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INSTABILITY OF STOKES WAVE, NLS equation, Benjamin Feir instability

We previously derived the Stokes wave to third order on infinitely deep water

$$\eta = a\cos\psi + \frac{1}{2}ka^2\cos 2\psi + \frac{3}{8}k^2a^3\cos 3\psi$$

$$\phi = \frac{ga}{\omega}e^{kz}\sin\psi$$

where $$\psi = kx - \omega\underbrace{\left(1+\frac{1}{2}(ka)^2\right)}_{\text{looks like a correction typical for Poincaré-Lindsted}}t$$

Now let us introduce slow modulation of the amplitudes, on scales

$$x_1 = \varepsilon x \quad \text{and} \quad t_1 = \varepsilon t$$

we could have done a two-scale analysis, or a WKB analysis with only the slow scale, but here we do a full three-scale analysis

$$x_2 = \varepsilon^2 x \quad \text{and} \quad t_2 = \varepsilon^2 t$$

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial t_1} + \varepsilon^2\frac{\partial}{\partial t_2}$$

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial x_1} + \varepsilon^2\frac{\partial}{\partial x_2}$$

However, we do not have any such cascade for z.

The equations become

$$\frac{\partial \eta}{\partial t} + \varepsilon \frac{\partial \eta}{\partial t_1} + \varepsilon^2 \frac{\partial \eta}{\partial t_2} + \varepsilon \left(\frac{\partial \phi}{\partial x} + \varepsilon \frac{\partial \phi}{\partial x_1} \right)\left(\frac{\partial \eta}{\partial x} + \varepsilon \frac{\partial \eta}{\partial x_1} \right)$$

$$+ \varepsilon^2 \eta \frac{\partial^2 \phi}{\partial x \partial z} \frac{\partial \eta}{\partial x} - \frac{\partial \phi}{\partial z} - \varepsilon \eta \frac{\partial^2 \phi}{\partial z^2} - \frac{1}{2} \varepsilon^2 \eta^2 \frac{\partial^3 \phi}{\partial z^3} = 0$$

$$\frac{\partial \phi}{\partial t} + \varepsilon \frac{\partial \phi}{\partial t_1} + \varepsilon^2 \frac{\partial \phi}{\partial t_2} + \varepsilon \eta \left(\frac{\partial^2 \phi}{\partial t \partial z} + \varepsilon \frac{\partial^2 \phi}{\partial t_1 \partial z} \right)$$

$$+ \frac{1}{2} \varepsilon^2 \eta^2 \frac{\partial^3 \phi}{\partial t \partial z^2} + g\eta$$

$$+ \frac{\varepsilon}{2} \left[\left(\frac{\partial \phi}{\partial x} \right)^2 + 2\varepsilon \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x_1} + \left(\frac{\partial \phi}{\partial z} \right)^2 \right] + \varepsilon^2 \eta \left[\frac{\partial \phi}{\partial x} \frac{\partial^2 \phi}{\partial x \partial z} + \frac{\partial \phi}{\partial z} \frac{\partial^2 \phi}{\partial z^2} \right] = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} + 2\varepsilon \frac{\partial^2 \phi}{\partial x \partial x_1} + \varepsilon^2 \frac{\partial^2 \phi}{\partial x_1^2} + 2\varepsilon^2 \frac{\partial^2 \phi}{\partial x \partial x_2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

for $-\infty < z < 0$

$$\frac{\partial \phi}{\partial z} \to 0 \quad \text{as} \quad z \to -\infty$$

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perturbation expansion

$$\eta = \eta_1 + \varepsilon \eta_2 + \varepsilon^2 \eta_3 + \cdots$$

$$\phi = \phi_1 + \varepsilon \phi_2 + \varepsilon^2 \phi_3 + \cdots$$

1 means linear
2 means quadratic
3 means cubic

$$\frac{\partial \eta_1}{\partial t} + \varepsilon \frac{\partial \eta_1}{\partial t_1} + \varepsilon^2 \frac{\partial \eta_1}{\partial t_2} + \varepsilon \frac{\partial \eta_2}{\partial t} + \varepsilon^2 \frac{\partial \eta_2}{\partial t_1} + \varepsilon^2 \frac{\partial \eta_3}{\partial t}$$

$$+ \varepsilon \left(\frac{\partial \phi_1}{\partial x} + \varepsilon \frac{\partial \phi_2}{\partial x} + \varepsilon \frac{\partial \phi_1}{\partial x_1} \right) \left(\frac{\partial \eta_1}{\partial x} + \varepsilon \frac{\partial \eta_2}{\partial x} + \varepsilon \frac{\partial \eta_1}{\partial x_1} \right)$$

$$+ \varepsilon^2 \eta_1 \frac{\partial^2 \phi_1}{\partial x \partial z} \frac{\partial \eta_1}{\partial x} - \frac{\partial \phi_1}{\partial z} - \varepsilon \frac{\partial \phi_2}{\partial z} - \varepsilon^2 \frac{\partial \phi_3}{\partial z}$$

$$- \varepsilon (\eta_1 + \varepsilon \eta_2) \left(\frac{\partial^2 \phi_1}{\partial z^2} + \varepsilon \frac{\partial^2 \phi_2}{\partial z^2} \right) - \frac{1}{2} \varepsilon^2 \eta_1^2 \frac{\partial^3 \phi_1}{\partial z^3} = 0$$

$$\frac{\partial \phi_1}{\partial t} + \varepsilon \frac{\partial \phi_2}{\partial t} + \varepsilon^2 \frac{\partial \phi_3}{\partial t} + \varepsilon \frac{\partial \phi_1}{\partial t_1} + \varepsilon^2 \frac{\partial \phi_2}{\partial t_1} + \varepsilon^2 \frac{\partial \phi_1}{\partial t_2}$$

$$+ \varepsilon (\eta_1 + \varepsilon \eta_2) \left(\frac{\partial^2 \phi_1}{\partial t \partial z} + \varepsilon \frac{\partial^2 \phi_2}{\partial t \partial z} + \varepsilon \frac{\partial^2 \phi_1}{\partial t_1 \partial z} \right)$$

$$+ \frac{1}{2} \varepsilon^2 \eta_1^2 \frac{\partial^3 \phi_1}{\partial t \partial z^2} + g \eta_1 + \varepsilon g \eta_2 + \varepsilon^2 g \eta_3$$

$$+ \frac{\varepsilon}{2} \left[\left(\frac{\partial \phi_1}{\partial x} + \varepsilon \frac{\partial \phi_2}{\partial x} \right)^2 + 2\varepsilon \left(\frac{\partial \phi_1}{\partial x} + \varepsilon \frac{\partial \phi_2}{\partial x} \right) \left(\frac{\partial \phi_1}{\partial x_1} + \varepsilon \frac{\partial \phi_2}{\partial x_1} \right) \right.$$

$$\left. + \left(\frac{\partial \phi_1}{\partial z} + \varepsilon \frac{\partial \phi_2}{\partial z} \right)^2 \right]$$

$$+ \varepsilon^2 \eta_1 \left[\frac{\partial \phi_1}{\partial x} \frac{\partial^2 \phi_1}{\partial x \partial z} + \frac{\partial \phi_1}{\partial z} \frac{\partial^2 \phi_1}{\partial z^2} \right] = 0$$

$$\frac{\partial^2 \phi_1}{\partial x^2} + \varepsilon \frac{\partial^2 \phi_2}{\partial x^2} + \varepsilon^2 \frac{\partial^2 \phi_3}{\partial x^2} + 2\varepsilon \left(\frac{\partial^2 \phi_1}{\partial x \partial x_1} + \varepsilon \frac{\partial^2 \phi_2}{\partial x \partial x_1} \right) + \varepsilon^2 \frac{\partial^2 \phi_1}{\partial x_1^2}$$

$$+ 2\varepsilon^2 \frac{\partial^2 \phi_1}{\partial x \partial x_2} + \frac{\partial^2 \phi_1}{\partial z^2} + \varepsilon \frac{\partial^2 \phi_2}{\partial z^2} + \varepsilon^2 \frac{\partial^2 \phi_3}{\partial z^2} = 0$$

$$\frac{\partial \phi_1}{\partial z} + \varepsilon \frac{\partial \phi_2}{\partial z} + \varepsilon^2 \frac{\partial \phi_3}{\partial z} \to 0$$

$O(\varepsilon^1)$:

$$\left.\begin{aligned} &\frac{\partial \eta_1}{\partial t} - \frac{\partial \phi_1}{\partial z} = 0 \\ &\frac{\partial \phi_1}{\partial t} + g\eta_1 = 0 \end{aligned}\right\} \text{ at } z=0$$

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial z^2} = 0 \qquad -h < z < 0$$

$$\frac{\partial \phi_1}{\partial z} \to 0 \qquad \text{as } z \to -h$$

Assume $\begin{pmatrix} \eta_1 \\ \phi_1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \hat{\eta}_1 \\ \hat{\phi}_1 \end{pmatrix} e^{i(kx-\omega t)} + \text{c.c.}$

$$\left\{\begin{aligned} &-i\omega \hat{\eta}_1 - \frac{\partial \hat{\phi}_1}{\partial z} = 0 \\ &-i\omega \hat{\phi}_1 + g\hat{\eta}_1 = 0 \end{aligned}\right\} \text{ at } z=0$$

$$\left\{\begin{aligned} &\frac{\partial^2 \hat{\phi}_1}{\partial z^2} - k^2 \hat{\phi}_1 = 0 \qquad \text{for } -h < z < 0 \\ &\frac{\partial \hat{\phi}_1}{\partial z} \to 0 \qquad \text{as } z \to -h \end{aligned}\right.$$

$$\hat{\phi}_1 = B(x_1, x_2, t_1, t_2)\, e^{kz}$$

$$-i\omega A - kB = 0$$
$$-i\omega B + gA = 0$$

$$\begin{pmatrix} -i\omega & -k \\ g & -i\omega \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-\omega^2 + gk = 0 \qquad \omega^2 = gk$ dispersion relation

$$B = -\frac{ig}{\omega} A$$

$$\hat{\eta}_1 = A \qquad \hat{\phi}_1 = -\frac{ig}{\omega} A\, e^{kz}$$

$O(\varepsilon^1)$:

$$\left.\begin{aligned}\frac{\partial \eta_2}{\partial t} - \frac{\partial \phi_2}{\partial z} &= -\frac{\partial \eta_1}{\partial t_1} - \frac{\partial \phi_1}{\partial x}\frac{\partial \eta_1}{\partial x} - \eta_1 \frac{\partial^2 \phi_1}{\partial z^2} \\ \frac{\partial \phi_2}{\partial t} + g\eta_2 &= -\frac{\partial \phi_1}{\partial t_1} - \eta_1 \frac{\partial^2 \phi_1}{\partial t \partial z} - \frac{1}{2}\left[\left(\frac{\partial \phi_1}{\partial x}\right)^2 + \left(\frac{\partial \phi_1}{\partial z}\right)^2\right]\end{aligned}\right\} \text{at } z=0$$

$$\frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial z^2} = -2\frac{\partial^2 \phi_1}{\partial x \partial x_1} \qquad -h < z < 0$$

new terms

$$\frac{\partial \phi_2}{\partial z} \to 0 \qquad \text{as } z \to -h$$

We already know that the "old" terms on the right-hand side will give 0th and 2nd harmonic forcing, only. The new terms will give first harmonic forcing, corresponding to the homogeneous solution, and will require a solvability condition to impede evanescent growth.

We anticipate response

$$\eta_2 = \hat{\eta}_{2,0} + \frac{1}{2}\left(\hat{\eta}_{2,1} e^{i()} + \hat{\eta}_{2,2} e^{2i()} + c.c.\right)$$

$$\phi_2 = \hat{\phi}_{2,0} + \frac{1}{2}\left(\hat{\phi}_{2,1} e^{i()} + \hat{\phi}_{2,2} e^{2i()} + c.c.\right)$$

Look at $O(\varepsilon^1)$ first harmonic problem :

$$\left\{\begin{array}{l|l} -1 & -i\omega\,\hat{\eta}_{2,1} - \dfrac{\partial \hat{\phi}_{2,1}}{\partial z} = -\dfrac{\partial A}{\partial t_1} \\[2ex] -\dfrac{i\omega}{g} & -i\omega\,\hat{\phi}_{2,1} + g\,\hat{\eta}_{2,1} = \dfrac{ig}{\omega}\dfrac{\partial A}{\partial t_1} \end{array}\right\} \quad \text{at } z=0$$

$$\frac{\partial^2 \hat{\phi}_{2,1}}{\partial z^2} - k^2 \hat{\phi}_{2,1} = -2\frac{gk}{\omega}\frac{\partial A}{\partial x_1} e^{kz} \qquad \text{for } -\infty<z<0$$

$$\frac{\partial \hat{\phi}_{2,1}}{\partial z} \to 0 \qquad \text{as } z \to -\infty$$

$$\frac{\partial \hat{\phi}_{2,1}}{\partial z} - \frac{\omega^2}{g}\hat{\phi}_{2,1} = 2\frac{\partial A}{\partial t_1}$$

The solvability condition is conveniently obtained by a version of Green's theorem

$$\int_{-\infty}^{0} \hat{\phi}_1\left(\frac{\partial^2 \hat{\phi}_{2,1}}{\partial z^2} - k^2\hat{\phi}_{2,1}\right) - \hat{\phi}_{2,1}\left(\frac{\partial^2 \hat{\phi}_1}{\partial z^2} - k^2\hat{\phi}_1\right) dz$$

$$= \int_{-\infty}^{0} \hat{\phi}_1 \frac{\partial^2 \hat{\phi}_{2,1}}{\partial z^2} - \hat{\phi}_{2,1}\frac{\partial^2 \hat{\phi}_1}{\partial z^2}\, dz$$

$$= \int_{-\infty}^{0} \frac{d}{dz}\left(\hat{\phi}_1 \frac{\partial \hat{\phi}_{2,1}}{\partial z} - \hat{\phi}_{2,1}\frac{\partial \hat{\phi}_1}{\partial z}\right) dz$$

$$= \left[\hat{\phi}_1\left(\frac{\partial \hat{\phi}_{2,1}}{\partial z} - \frac{\omega^2}{g}\hat{\phi}_{2,1}\right) - \hat{\phi}_{2,1}\left(\frac{\partial \hat{\phi}_1}{\partial z} - \frac{\omega^2}{g}\hat{\phi}_1\right)\right]_{-\infty}^{0} = \hat{\phi}_1\left(\frac{\partial \hat{\phi}_{2,1}}{\partial z} - \frac{\omega^2}{g}\hat{\phi}_{2,1}\right)\Bigg|_{z=0}$$

Plug in

$$\int_{-\infty}^{0}\left(-\frac{ig}{\omega} A e^{kz}\right)\left(-2\frac{gk}{\omega}\frac{\partial A}{\partial x_1} e^{kz}\right) dz = 2\frac{ig^2 k}{\omega^2} A \frac{\partial A}{\partial x_1}\frac{1}{2k}$$

$$= \left(-\frac{ig}{\omega} A\right)\left(2\frac{\partial A}{\partial t_1}\right)$$

$$\frac{\partial A}{\partial t_1} + \underbrace{\frac{g}{2\omega}}_{= c_g}\frac{\partial A}{\partial x_1} = 0$$

$$\omega^2 = gk \qquad 2\omega\frac{d\omega}{dk} = g \qquad \frac{d\omega}{dk} = \frac{g}{2\omega} = c_g$$

$$\frac{\partial A}{\partial t_1} + c_g\frac{\partial A}{\partial x_1} = 0$$

Conclusion from $\mathcal{O}(\varepsilon^1)$ first harmonic:

modulations move with the group velocity.

$O(\varepsilon^2)$ first harmonic

we just state the final result:

$$\frac{\partial A}{\partial t_1} + c_g \frac{\partial A}{\partial x_1} + \frac{i\omega}{8k^2}\frac{\partial^2 A}{\partial x_1^2} + \frac{ik^2\omega}{2}|A|^2 A = 0$$

This is known as the cubic nonlinear Schrödinger equation.

The Stokes wave corresponds to the solution (similar to the one previously found)

$$A = a\, e^{-\frac{i}{2}\omega k^2 a^2 t_1}$$

(From here we don't bother to write index 1).

Let us check if the Stokes wave is stable or unstable. Consider a small perturbation

$$A = a\,(1 + \alpha + i\beta)\, e^{-\frac{i}{2}\omega k^2 a^2 t}$$

where α and β are real, small perturbations, functions of x and t.

Therefore we can linearize in α and β

$$|A|^2 = a^2(1+\alpha+i\beta)(1+\alpha-i\beta) \approx a^2(1+2\alpha)$$

$$|A|^2A \approx a^3(1+2\alpha)(1+\alpha+i\beta)\, e^{-\frac{i}{2}\omega k^2a^2t}$$

$$\approx a^3(1+3\alpha+i\beta)\, e^{-\frac{i}{2}\omega k^2a^2t}$$

We got

$$-\frac{i}{2}\omega k^2a^2(1+\alpha+i\beta) + \frac{\partial\alpha}{\partial t} + i\frac{\partial\beta}{\partial t}$$

$$+ c_g\frac{\partial\alpha}{\partial x} + ic_g\frac{\partial\beta}{\partial x} + \frac{i\omega}{8k^2}\frac{\partial^2\alpha}{\partial x^2} - \frac{\omega}{8k^2}\frac{\partial^2\beta}{\partial x^2}$$

$$+ \frac{i}{2}\omega k^2a^2(1+3\alpha+i\beta) = 0$$

$$\frac{\partial\alpha}{\partial t} + i\frac{\partial\beta}{\partial t} + c_g\frac{\partial\alpha}{\partial x} + ic_g\frac{\partial\beta}{\partial x} + \frac{i\omega}{8k^2}\frac{\partial^2\alpha}{\partial x^2} - \frac{\omega}{8k^2}\frac{\partial^2\beta}{\partial x^2}$$

$$+ i\omega k^2a^2\alpha = 0$$

Split into two real equations

$$\frac{\partial\alpha}{\partial t} + c_g\frac{\partial\alpha}{\partial x} - \frac{\omega}{8k^2}\frac{\partial^2\beta}{\partial x^2} = 0$$

$$\frac{\partial\beta}{\partial t} + c_g\frac{\partial\beta}{\partial x} + \frac{\omega}{8k^2}\frac{\partial^2\alpha}{\partial x^2} + \omega k^2a^2\alpha = 0$$

Assume plane wave solution

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\hat\alpha\\ \hat\beta\end{pmatrix} e^{i(\lambda x - \Omega t)}$$

$$-i\Omega\hat{\alpha} + i\lambda c_g\hat{\alpha} + \lambda^2\frac{\omega}{8k^2}\hat{\beta} = 0$$

$$-i\Omega\hat{\beta} + i\lambda c_g\hat{\beta} - \lambda^2\frac{\omega}{8k^2}\hat{\alpha} + \omega k^2 a^2\hat{\alpha} = 0$$

$$\begin{pmatrix} -i\Omega + i\lambda c_g & \lambda^2\frac{\omega}{8k^2} \\ -\lambda^2\frac{\omega}{8k^2} + \omega k^2 a^2 & -i\Omega + i\lambda c_g \end{pmatrix}\begin{pmatrix}\hat{\alpha} \\ \hat{\beta}\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\det(\,) = -(-\Omega + \lambda c_g)^2 - \lambda^2\frac{\omega}{8k^2}\left(\omega k^2 a^2 - \lambda^2\frac{\omega}{8k^2}\right) = 0$$

$$\Omega = \lambda c_g \pm i\sqrt{\lambda^2\frac{\omega}{8k^2}\left(\omega k^2 a^2 - \lambda^2\frac{\omega}{8k^2}\right)}$$

growth rate $= \sqrt{\lambda^2\frac{\omega}{8k^2}\left(\omega k^2 a^2 - \lambda^2\frac{\omega}{8k^2}\right)}$

> 0 when radicand is positive

$$k^2 a^2 - \frac{\lambda^2}{8k^2} > 0$$

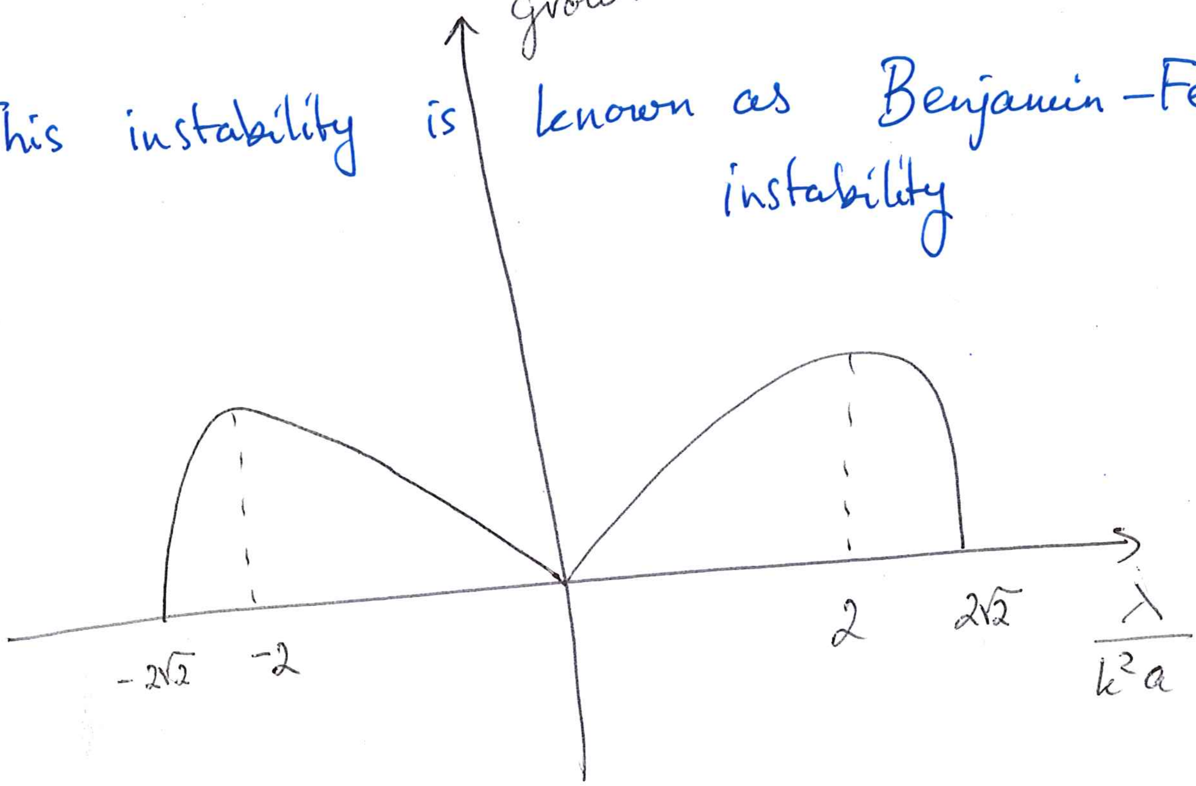
$$|\lambda| < \sqrt{8}\, k^2 a$$

maximum growth rate $\quad \frac{d}{d\lambda}\left\{\lambda^2\frac{\omega^2}{8k^2}\left(k^2a^2 - \frac{\lambda^2}{8k^2}\right)\right\}$

$$= 2\lambda\frac{\omega^2}{8k^2}k^2a^2 - \frac{4\lambda^3}{(8k^2)^2}\omega^2 = 0$$

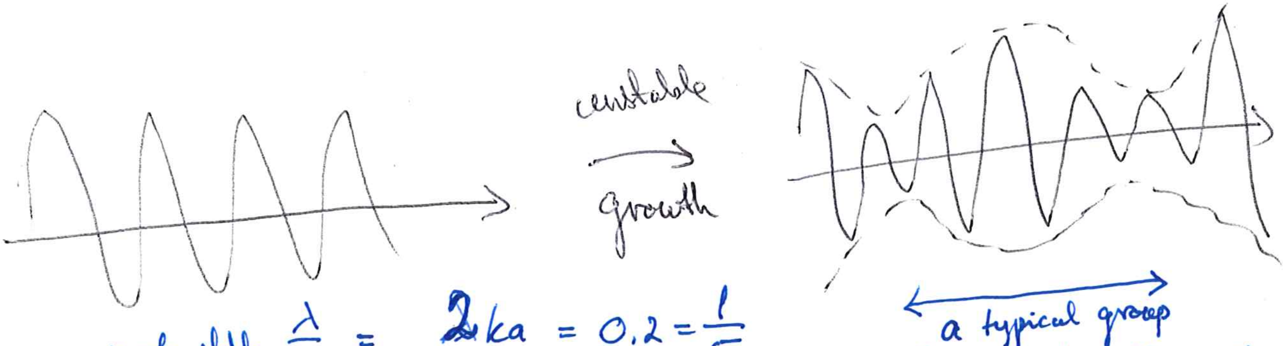
$$|\lambda| = \sqrt{\tfrac{1}{4}k^2a^2\,8k^2} = \sqrt{2}\,k^2 a$$

This instability is known as Benjamin–Feir instability



Note: The Stokes wave corresponds to no perturbation, with all energy at $\lambda = 0$.

The Stokes wave is unstable to slowly varying modulations ($|\lambda|$ small) and thus there will be a natural tendency for wave grouping!



most unstable $\frac{\lambda}{k} = 2ka = 0.2 = \frac{1}{5}$

a typical group should contain 5 waves