

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MEK4100 — Mathematical Methods in Mechanics

Day of examination: Tuesday 9. December 2025

Examination hours: 09.00–13.00

This problem set consists of 8 pages.

Appendices: None

Permitted aids: K. Rottmann: Matematiske Formelsamling.
Approved calculator.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Suggested solution in blue.

There are 10 individual questions. Each individual question is given a score from 0 to 10 points (10 for complete answer, 0 for blank).

Problem 1

A point mass m moves along a horizontal axis with position x and velocity v . The mass is attached to a spring with spring constant k , such that if the spring suffers an elongation Δx from equilibrium there will be a restoring force $-k\Delta x$. The motion of the mass is limited by friction that depends on the velocity through two different friction coefficients ν and μ , the friction force is $-\nu v - \mu v^2$. We do not consider gravitational pull.

Determine the minimal number of dimensionless parameters needed to describe the motion of the mass. Find a set of such dimensionless parameters.

The problem takes for granted that the dimensions of mass, length, time, velocity and force are known: $[m] = M$, $[x] = [\Delta x] = L$, $[t] = T$, $[v] = L/T$, $[F] = ML/T^2$. It can then be deduced that the dimensions of the other dimensional parameters are $[k] = M/T^2$, $[\nu] = M/T$, $[\mu] = M/L$.

We simplify the problem by x having origin at equilibrium, then $x = \Delta x$.

The problem can be solved either by the Pi theorem [I] or by scaling [II].

[I] The Pi theorem can be employed either on the parameters $\{m, k, \nu, \mu\}$ only, or on those parameters together with position x and time t . The dimension matrices are then, respectively,

$$\begin{array}{c|cccc} & m & k & \nu & \mu \\ \hline M & 1 & 1 & 1 & 1 \\ T & 0 & -2 & -1 & 0 \\ L & 0 & 0 & 0 & -1 \end{array} \quad \text{or} \quad \begin{array}{c|cccccc} & m & k & \nu & \mu & t & x \\ \hline M & 1 & 1 & 1 & 1 & 0 & 0 \\ T & 0 & -2 & -1 & 0 & 1 & 0 \\ L & 0 & 0 & 0 & -1 & 0 & 1 \end{array}$$

both of which have rank 3.

(Continued on page 2.)

In the first case the nullspace has dimension $4 - 3 = 1$. The nullspace is spanned by $(1, 1, -2, 0)^T$. Thus there is one dimensionless parameter that can be expressed by $\pi = mk/\nu^2$.

In the second case the nullspace has dimension $6 - 3 = 3$. Thus there are three dimensionless parameter, for example $\pi_1 = mk/\nu^2$, $\pi_2 = \mu x/m$ and $\pi_3 = \nu t/m$.

[III] Using scaling, we first state the governing equation (Newton's 2nd law)

$$m \frac{d^2 x}{dt^2} + kx + \nu \frac{dx}{dt} + \mu \left(\frac{dx}{dt} \right)^2 = 0$$

and then introduce dimensionless position and time, e.g.

$$\xi = \frac{\mu x}{m} \quad \tau = \frac{\nu t}{m}$$

and express the equation in terms of the scaled variables

$$\frac{d^2 \xi}{d\tau^2} + \frac{km}{\nu^2} \xi + \frac{d\xi}{d\tau} + \left(\frac{d\xi}{d\tau} \right)^2 = 0$$

from which we clearly see that the interesting dimensionless parameter is km/ν^2 .

Problem 2

Given the cubic polynomial equation

$$u^3 + \epsilon u^2 - u - \epsilon^2 = 0, \quad 0 < \epsilon \ll 1.$$

a

Find two-term perturbation expansions for the roots that are nonzero in the limit $\epsilon \rightarrow 0$.

Assume perturbation expansion $u = u_0 + \epsilon u_1 + \dots$.

At order $\mathcal{O}(\epsilon^0)$: $u_0^3 - u_0 = 0$ so $u_0 = \pm 1$ for the roots that are nonzero for vanishing ϵ .

At order $\mathcal{O}(\epsilon^1)$: $3u_0^2 u_1 + u_0^2 - u_1 = 0$ so $u_1 = -1/2$ in both cases.

Two-term solution

$$u = \pm 1 - \frac{\epsilon}{2} + \dots$$

b

Find a two-term perturbation expansion for the root that is zero in the limit $\epsilon \rightarrow 0$. Note that by "two-term" expansion we want both terms to be nonzero.

By brute force, or by dominant balance, the appropriate perturbation expansion turns out to be $u = \epsilon^2 u_2 + \epsilon^5 u_5 + \dots$.

At order $\mathcal{O}(\epsilon^2)$: $-u_2 - 1 = 0$ so $u_2 = -1$.

At order $\mathcal{O}(\epsilon^5)$: $u_2^2 - u_5 = 0$ so $u_5 = 1$.

Two-term solution

$$u = -\epsilon^2 + \epsilon^5 + \dots$$

(Continued on page 3.)

Note that problems 2ab can also be solved by perturbation expansions of the analytical roots (by Cardano's formula) of the cubic polynomial equation.

Problem 3

Find a two-term perturbation solution to the problem

$$\frac{du}{dt} + u = \frac{1}{1 - \epsilon u}, \quad u(0) = 1, \quad 0 < \epsilon \ll 1.$$

We may use the well-known formula $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \dots$ to rewrite $u' + u = 1 + \epsilon u + \epsilon^2 u^2 + \dots$.

Assume $u = u_0 + \epsilon u_1 + \dots$.

At order $\mathcal{O}(\epsilon^0)$: $u'_0 + u_0 = 1$, which has solution $u_0 = 1 + c_0 e^{-t}$, and with the initial condition $u_0(0) = 1$, we determine $c_0 = 0$, so $u_0 = 1$.

At order $\mathcal{O}(\epsilon^1)$: $u'_1 + u_1 = u_0 = 1$, which has solution $u_1 = 1 + c_1 e^{-t}$, and with the initial condition $u_1(0) = 0$, we determine $c_1 = -1$, so $u_1 = 1 - e^{-t}$.

Two-term solution $u = u_0 + \epsilon u_1 = 1 + \epsilon(1 - e^{-t})$.

Problem 4

Consider the boundary value problem

$$\epsilon \frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + y^2 = 0, \quad y(0) = -1, \quad y(1) = 1, \quad 0 < \epsilon \ll 1.$$

a

The problem has a boundary layer. Discuss where it is located. Find the inner approximation of the solution in the boundary layer.

Anticipating the boundary layer has thickness ϵ and is near $x = 1$, introduce inner coordinate $\xi = \epsilon^{-1}(x - 1)$ and obtain

$$\epsilon^{-1} \frac{d^2 y}{d\xi^2} - 2\epsilon^{-1} \frac{dy}{d\xi} + y^2 = 0$$

At leading order we have $\frac{d^2 y}{d\xi^2} - 2 \frac{dy}{d\xi} = 0$, with solution $y(\xi) = \alpha + \beta e^{2\xi}$, which decays exponentially to the finite value α away from the boundary layer for $\xi \rightarrow -\infty$.

If we had assumed the boundary layer near $x = 0$ we would have found the same solution, but growing exponentially away from the boundary layer, which is not desired. This justifies the assumption of a boundary layer near $x = 1$.

With the boundary condition $y(1) = 1$ we get $\alpha + \beta = 1$, so the boundary layer solution is $y(\xi) = \alpha + (1 - \alpha)e^{2\xi}$ or $y_{\text{inner}}(x) = \alpha + (1 - \alpha)e^{2(x-1)/\epsilon}$.

b

Find the outer approximation of the solution.

At leading order we have $-2 \frac{dy}{dx} + y^2 = 0$ which is separable and has solution $y = \frac{2}{c-x}$. With the boundary condition $y(0) = -1$ (which is not at the boundary layer) we determine $c = -2$, so the outer solution is $y_{\text{outer}}(x) = -\frac{2}{2+x}$.

(Continued on page 4.)

c

Obtain the matching and determine a unified solution. Sketch the inner, outer and unified approximations.

Maybe we can imagine two alternative solutions, the first being just OK but not excellent, the second one being excellent:

Alternative 1:

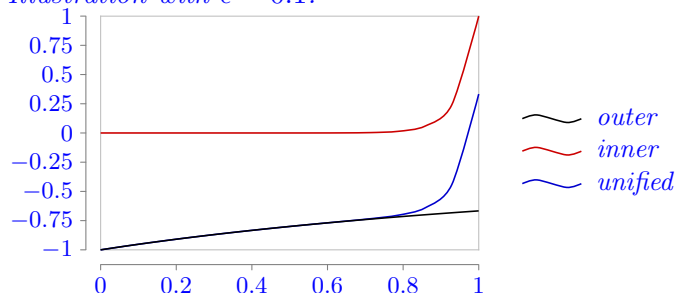
For $x \rightarrow \infty$ the outer solution becomes $y_{\text{outer}}(\infty) = 0$.

For $x \rightarrow -\infty$ the inner solution becomes $y_{\text{inner}}(0) = \alpha$.

Setting these equal, we get $\alpha = 0$.

Unified solution $y_{\text{unified}}(x) = y_{\text{outer}}(x) + y_{\text{inner}}(x) - 0 = -\frac{2}{2+x} + e^{2(x-1)/\epsilon}$.

Illustration with $\epsilon = 0.1$:



The problem with alternative 1 is that the matching takes place outside the domain of the solution, therefore the unified solution does not satisfy the boundary condition at the boundary layer.

Alternative 2:

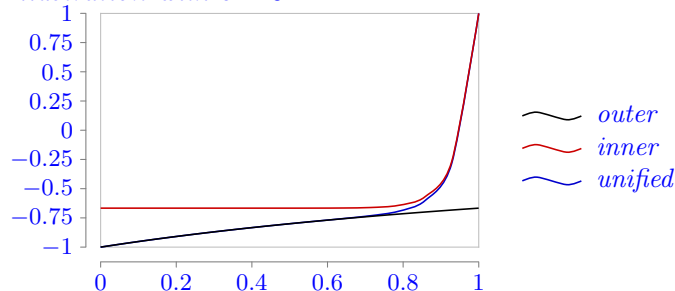
For $x \rightarrow 1$ the outer solution becomes $y_{\text{outer}}(1) = -2/3$.

For $x \rightarrow -\infty$ the inner solution becomes $y_{\text{inner}}(x) = \alpha$.

Setting these equal, we get $\alpha = -2/3$.

Unified solution $y_{\text{unified}}(x) = y_{\text{outer}}(x) + y_{\text{inner}}(x) + \frac{2}{3} = -\frac{2}{2+x} + \frac{5}{3}e^{2(x-1)/\epsilon}$.

Illustration with $\epsilon = 0.1$:



Now the unified solution satisfies all required boundary conditions.

Problem 5

With $y(x)$ a continuously differentiable function, consider the functional

$$J(y) = \int_0^1 (y')^2 - 2xyy' \, dx, \quad y(0) = 1, \quad y(1) \text{ is free}$$

Find an extremal solution of the functional.

(Continued on page 5.)

Let $L = (y')^2 - 2xyy'$.

We need to solve the Euler–Lagrange equation $\frac{\partial L}{\partial y} - \frac{d}{dx}(\frac{\partial L}{\partial y'}) = 0$, together with the free boundary condition $\frac{\partial L}{\partial y'} = 0$ at $x = 1$, and with the given boundary condition $y(0) = 1$ at $x = 0$.

The Euler–Lagrange equation is $-2xy' - \frac{d}{dx}(2y' - 2xy) = -2xy' - 2y'' + 2y + 2xy' = -2(y'' - y) = 0$, which has solution $y(x) = ae^x + be^{-x}$.

The given boundary condition at $x = 0$ is $y(0) = a + b = 1$.

The free boundary condition at $x = 1$ is $2y'(1) - 2xy(1) = 2(y'(1) - y(1)) = 0$, which gives $b = 0$.

Thus $a = 1$ and the extremal solution is $y(x) = e^x$.

Problem 6

Solve the problem

$$Lu \equiv \frac{d^2u}{dt^2} - u = 0 \quad \text{for } t \geq 0$$

where u satisfies the boundary conditions

$$\frac{du}{dt}(0) + u(0) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} u(t) = 0$$

Find the adjoint operator $L^\dagger$ and find the nullspace of $L^\dagger$.

The problem $Lu = 0$ has general solution $u(x) = \alpha e^{-t} + \beta e^t$. Both boundary conditions require $\beta = 0$. The solution of the boundary value problem is therefore $u(t) = \alpha e^{-t}$ where α can be any number.

In order to find the adjoint operator, we first have to define an appropriate inner product, the most obvious choice is $\langle u, v \rangle = \int_0^\infty u(t)\bar{v}(t) dt$. Then, carrying out two integrations by parts, we have

$$\begin{aligned} \langle Lu, v \rangle &= \int_0^\infty \left(\frac{d^2u}{dt^2} - u \right) \bar{v} dt = \left[\frac{du}{dt} \bar{v} \right]_0^\infty + \int_0^\infty \frac{du}{dt} \frac{d\bar{v}}{dt} - u \bar{v} dt \\ &= \left[\frac{du}{dt} \bar{v} - u \frac{d\bar{v}}{dt} \right]_0^\infty + \int_0^\infty u \left(\frac{d^2\bar{v}}{dt^2} - \bar{v} \right) dt = \langle u, Lv \rangle \end{aligned}$$

We conclude that the operator is self-adjoint, $L^\dagger = L$, and we have already found that its nullspace is spanned by e^{-t} .

Note: We applied the same boundary conditions for u and v , and that made the boundary contributions disappear. In particular, at $t = 0$ we have $u'\bar{v} - u\bar{v}' = (u' + u)\bar{v} - u(\bar{v}' + \bar{v}) = 0$.

Problem 7

Consider the problem

$$\frac{d^2u}{dt^2} - u = -\epsilon q(\epsilon t)u \quad \text{for } t \geq 0$$

where $0 < \epsilon \ll 1$ and $q(0) = 0$ and where u satisfies the boundary conditions

$$\frac{du}{dt}(0) + u(0) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} u(t) = 0$$

(Continued on page 6.)

Find an approximate solution for u describing the combined behaviour on the fast scale t and the slow scale $\tau = \epsilon t$.

The problem can be solved in several different ways, here we outline three different methods, with increasing level of complication:

First and easiest method: WKB with exponential substitution

Focus on the slow scale only, substitute t with $\tau = \epsilon t$, $\frac{d}{dt} = \frac{d\tau}{dt} \frac{d}{d\tau} = \epsilon \frac{d}{d\tau}$. The problem becomes

$$\epsilon^2 \frac{d^2 u}{d\tau^2} - u + \epsilon q(\tau)u = 0, \quad \tau \geq 0$$

with boundary conditions

$$\epsilon \frac{du}{d\tau}(0) + u(0) = 0 \quad \text{and} \quad \lim_{\tau \rightarrow \infty} u(\tau) = 0$$

Substitute $u = e^{S/\epsilon}$, $u' = \epsilon^{-1} S' e^{S/\epsilon}$, $u'' = [\epsilon^{-2} (S')^2 + \epsilon^{-1} S''] e^{S/\epsilon}$. The problem becomes

$$(S')^2 + \epsilon S'' - 1 + \epsilon q(\tau) = 0, \quad \tau \geq 0$$

with boundary conditions

$$S'(0) + 1 = 0 \quad \text{and} \quad \lim_{\tau \rightarrow \infty} S(\tau) = -\infty$$

Introduce the perturbation expansion $S = S_0 + \epsilon S_1 + \dots$. The problem at order $\mathcal{O}(\epsilon^0)$ is

$$(S'_0)^2 - 1 = 0, \quad \tau \geq 0$$

with boundary conditions

$$S'_0(0) + 1 = 0 \quad \text{and} \quad \lim_{\tau \rightarrow \infty} S_0(\tau) = -\infty$$

which has solution

$$S_0(\tau) = -\tau + c_0$$

The problem at order $\mathcal{O}(\epsilon^1)$ is

$$2S'_0 S'_1 + S''_0 + q(\tau) = 0, \quad \tau \geq 0$$

with boundary conditions

$$S'_1(0) = 0 \quad \text{and} \quad \lim_{\tau \rightarrow \infty} (S_0(\tau) + \epsilon S_1(\tau)) = -\infty$$

which has solution

$$S_1(\tau) = \frac{1}{2} \int_0^\tau q(\tau') d\tau' + c_1$$

The solution for u is therefore

$$u(\tau) \approx u(0) e^{-\frac{\tau}{\epsilon} + \frac{1}{2} \int_0^\tau q(\tau') d\tau'}$$

where $u(0) = e^{c_0 + \epsilon c_1}$.

(Continued on page 7.)

Second method: WKB with slowly varying amplitude of the basic exponential

From the previous problem 6 we already know that in the limit $\epsilon = 0$ the solution is $u = \alpha e^{-t}$. For small nonzero ϵ we may assume the rapid exponential decay should be multiplied by a slowly varying amplitude. Thus we may start with the above equations with focus on the slow scale only

$$\epsilon^2 \frac{d^2 u}{d\tau^2} - u + \epsilon q(\tau)u = 0, \quad \tau \geq 0$$

with boundary conditions

$$\epsilon \frac{du}{d\tau}(0) + u(0) = 0 \quad \text{and} \quad \lim_{\tau \rightarrow \infty} u(\tau) = 0$$

and assume

$$u = A(\tau)e^{-\tau/\epsilon}$$

Substitution into the equations gives

$$A - 2\epsilon A' + \epsilon^2 A'' - A = -\epsilon q(\tau)A, \quad \tau \geq 0$$

with boundary conditions

$$-A(0) + \epsilon A'(0) + A(0) = 0 \quad \text{and} \quad \lim_{\tau \rightarrow \infty} A(\tau)e^{-\tau/\epsilon} = 0$$

At leading order we solve

$$-2A' = -q(\tau)A \quad \text{with} \quad A'(0) = 0$$

which leads to

$$A(\tau) = A(0)e^{\frac{1}{2} \int_0^\tau q(\tau') d\tau'}$$

which gives us the same solution as before

$$u(\tau) \approx u(0)e^{-\frac{\tau}{\epsilon} + \frac{1}{2} \int_0^\tau q(\tau') d\tau'}$$

Third method: Multiple scales perturbation expansion

Warning!!! This is very cumbersome!!!

With simultaneous fast and slow scales, $u(t) \rightarrow u(t, \tau)$ and $\frac{du}{dt} \rightarrow \frac{\partial u}{\partial t} + \epsilon \frac{\partial u}{\partial \tau}$ and $\frac{d^2 u}{dt^2} \rightarrow \frac{\partial^2 u}{\partial t^2} + 2\epsilon \frac{\partial^2 u}{\partial t \partial \tau} + \epsilon^2 \frac{\partial^2 u}{\partial \tau^2}$, the problem becomes

$$\frac{\partial^2 u}{\partial t^2} + 2\epsilon \frac{\partial^2 u}{\partial t \partial \tau} + \epsilon^2 \frac{\partial^2 u}{\partial \tau^2} - u + \epsilon q(\tau)u = 0 \quad \text{for } t \geq 0 \quad \text{and} \quad \tau \geq 0$$

with boundary conditions

$$\frac{\partial u}{\partial t} + \epsilon \frac{\partial u}{\partial \tau} + u = 0 \quad \text{at } t = \tau = 0 \quad \text{and} \quad u \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad \text{and} \quad \tau \rightarrow \infty$$

Employ the perturbation series $u = u_0 + \epsilon u_1 + \dots$

The problem at order $\mathcal{O}(\epsilon^0)$ is

$$\frac{\partial^2 u_0}{\partial t^2} - u_0 = 0 \quad \text{for } t \geq 0 \quad \text{and} \quad \tau \geq 0$$

(Continued on page 8.)

with boundary conditions

$$\frac{\partial u_0}{\partial t} + u_0 = 0 \quad \text{at} \quad t = \tau = 0 \quad \text{and} \quad u_0 \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty \quad \text{and} \quad \tau \rightarrow \infty$$

which has solution

$$u_0(t, \tau) = a(\tau)e^{-t}$$

The problem at order $\mathcal{O}(\epsilon^1)$ is

$$\frac{\partial^2 u_1}{\partial t^2} - u_1 = -2 \frac{\partial^2 u_0}{\partial t \partial \tau} - q(\tau)u_0 = \left(2 \frac{\partial a}{\partial \tau} - q(\tau)a\right) e^{-t} \quad \text{for} \quad t \geq 0 \quad \text{and} \quad \tau \geq 0$$

with boundary conditions

$$\frac{\partial u_1}{\partial t} + u_1 = -\frac{\partial u_0}{\partial \tau} = -\frac{\partial a}{\partial \tau} \quad \text{at} \quad t = \tau = 0 \quad \text{and} \quad u_1 \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty \quad \text{and} \quad \tau \rightarrow \infty$$

Now we need to impose a solvability condition: The inhomogeneity must be orthogonal to the nullspace of the adjoint operator $L^\dagger$, with appropriate consideration of the boundary conditions. Suppose the problem is $Lu_1 = b$ with boundary conditions. Thus with v in the nullspace of $L^\dagger$ we require

$$\langle Lu_1, v \rangle = \langle b, v \rangle = \left[\frac{\partial u_1}{\partial t} \bar{v} - u_1 \frac{\partial \bar{v}}{\partial t} \right]_{t=0}^\infty$$

From the previous problem we already know that the nullspace of $L^\dagger$, with the relevant boundary conditions, is spanned by the function $v = e^{-t}$. Thus we require

$$\int_0^\infty \left(2 \frac{\partial a}{\partial \tau} - q(\tau)a\right) e^{-t} e^{-t} dt = \frac{\partial u_0}{\partial \tau} \bar{v} \Big|_{t=0} = \frac{\partial a}{\partial \tau}(0)$$

Warning!!! This is very cumbersome!!!

We did not spend much time learning about how to solve such integral equations, however, it is nice to observe that if we set

$$2 \frac{\partial a}{\partial \tau} - q(\tau)a = 0$$

we get precisely

$$a(\tau) = a(0)e^{\frac{1}{2} \int_0^\tau q(\tau') d\tau'}$$

and then $\frac{\partial a}{\partial \tau}(0) = 0$ due to the requirement that $q(0) = 0$.

So, in principle, the third method also works.