in  $\mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}^3 \oplus \mathcal{O}(1,1))$ .

$\rightsquigarrow X_1$  is stably irrational by [Schreieder 2017].

**The strata  $X_2 = E \cap C$  and  $X_{12} = E \cap Q_0 \cap C$**

$C$  restricts to a  $(1, 2)$ -divisor on  $E \simeq \mathbb{P}^2 \times \mathbb{P}^3$

$Q_0$  restricts to a  $(0, 2)$ -divisor on  $E \simeq \mathbb{P}^2 \times \mathbb{P}^3$ .

$\rightsquigarrow$   $X_2$  and  $X_{12}$  are both rational.

By the motivic volume formula:

$$\begin{aligned}\mathrm{Vol}([\mathcal{X}]_{\mathrm{sb}}) &= [X_1]_{\mathrm{sb}} + [X_2]_{\mathrm{sb}} - [X_{12}]_{\mathrm{sb}} \\ &= [X_1]_{\mathrm{sb}} + [\mathrm{Spec} \mathbb{C}]_{\mathrm{sb}} - [\mathrm{Spec} \mathbb{C}]_{\mathrm{sb}} \\ &\neq [\mathrm{Spec} \mathbb{C}]_{\mathrm{sb}}\end{aligned}$$

This implies that a very general  $X$  is stably irrational.

□