

Winter school characteristic classes schedule

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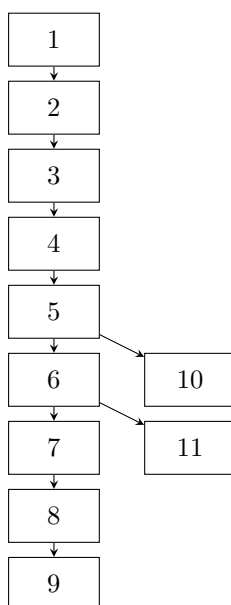


Figure 1: Dependency graph of talks

Talks 1-9 build more-or-less linearly upon each other, and introduce vector bundles, different flavors of characteristic classes, cobordism and characteristic numbers as cobordism invariants, the Hirzebruch signature theorem and the existence of exotic spheres as an application. Talks 10 and 11 present alternative applications/perspectives.

The dependencies in Figure 1 of course don't mean that preparing talk 9 requires a full overview of all details from earlier talks, but of course for the later talks some prior familiarity with some of the material from the earlier talks is helpful.

We encourage in particular the participants with some prior familiarity with vector bundles and characteristic classes to volunteer for the later talks. Prior to the school, the organizers will be available for aiding in the preparation.

1. **Vector bundles and Stiefel-Whitney classes.** Follow [5, Section 2-3] for the definition of vector bundles, their direct sums and their pullbacks, then introduce Stiefel-Whitney classes axiomatically as in Section 4, and discuss some of the applications there (skip 4.4: Stiefel-Whitney numbers)
2. **Classifying spaces of vector bundles.** Following [2, Section 2.2] prove homotopy invariance of pullbacks, then introduce Grassmannians and their universal bundle, and prove Theorem 1.16. (This is [5, Theorem 5.6 + Theorem 5.7], but there it is stated in a weaker form).
3. **The Leray-Hirsch theorem, construction of Stiefel-Whitney and Chern classes, and the splitting principle.** Following [hatcher-vbtk], construct Stiefel-Whitney and Chern classes (the constructions are completely analogous, so focus on one of them and just explain what to modify for the other). Make sure to explain the splitting principle (Proposition 3.3). Both the splitting principle and the construction of the characteristic classes themselves rely on the *Leray-Hirsch theorem*, of which you can find a statement (and proof, which you should only do if time permits) in [1, Theorem 4D.1. and a relative version in Theorem 4D.8.].
4. **The Thom isomorphism, Euler classes and the Gysin sequence.** Following [5, Section 9 and 10] or [1, Section 4D], state the Thom isomorphism (explain the concept of oriented vector bundles) and maybe sketch a proof. (This is a combination of a relative version of the Leray-Hirsch theorem from the previous talk and the existence of a so-called *Thom class*. Depending on the exact technical setup (Hatcher does everything for CW complex base spaces), the latter is some inductive argument.)

Following [2, Section 3.2], construct the Euler class of an oriented vector bundle and prove its basic properties (Proposition 3.13). If time permits, do the example of the tangent bundle of S^n (Proposition 3.14).
5. **The cohomology of Grassmannians.** The goal of this talk is the calculation of the cohomology ring of Grassmannians ([2, Theorem 3.9]). In [2] and [5, Section 7], this is done using a cell structure on Grassmannians. In [5, Theorem 14.5], the proof is instead done more cleanly using the Gysin sequence and the relation of Euler classes to Chern classes. The same argument can be done in the real case using a $\mathbb{Z}/2$ coefficient version of the Gysin sequence, arising analogously from the $\mathbb{Z}/2$ coefficient Thom class which exists without orientability assumptions (and yields w_n as “ $\mathbb{Z}/2$ Euler class”).

The plan for this talk is to introduce the Gysin sequence (following for example [2]), also noting the $\mathbb{Z}/2$ -coefficient version which exists without orientability assumptions, and to calculate the cohomology ring of Grassmannians in both the real ($\mathbb{Z}/2$ -coefficient) case and the complex case, following this strategy. For the real version of the proof, we don't know a reference, but it isn't hard to adapt from the proof of [5, Theorem 14.5].

6. **Pontryagin classes and characteristic numbers.** Following [5, Section 15] and/or [2, pp 94–97], introduce Pontryagin classes, discuss the relationship to Stiefel-Whitney and Euler classes, and sketch a computation of (or at least state the result for) the $\mathbb{Z}[1/2]$ -coefficient cohomology of the oriented Grassmannian. Then, following [5, Section 4.4, Section 16.3], introduce Stiefel-Whitney numbers, Pontryagin numbers and Chern numbers uniformly. Prove Lemma 4.9 and give the oriented version mentioned at the end of Section 16.3.
7. **Cobordisms and the Pontryagin-Thom theorem.** Define the oriented cobordism ring following [5, Section 17]. Sketch the Pontryagin-Thom theorem expressing the cobordism ring in terms of homotopy groups of Thom spaces, for example from [6, Section 2]. (The emphasis here should be on explaining what the spaces are, and how the bijection works both ways, not so much on proving that the constructions are well-defined and inverse to each other.) Then state (not prove) [5, Theorem 4.10] and the oriented analogue mentioned after Corollary 18.10 in the text, and Corollary 18.9
8. **The Hirzebruch signature theorem.** Follow [5, Section 19] to formulate and prove the Signature Theorem.
9. **Milnor’s discovery of an exotic 7-sphere.** The goal of this talk is to discuss [5, Section 20.3, Example 2] of a smooth manifold homeomorphic but not diffeomorphic to S^7 . The fact that it is homeomorphic to S^7 is black-boxed here, but you can check the original paper [4] for a direct proof. If time permits you can state some of the results from the later [3] for context.
10. **Topological K -theory and the Chern character.** Introduce topological K -theory as a generalized cohomology theory following [2, Sections 2.1-2.2], black-boxing Bott periodicity. Then go through the section titled “The Chern character”, pp 109-111.
11. **Chern-Weil theory.** Follow [5, Appendix C] to sketch an interpretation of characteristic classes in terms of curvature forms.

References

- [1] Allen Hatcher. *Algebraic Topology*. Cambridge: Cambridge University Press, 2002. ISBN: 9780521795401. URL: <https://pi.math.cornell.edu/~hatcher/AT/ATpage.html>.
- [2] Allen Hatcher. *Vector Bundles and K-Theory*. Unpublished lecture notes / online draft, 2017. URL: <https://pi.math.cornell.edu/~hatcher/VBKT/VBpage.html>.

- [3] Michel A. Kervaire and John W. Milnor. “Groups of Homotopy Spheres: I”. In: *Annals of Mathematics* 77.3 (1963), pp. 504–537. ISSN: 0003486X, 19398980. URL: <http://www.jstor.org/stable/1970128> (visited on 09/13/2026).
- [4] John Milnor. “On Manifolds Homeomorphic to the 7-Sphere”. In: *Annals of Mathematics* 64.2 (1956), pp. 399–405. ISSN: 0003486X, 19398980. URL: <http://www.jstor.org/stable/1969983> (visited on 09/13/2026).
- [5] John W. Milnor and James D. Stasheff. *Characteristic classes*. Vol. No. 76. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1974, pp. vii+331.
- [6] Robert E. Stong. *Notes on Cobordism Theory*. Mathematical Notes. Princeton, N.J.: Princeton University Press, 1968.