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$\boxed{\text{Stokes waves}}$ - nonlinear modification of the linear monochromatic wave

Exact kinematic surface condition

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} - \frac{\partial \phi}{\partial z} = 0 \qquad \text{at } z = \eta$$

Exact dynamic surface condition

$$\frac{\partial \phi}{\partial t} + g\eta + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] = 0 \qquad \text{at } z = \eta$$

Waves with typical wavenumber k_c, angular frequency ω_c, and typical amplitude a_c.

Normalize

$$t' = \omega_c t$$

$$x' = k_c x \qquad z' = k_c z$$

$$\eta = a_c \eta' \qquad \phi = \frac{\omega_c a_c}{k_c}\phi'$$

We have

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} = k_c \frac{\partial}{\partial x'} \qquad \frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} = \omega_c \frac{\partial}{\partial t'}$$

Plug in

$$\omega_c a_c \frac{\partial \eta'}{\partial t'} + k_c \frac{\omega_c a_c}{k_c}\frac{\partial \phi'}{\partial x'} k_c a_c \frac{\partial \eta'}{\partial x'} - k_c \frac{\omega_c a_c}{k_c}\frac{\partial \phi'}{\partial z'} = 0$$

$$\omega_c \frac{\omega_c a_c}{k_c}\frac{\partial \phi'}{\partial t'} + g a_c \eta' + \frac{1}{2}\left(k_c \frac{\omega_c a_c}{k_c}\right)^2 \left[\left(\frac{\partial \phi'}{\partial x'}\right)^2 + \left(\frac{\partial \phi'}{\partial z'}\right)^2\right] = 0$$

both at $z' = k_c a_c \eta'$

Introduce steepness $\varepsilon \equiv k_c a_c \ll 1$
and drop the primes

$$\left.\begin{aligned} &\frac{\partial \eta}{\partial t} + \varepsilon \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} - \frac{\partial \phi}{\partial z} = 0 \\ &\frac{\partial \phi}{\partial t} + \underbrace{\left(\frac{g k_c}{\omega_c^2}\right)}_{=1} \eta + \frac{\varepsilon}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] = 0 \end{aligned}\right\} \text{at } z = \varepsilon \eta$$

$=1$ by virtue of the linear dispersion relation $\omega_c^2 = g k_c$

Since $\varepsilon \ll 1$ is small we can Taylor-expand around $z = 0$ up to cubic nonlinear terms

$$\frac{\partial \eta}{\partial t} + \varepsilon \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} + \varepsilon^2 \eta \frac{\partial^2 \phi}{\partial x \partial z}\frac{\partial \eta}{\partial x} - \frac{\partial \phi}{\partial z} - \varepsilon \eta \frac{\partial^2 \phi}{\partial z^2} - \frac{\varepsilon^2}{2}\eta^2 \frac{\partial^3 \phi}{\partial z^3} = 0$$

$$\frac{\partial \phi}{\partial t} + \varepsilon \eta \frac{\partial^2 \phi}{\partial t \partial z} + \frac{1}{2}\varepsilon^2 \eta^2 \frac{\partial^3 \phi}{\partial t \partial z^2} + \eta + \frac{\varepsilon}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] + \frac{\varepsilon^2}{2}\eta\left[2\frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x \partial z} + 2\frac{\partial \phi}{\partial z}\frac{\partial^2 \phi}{\partial z^2}\right] = 0$$

at $z = 0$

In addition we have Laplace equation $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial z^2} = 0 \qquad -\infty < z < 0$

bottom "boundary" condition $\dfrac{\partial \phi}{\partial z} \to 0$ for $z \to -\infty$

It will turn out that we will need a slow scale response at $t_2 = \varepsilon^2 t$

thus

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} + \frac{\partial t_2}{\partial t}\frac{\partial}{\partial t_2} = \frac{\partial}{\partial t} + \varepsilon^2 \frac{\partial}{\partial t_2}$$

$$\frac{\partial \eta}{\partial t} + \varepsilon^2 \frac{\partial \eta}{\partial t_2} + \varepsilon \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} + \varepsilon^2 \eta \frac{\partial^2 \phi}{\partial x \partial z}\frac{\partial \eta}{\partial x} - \frac{\partial \phi}{\partial z} - \varepsilon \eta \frac{\partial^2 \phi}{\partial z^2} - \frac{1}{2}\varepsilon^2 \eta^2 \frac{\partial^3 \phi}{\partial z^3} = 0$$

$$\frac{\partial \phi}{\partial t} + \varepsilon^2 \frac{\partial \phi}{\partial t_2} + \varepsilon \eta \frac{\partial^2 \phi}{\partial t \partial z} + \frac{1}{2}\varepsilon^2 \eta^2 \frac{\partial^3 \phi}{\partial t \partial z^2} + \eta + \frac{\varepsilon}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] + \varepsilon^2 \eta \left[\frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x \partial z} + \frac{\partial \phi}{\partial z}\frac{\partial^2 \phi}{\partial z^2}\right] = 0$$

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perturbation expansion

1 means linear

2 means quadratic

3 means cubic

$$\eta = \eta_1 + \varepsilon \eta_2 + \varepsilon^2 \eta_3 + \cdots$$

$$\phi = \phi_1 + \varepsilon \phi_2 + \varepsilon^2 \phi_3 + \cdots$$

$$\frac{\partial \eta_1}{\partial t} + \varepsilon \frac{\partial \eta_2}{\partial t} + \varepsilon^2 \frac{\partial \eta_3}{\partial t} + \varepsilon^2 \frac{\partial \eta_1}{\partial t_2} + \varepsilon \frac{\partial \phi_1}{\partial x}\frac{\partial \eta_1}{\partial x} + \varepsilon^2 \frac{\partial \phi_2}{\partial x}\frac{\partial \eta_1}{\partial x} + \varepsilon^2 \frac{\partial \phi_1}{\partial x}\frac{\partial \eta_2}{\partial x}$$

$$+ \varepsilon^2 \eta_1 \frac{\partial^2 \phi_1}{\partial x \partial z}\frac{\partial \eta_1}{\partial x} - \frac{\partial \phi_1}{\partial z} - \varepsilon \frac{\partial \phi_2}{\partial z} - \varepsilon^2 \frac{\partial \phi_3}{\partial z}$$

$$- \varepsilon \eta_1 \frac{\partial^2 \phi_1}{\partial z^2} - \varepsilon^2 \eta_2 \frac{\partial^2 \phi_1}{\partial z^2} - \varepsilon^2 \eta_1 \frac{\partial^2 \phi_2}{\partial z^2} - \frac{1}{2}\varepsilon^2 \eta_1^2 \frac{\partial^3 \phi_1}{\partial z^3} = 0$$

Let us put g back in, so we recognize where it appears

$$\frac{\partial \phi_1}{\partial t} + \varepsilon \frac{\partial \phi_2}{\partial t} + \varepsilon^2 \frac{\partial \phi_3}{\partial t} + \varepsilon^2 \frac{\partial \phi_1}{\partial t_2}$$

$$+ \varepsilon \eta_1 \frac{\partial^2 \phi_1}{\partial t \partial z} + \varepsilon^2 \eta_2 \frac{\partial^2 \phi_1}{\partial t \partial z} + \varepsilon^2 \eta_1 \frac{\partial^2 \phi_2}{\partial t \partial z} + \frac{1}{2}\varepsilon^2 \eta_1^2 \frac{\partial^3 \phi_1}{\partial t \partial z^2}$$

$$+ g\eta_1 + \varepsilon g \eta_2 + \varepsilon^2 g \eta_3$$

$$+ \frac{\varepsilon}{2}\left[\left(\frac{\partial \phi_1}{\partial x}\right)^2 + \left(\frac{\partial \phi_1}{\partial z}\right)^2\right] + \varepsilon^2\left[\frac{\partial \phi_1}{\partial x}\frac{\partial \phi_2}{\partial x} + \frac{\partial \phi_1}{\partial z}\frac{\partial \phi_2}{\partial z}\right]$$

$$+ \varepsilon^2 \eta_1 \left[\frac{\partial \phi_1}{\partial x}\frac{\partial^2 \phi_1}{\partial x \partial z} + \frac{\partial \phi_1}{\partial z}\frac{\partial^2 \phi_1}{\partial z^2}\right] = 0$$

$O(\varepsilon^0)$

$$\left.\begin{aligned} \frac{\partial \eta_1}{\partial t} - \frac{\partial \phi_1}{\partial z} &= 0 \\ \frac{\partial \phi_1}{\partial t} + g\eta_1 &= 0 \end{aligned}\right\} z = 0$$

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial z^2} = 0 \qquad z < 0$$

$$\frac{\partial \phi_1}{\partial z} \to 0 \qquad z \to -\infty$$

Assume

$$\begin{pmatrix} \eta_1 \\ \phi_1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \hat{\eta}_1 \\ \hat{\phi}_1 \end{pmatrix} e^{i(kx - \omega t)} + \text{c.c.}$$

$$\left.\begin{aligned} -i\omega \hat{\eta}_1 - \frac{\partial \hat{\phi}_1}{\partial z} &= 0 \\ -i\omega \hat{\phi}_1 + g\hat{\eta}_1 &= 0 \end{aligned}\right\} z = 0 \qquad \hat{\eta}_1 = A$$

$$\left.\begin{aligned} -k^2 \hat{\phi}_1 + \frac{\partial^2 \hat{\phi}_1}{\partial z^2} &= 0 \qquad z < 0 \\ \frac{\partial \hat{\phi}_1}{\partial z} &\to 0 \qquad z \to -\infty \end{aligned}\right\} \Rightarrow \hat{\phi}_1 = Be^{kz}$$

$$-i\omega A - kB = 0$$

$$-i\omega B + gA = 0$$

$$\begin{pmatrix} -i\omega & -k \\ g & -i\omega \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-\omega^2 + gk = 0 \qquad B = -\frac{ig}{\omega} A$$

$$\hat{\eta}_1 = A \qquad \hat{\phi}_1 = -\frac{ig}{\omega} A e^{kz}$$

$$\begin{pmatrix} \eta_1 \\ \phi_1 \end{pmatrix} = A \begin{pmatrix} 1 \\ -\frac{ig}{\omega} e^{kz} \end{pmatrix} e^{i(kx-\omega t)} + \text{c.c.}$$

$O(\varepsilon^1)$

$$\left.\begin{aligned} \frac{\partial \eta_2}{\partial t} - \frac{\partial \phi_2}{\partial z} &= -\frac{\partial \phi_1}{\partial x}\frac{\partial \eta_1}{\partial x} + \eta_1 \frac{\partial^2 \phi_1}{\partial z^2} \\ \frac{\partial \phi_2}{\partial t} + g\eta_2 &= -\eta_1 \frac{\partial^2 \phi_1}{\partial t \partial z} - \frac{1}{2}\left[\left(\frac{\partial \phi_1}{\partial x}\right)^2 + \left(\frac{\partial \phi_1}{\partial z}\right)^2\right] \end{aligned}\right\} z=0$$

$$\frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial z^2} = 0 \qquad z < 0$$

$$\frac{\partial \phi_2}{\partial z} \to 0 \qquad z \to -\infty$$

Inhomogeneous terms (right hand side) are quadratic

$$\left(e^{i(kx-\omega t)}, e^{-i(kx-\omega t)}\right) \cdot \left(e^{i(kx-\omega t)}, e^{-i(kx-\omega t)}\right) = \left(e^0, e^{2i()}, e^{-2i()}\right)$$

Forcing will be zeroth or second harmonic

$$\eta_2 = \hat{\eta}_{2,0} + \frac{1}{2}\left(\hat{\eta}_{2,2}\, e^{2i(\,)} + \hat{\eta}_{2,-2}\, e^{-2i(\,)}\right)$$

$$\phi_2 = \hat{\phi}_{2,0} + \frac{1}{2}\left(\hat{\phi}_{2,2}\, e^{2i(\,)} + \hat{\phi}_{2,-2}\, e^{-2i(\,)}\right)$$

$O(\varepsilon^1)$ zeroth harmonic

$$\left.\begin{aligned} \frac{\partial \hat{\phi}_{2,0}}{\partial z} &= 0 \\ \hat{\eta}_{2,0} &= 0 \end{aligned}\right\} \text{at } z=0$$

$$\frac{\partial^2 \hat{\phi}_{2,0}}{\partial z^2} = 0 \qquad z<0$$

$$\frac{\partial \hat{\phi}_{2,0}}{\partial z} \to 0 \quad \text{as } z \to -\infty$$

Solution $\quad \hat{\eta}_{2,0} = 0 \qquad \hat{\phi}_{2,0} = \Phi(x, t, t_2)$

$\uparrow$

arbitrary horizontal current

$O(\varepsilon^1)$ second harmonic $\quad e^{2i(kx-\omega t)}$

$$\left.\begin{aligned} -2i\omega\, \hat{\eta}_{2,2} - \frac{\partial \hat{\phi}_{2,2}}{\partial z} &= -\frac{igk^2}{\omega} A^2 \\ -2i\omega\, \hat{\phi}_{2,2} + g\, \hat{\eta}_{2,2} &= \frac{gk}{2} A^2 \end{aligned}\right\} \text{at } z=0$$

$$\frac{\partial^2 \hat{\phi}_{2,2}}{\partial z^2} - 4k^2\, \hat{\phi}_{2,2} = 0 \qquad z<0$$

$$\frac{\partial \hat{\phi}_{2,2}}{\partial z} \to 0 \quad \text{as } z \to -\infty$$

Easy to find solution

$$\hat{\eta}_{2,2} = \frac{k}{2} A^2 \qquad \hat{\phi}_{2,2} = 0$$

It is remarkable that $\hat{\phi}_{2,2}$ vanishes!
This only happens for infinite depth!

$O(\varepsilon^2)$

$$\frac{\partial \eta_3}{\partial t} - \frac{\partial \phi_3}{\partial z} = -\frac{\partial \eta_1}{\partial t_2} - \frac{\partial \phi_1}{\partial x}\frac{\partial \eta_2}{\partial x} - \frac{\partial \phi_2}{\partial x}\frac{\partial \eta_1}{\partial x}$$

$$- \eta_1 \frac{\partial^2 \phi_1}{\partial x \partial z}\frac{\partial \eta_1}{\partial x} + \eta_1 \frac{\partial^2 \phi_2}{\partial z^2} + \eta_2 \frac{\partial^2 \phi_1}{\partial z^2}$$

$$+ \frac{1}{2}\eta_1^2 \frac{\partial^3 \phi_1}{\partial z^3}$$

$$\frac{\partial \phi_3}{\partial t} + g\,\eta_3 = -\frac{\partial \phi_1}{\partial t_2} - \eta_1 \frac{\partial^2 \phi_2}{\partial z \partial t} - \eta_2 \frac{\partial^2 \phi_1}{\partial z \partial t}$$

$$- \frac{1}{2}\eta_1^2 \frac{\partial^3 \phi_1}{\partial z^2 \partial t} - \frac{\partial \phi_1}{\partial x}\frac{\partial \phi_2}{\partial x} - \frac{\partial \phi_1}{\partial z}\frac{\partial \phi_2}{\partial z}$$

$$- \eta_1 \left[\frac{\partial \phi_1}{\partial x}\frac{\partial^2 \phi_1}{\partial x \partial z} + \frac{\partial \phi_1}{\partial z}\frac{\partial^2 \phi_1}{\partial z^2} \right]$$

$$\frac{\partial^2 \phi_3}{\partial x^2} + \frac{\partial^2 \phi_3}{\partial z^2} = 0 \qquad z < 0$$

$$\frac{\partial \phi_3}{\partial z} \to 0 \qquad \text{as } z \to -\infty$$

~~Third order~~

The inhomogeneous terms (right hand side) are linear, quadratic or cubic,

they are ① or ①·② or ①·①·①

first harmonic; first, second or zeroth; first harmonic

$$(+1,-1)+(+2,0,-2) = (+3,+1,-1,-3)$$

$$(+1,-1)+(+1,-1)+(+1,-1) = (+3,+1,-1,-3)$$

The inhomogeneous terms are first or third harmonic

The response should therefore be

$$\eta_3 = \frac{1}{2}\left(\hat{\eta}_{3,3}\, e^{3i(\,)} + \hat{\eta}_{3,1}\, e^{i(\,)} + \hat{\eta}_{3,-1}\, e^{-i(\,)} + \hat{\eta}_{3,-3}\, e^{-3i(\,)}\right)$$

$$\phi_3 = \frac{1}{2}\left(\hat{\phi}_{3,3}\, e^{3i(\,)} + \cdots\right.$$

Third order first harmonic

$$\begin{cases} -i\omega\hat{\eta}_{3,1} - \dfrac{\partial\hat{\phi}_{3,1}}{\partial z} = -\dfrac{\partial A}{\partial t_2} - \dfrac{5igk^3}{8\omega}|A|^2A \equiv F \\ -i\omega\hat{\phi}_{3,1} + g\hat{\eta}_{3,1} = \dfrac{ig}{\omega}\dfrac{\partial A}{\partial t_2} - \dfrac{3gk^2}{8}|A|^2A \equiv G \end{cases}$$

$$\begin{cases} \dfrac{\partial^2\hat{\phi}_{3,1}}{\partial z^2} - k^2\hat{\phi}_{3,1} = 0 & z<0 \\ \dfrac{\partial\hat{\phi}_{3,1}}{\partial z} \to 0 & z\to-\infty \end{cases}$$

$$\hat{\phi}_{3,1} = Be^{kz}$$

$$-i\omega F + kG = 0$$

$$\begin{array}{r|l} -i\omega & -i\omega\hat{\eta}_{3,1} - kB = F \\ k & g\hat{\eta}_{3,1} - i\omega B = G \end{array}$$

$$0 = -i\omega F + kG$$

$$-i\omega F + kG = 0$$

$$\Rightarrow \quad \frac{\partial A}{\partial t_2} + \frac{i\omega k^2}{2} |A|^2 A = 0$$

$$A = a\, e^{-i \frac{\omega k^2 a^2}{2} t_2 + i\theta}$$

where a and θ are two real constants

Third order third harmonic

$$\left.\begin{aligned} -3i\omega \hat{\eta}_{3,3} - \frac{\partial \hat{\phi}_{3,3}}{\partial z} &= -\frac{9igk^3}{8\omega} A^3 \\ -3i\omega \hat{\phi}_{3,3} + g\hat{\eta}_{3,3} &= \frac{3gk^2}{8} A^3 \end{aligned}\right\} \text{at } z=0$$

$$\frac{\partial^2 \hat{\phi}_{3,3}}{\partial z^2} - 9k^2 \hat{\phi}_{3,3} = 0 \quad \text{for } z<0$$

$$\frac{\partial \hat{\phi}_{3,3}}{\partial z} \to 0 \quad \text{as } z \to -\infty$$

$$\Rightarrow \quad \hat{\eta}_{3,3} = \frac{3k^2}{8} A^3 \qquad \hat{\phi}_{3,3} = 0$$

It is remarkable that $\hat{\phi}_{3,3}$ vanishes!
This only happens for infinite depth!

Summary:

$$\eta = a\cos\psi + \frac{1}{2}ka^2\cos 2\psi + \frac{3}{8}k^2a^3\cos 3\psi$$

$$\phi = \frac{ga}{\omega} e^{kz} \sin\psi$$

$$\psi = kx - \omega\left(\underbrace{1 + \frac{1}{2}(ka)^2}\right)t + \Theta$$

FOR A GIVEN WAVE LENGTH THE FREQUENCY IS SLIGHTLY LARGER

where a is the linear amplitude and Θ is constant phase angle.

Kinematics field (water velocity) is linear!

Crest = kam: $\psi = 2\pi n$, $n = 0, \pm 1, \pm 2, \ldots$

$$\eta_c = a\left(1 + \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2\right)$$

Crested lifted slightly

trough = bunn: $\psi = \pi + 2\pi n$, $n = 0, \pm 1, \pm 2, \ldots$

$$\eta_t = -a\left(1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2\right)$$

trough lifted slightly

wave height = bølgehøyde: $H = \eta_c - \eta_t = 2a\left(1 + \frac{3}{8}\varepsilon^2\right)$

Second-order waveheight = first-order waveheight!

third-order waveheight is slightly higher.

Profile is "pointed upward" with peaked crest and shallow trough

Suppose we fix the wavelength $= \frac{2\pi}{k}$

Linear wave without frequency modification

Linear wave with frequency modification (larger frequency)

Second order nonlinear profile with higher more peaked crest, shallower and less peaked trough.