

Problem 1a)

|   | k  | $\omega$ | U  | g  | $\sigma$ | $\rho$ |
|---|----|----------|----|----|----------|--------|
| L | -1 | 0        | 1  | 1  | 0        | -3     |
| T | 0  | -1       | -1 | -2 | -2       | 0      |
| M | 0  | 0        | 0  | 0  | 1        | 1      |

6 variable, rang 3,  $6-3=3$  dimensjonsløse grupper

For eksempel

$$\pi_1 = \frac{Uk}{\omega}$$

$$\pi_2 = \frac{gk}{\omega^2}$$

$$\pi_3 = \frac{\sigma k^3}{\rho \omega^2}$$

eller

$$\pi_1 = U \sqrt{\frac{k}{g}}$$

$$\pi_2 = \frac{gk}{\omega^2}$$

$$\pi_3 = \frac{\sigma k^2}{\rho g}$$

eller etc.

Problem 1b)

$$F(\pi_1, \pi_2, \pi_3) = 0$$

$$\pi_2 = f(\pi_1, \pi_3)$$

$$\omega^2 = gk f\left(U \sqrt{\frac{k}{g}}, \frac{\sigma k^2}{\rho g}\right)$$

eller

$$\omega = \sqrt{gk} f\left(U \sqrt{\frac{k}{g}}, \frac{\sigma k^2}{\rho g}\right)$$

(The dispersion relation is actually  $(\omega - kU)^2 = gk + \frac{\sigma k^3}{\rho}$ , but you are not supposed to know that.)

Problem 2)

$$y(x) = \int_0^x (x-s) y(s) ds + 1$$

$$y(0) = 1 \quad (\text{initial condition})$$

$$y'(x) = \int_0^x y(s) ds$$

$$y'(0) = 0 \quad (\text{another initial condition})$$

$$y''(x) = y \quad (\text{differential equation})$$

So we solve the initial value problem

$$y'' - y = 0$$

$$y(0) = 1 \quad y'(0) = 0$$

$$y = a e^x + b e^{-x}$$

$$y' = a e^x - b e^{-x}$$

$$y(0) = a + b = 1$$

$$y'(0) = a - b = 0$$

$$\Rightarrow a = b = \frac{1}{2}$$

$$y = \frac{1}{2} (e^x + e^{-x}) = \cosh x$$

Problem 3)



$$\underline{r} = l \underline{i}_r$$

$$\underline{v} = \dot{l} \underline{i}_r + l \dot{\underline{i}}_r$$

$$= \dot{l} \underline{i}_r + l \dot{\theta} \underline{i}_\theta$$

$$v^2 = \dot{l}^2 + l^2 \dot{\theta}^2$$

$$T = \frac{1}{2} m (\dot{l}^2 + l^2 \dot{\theta}^2)$$

$$V = -mg l \cos \theta$$

$$L = T - V = \frac{1}{2} m (\dot{l}^2 + l^2 \dot{\theta}^2) + mg l \cos \theta$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) = -mg l \sin \theta - \frac{d}{dt} (m l^2 \dot{\theta}) = 0$$

$$m \left[ -2l \dot{l} \dot{\theta} - l^2 \ddot{\theta} - g l \sin \theta \right] = 0$$

$$\ddot{\theta} + 2 \frac{\dot{l}}{l} \dot{\theta} + \frac{g}{l} \sin \theta = 0$$

# Problem 4)

$$\sqrt{x} = \cos(\varepsilon x)$$

$$t = \sqrt{x}$$

$$t = \cos(\varepsilon t^2) = 1 - \frac{1}{2} \varepsilon^2 t^4 + \dots$$

Assume

$$t = t_0 + \varepsilon^2 t_1 + \dots$$

$$t^4 = t_0^4 + 4\varepsilon^2 t_1 t_0^3 + \dots$$

$$\rightarrow t_0 + \varepsilon^2 t_1 + \dots = 1 - \frac{1}{2} \varepsilon^2 t_0^4 + \dots$$

$$\mathcal{O}(\varepsilon^0) : t_0 = 1$$

$$\mathcal{O}(\varepsilon^2) : t_1 = -\frac{1}{2} t_0^4 = -\frac{1}{2}$$

$$t = 1 - \frac{1}{2} \varepsilon^2 + \dots$$

$$x = t^2 = \left(1 - \frac{1}{2} \varepsilon^2 + \dots\right)^2$$

$$= 1 - \varepsilon^2 + \dots$$

# Problem 5)

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$$\varepsilon y'' + q(x)(y' + y) = 0$$

$$y(0) = 1 \quad y'(0) = \varepsilon^{-1}$$

$$y = e^s$$

$$y' = s' e^s$$

$$y'' = [s'' + (s')^2] e^s$$

$$\varepsilon s'' + \varepsilon (s')^2 + q s' + q = 0$$

Try dominant balance  $q s' \sim q$

$$s' = -1 + O(\varepsilon)$$

$$s = -x + O(\varepsilon)$$

Try dominant balance  $\varepsilon (s')^2 \sim q s'$

$$s' \sim \varepsilon^{-1}$$

$$\varepsilon (s')^2 + q s' = 0$$

$$s' = -\frac{q}{\varepsilon} + O(\varepsilon^0)$$

$$s = -\varepsilon^{-1} \int q(s) ds + O(\varepsilon^0)$$

Both of these dominant balances are good (\*) and represent the fundamental solutions of the equation.

(\*) Show that!

Use the two fundamental solutions found 6

$$y = \alpha e^{-x + o(\varepsilon)} + \beta e^{-\varepsilon^{-1} \int_0^x q(s) ds + o(1)}$$
$$y' = -\alpha e^{-x} - \varepsilon^{-1} \beta q(x) e^{-\varepsilon^{-1} \int_0^x q(s) ds}$$

$$y(0) = \alpha + \beta = 1$$

$$y'(0) = -\alpha - \varepsilon^{-1} q(0) \beta = \varepsilon^{-1}$$

$$\Rightarrow \beta = -\frac{1}{q(0)} \quad \alpha = 1 + \frac{1}{q(0)}$$
$$y(x) = \left(1 + \frac{1}{q(0)}\right) e^{-x} - \frac{1}{q(0)} e^{-\varepsilon^{-1} \int_0^x q(s) ds}$$

# Problem 6)

$$y'' + y = \varepsilon y^3$$

$$y(0) = 0 \quad y'(0) = 1$$

$$y = y_0 + \varepsilon y_1 + \dots$$

$$y_0'' + \varepsilon y_1'' + \dots + y_0 + \varepsilon y_1 + \dots = \varepsilon y_0^3$$

$$O(\varepsilon^0): \quad y_0'' + y_0 = 0 \quad y_0(0) = 0 \quad y_0'(0) = 1$$

$$y_0 = a \cos t + b \sin t$$

$$y_0(0) = a = 0$$

$$y_0' = b \cos t$$

$$y_0'(0) = b = 1$$

$$y_0(t) = \sin t$$

$$O(\varepsilon^1): \quad y_1'' + y_1 = \sin^3 t = -\frac{1}{4} \sin 3t + \frac{3}{4} \sin t$$

↑  
gir resonans

Prøve derfor Poincaré-Lindsted

$$\tau = (1 + \varepsilon \omega_1 + \dots) t$$

$$\frac{d}{dt} = \frac{d\tau}{dt} \frac{d}{d\tau} = (1 + \varepsilon \omega_1 + \dots) \frac{d}{d\tau}$$

$$\frac{d^2}{dt^2} = (1 + 2\varepsilon \omega_1 + \dots) \frac{d^2}{d\tau^2}$$

Nytt problem til orden  $O(\varepsilon)$

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$$2\omega_1 \underbrace{\frac{d^2 y_0}{d\tau^2}}_{=-\sin\tau} + \frac{d^2 y_1}{d\tau^2} + y_1 = -\frac{1}{4} \sin 3\tau + \frac{3}{4} \sin \tau$$

Set  $2\omega_1 = \frac{3}{4} \Rightarrow \omega_1 = -\frac{3}{8}$ , så blir resonant vekst borte.

partikulær løsning

$$y_{1,p} = A \sin 3\tau$$

$$-9A + A = -\frac{1}{4} \Rightarrow A = \frac{1}{32}$$

generell løsning

$$y_1 = \frac{1}{32} \sin 3\tau + \alpha \sin \tau + \beta \cos \tau$$

Initialbetingelser

$$y_1(0) = 0 \quad \text{og} \quad \omega_1 y_0'(0) + y_1'(0) = 0$$

$$\Rightarrow \beta = 0$$

$$\omega_1 y_0'(0) + y_1'(0) = 0$$

$$-\frac{3}{8} \cdot 1 + \frac{3}{32} + \alpha = 0 \Rightarrow \alpha = \frac{3}{8} - \frac{3}{32} = \frac{9}{32}$$

$$y_1 = \frac{1}{32} \sin 3\tau + \frac{9}{32} \sin \tau$$

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Løsning  $y(\tau) = \sin \tau + \frac{\varepsilon}{32} (\sin 3\tau + 9 \sin \tau)$

$$\text{hvor } \tau = \left(1 - \frac{3}{8} \varepsilon\right) t$$