Stokes waves

presentation for MEK4320

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The following exposition is not the most compact or sophisticated treatment, but a completely systematic approach making a minimum of assumptions or "clever" tricks.

A very comprehensive presentation of Stokes waves can be found on Wikipedia Stokes wave. As all relevant references can be found on that Wikipedia page, they are not mentioned here.

Consider monochromatic gravity waves on the water surface. For simplicity assume the water is of infinite depth. Neglect surface tension, compressibility and viscosity. Assume the fluid motion is irrotational such that the velocity can be represented by a potential $\boldsymbol{v} = \nabla\phi$. We furthermore limit to one horizontal dimension x. The vertical axis z points upward and t is time. The surface elevation is η and the acceleration of gravity is g. We start with the governing equations on the form

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} - \frac{\partial \phi}{\partial z} = 0 \qquad \text{at } z = \eta \tag{1}$$

$$\frac{\partial \phi}{\partial t} + g\eta + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] = 0 \qquad \text{at } z = \eta \tag{2}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad \text{for } -\infty < z < \eta \tag{3}$$

$$\frac{\partial \phi}{\partial z} = 0 \qquad \text{at } z \to -\infty \tag{4}$$

We normalize the equations assuming the spatial scale is given by a characteristic wavenumber k_c, the temporal scale is given by a characteristic angular frequency ω_c, and the amplitude is given by a characteristic amplitude a_c. We can then write

$$x' = k_c x, \qquad z' = k_c z, \qquad t' = \omega_c t, \qquad \eta = a_c \eta', \qquad \phi = \frac{a_c \omega_c}{k_c}\phi' \tag{5}$$

where the primed quantities are non-dimensional. The derivatives will transform as

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} = k_c \frac{\partial}{\partial x'}, \qquad \frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} = \omega_c \frac{\partial}{\partial t'} \tag{6}$$

etc.

After introducing the non-dimensional quantities, the governing equations take the form

$$\frac{\partial \eta'}{\partial t'} + \epsilon \frac{\partial \phi'}{\partial x'}\frac{\partial \eta'}{\partial x'} - \frac{\partial \phi'}{\partial z'} = 0 \qquad \text{at } z' = \epsilon \eta' \tag{7}$$

$$\frac{\partial \phi'}{\partial t'} + \frac{g k_c}{\omega_c^2}\eta' + \epsilon \frac{1}{2}\left[\left(\frac{\partial \phi'}{\partial x'}\right)^2 + \left(\frac{\partial \phi'}{\partial z'}\right)^2\right] = 0 \qquad \text{at } z' = \epsilon \eta' \tag{8}$$

$$\frac{\partial^2 \phi'}{\partial x'^2} + \frac{\partial^2 \phi'}{\partial z'^2} = 0 \qquad \text{for } -\infty < z' < \epsilon \eta' \tag{9}$$

$$\frac{\partial \phi'}{\partial z'} = 0 \qquad \text{at } z' \to -\infty \tag{10}$$

where $\epsilon \equiv k_c a_c$ is known as the steepness.

In the above normalized equations we recognize two dimensionless groups: The steepness which we shall assume is small, $\epsilon \ll 1$, and the combination $g k_c/\omega_c^2$ which will be allowed to have order unity by virtue of relating k_c and ω_c by the linear dispersion relation.

Having identified the steepness ϵ as the single small non-dimensional parameter which will allow us to carry out a perturbation analysis, we go back to the original dimensional equations with the steepness ϵ added as a tag for ordering purposes.

$$\frac{\partial \eta}{\partial t} + \epsilon \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} - \frac{\partial \phi}{\partial z} = 0 \qquad \text{at } z = \epsilon \eta \tag{11}$$

$$\frac{\partial \phi}{\partial t} + g\eta + \frac{\epsilon}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] = 0 \qquad \text{at } z = \epsilon \eta \tag{12}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad \text{for } -\infty < z < \epsilon \eta \tag{13}$$

$$\frac{\partial \phi}{\partial z} = 0 \qquad \text{at } z \to -\infty \tag{14}$$

We Taylor-expand the surface conditions around the quiescent surface $z = 0$ up to cubic nonlinear terms

$$\frac{\partial \eta}{\partial t} + \epsilon \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} + \epsilon^2 \eta \frac{\partial^2 \phi}{\partial x \partial z}\frac{\partial \eta}{\partial x} - \frac{\partial \phi}{\partial z} - \epsilon \eta \frac{\partial^2 \phi}{\partial z^2} - \frac{\epsilon^2}{2}\eta^2 \frac{\partial^3 \phi}{\partial z^3} = 0 \qquad \text{at } z = 0 \tag{15}$$

$$\frac{\partial \phi}{\partial t} + \epsilon\eta\frac{\partial^2\phi}{\partial z\partial t} + \frac{\epsilon^2}{2}\eta^2\frac{\partial^3\phi}{\partial z^2\partial t} + g\eta + \frac{\epsilon}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right]$$
$$+ \epsilon^2\eta\left[\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x\partial z} + \frac{\partial\phi}{\partial z}\frac{\partial^2\phi}{\partial z^2}\right] = 0 \qquad \text{at } z = 0 \quad (16)$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad \text{for } -\infty < z < 0 \quad (17)$$

$$\frac{\partial\phi}{\partial z} = 0 \qquad \text{at } z \to -\infty \quad (18)$$

We are going to insist on simple-harmonic oscillations in x, then it will turn out to be necessary to include slow modulation in time balancing cubic nonlinear terms, therefore we introduce the slow time

$$t_2 = \epsilon^2 t \quad (19)$$

and assume dependencies $\eta(x, t, t_2)$ and $\phi(x, z, t, t_2)$. We notice that the time derivative now becomes

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \frac{\partial t_2}{\partial t}\frac{\partial}{\partial t_2} = \frac{\partial}{\partial t} + \epsilon^2\frac{\partial}{\partial t_2} \quad (20)$$

and the surface conditions therefore become

$$\frac{\partial\eta}{\partial t} + \epsilon^2\frac{\partial\eta}{\partial t_2} + \epsilon\frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x} + \epsilon^2\eta\frac{\partial^2\phi}{\partial x\partial z}\frac{\partial\eta}{\partial x} - \frac{\partial\phi}{\partial z} - \epsilon\eta\frac{\partial^2\phi}{\partial z^2} - \frac{\epsilon^2}{2}\eta^2\frac{\partial^3\phi}{\partial z^3} = 0 \qquad \text{at } z = 0 \quad (21)$$

$$\frac{\partial\phi}{\partial t} + \epsilon^2\frac{\partial\phi}{\partial t_2} + \epsilon\eta\frac{\partial^2\phi}{\partial z\partial t} + \frac{\epsilon^2}{2}\eta^2\frac{\partial^3\phi}{\partial z^2\partial t} + g\eta + \frac{\epsilon}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right]$$
$$+ \epsilon^2\eta\left[\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x\partial z} + \frac{\partial\phi}{\partial z}\frac{\partial^2\phi}{\partial z^2}\right] = 0 \qquad \text{at } z = 0 \quad (22)$$

Now let us introduce regular perturbation expansions for η and ϕ

$$\eta = \eta_1 + \epsilon\eta_2 + \epsilon^2\eta_3 + \ldots \quad (23)$$

$$\phi = \phi_1 + \epsilon\phi_2 + \epsilon^2\phi_3 + \ldots \quad (24)$$

First-order problem at order $\mathcal{O}(\epsilon^0)$

$$\frac{\partial\eta_1}{\partial t} - \frac{\partial\phi_1}{\partial z} = 0 \qquad \text{at } z = 0 \quad (25)$$

$$\frac{\partial \phi_1}{\partial t} + g\eta_1 = 0 \qquad \text{at } z = 0 \tag{26}$$

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial z^2} = 0 \qquad \text{for } -\infty < z < 0 \tag{27}$$

$$\frac{\partial \phi_1}{\partial z} = 0 \qquad \text{at } z \to -\infty \tag{28}$$

This problem should be familiar to us, and we assume a monochromatic wave

$$\eta_1 = \frac{1}{2}\left(\hat{\eta}_{1,1}\mathrm{e}^{\mathrm{i}\chi} + \text{c.c.}\right) \tag{29}$$

$$\phi_1 = \frac{1}{2}\left(\hat{\phi}_{1,1}\mathrm{e}^{\mathrm{i}\chi} + \text{c.c.}\right) \tag{30}$$

where $\chi = kx - \omega t$ and "c.c." signifies the complex conjugate of the expression in front of it.

The vertical structure of the solution is e^{kz}, and we may set

$$\hat{\eta}_{1,1} = A \qquad \text{and} \qquad \hat{\phi}_{1,1} = B\mathrm{e}^{kz} = -\frac{\mathrm{i}g}{\omega}A\mathrm{e}^{kz} \tag{31}$$

and the linear dispersion relation

$$\omega^2 = gk. \tag{32}$$

It is natural to call the first-order solution "first-harmonic" because the rapid phase oscillations are of the kind $\mathrm{e}^{\pm\mathrm{i}\chi}$.

We employ the notation $\hat{\eta}_{m,-n} = \hat{\eta}^*_{m,n}$ and $\hat{\phi}_{m,-n} = \hat{\phi}^*_{m,n}$.

Second-order problem at order $\mathcal{O}(\epsilon^1)$

$$\frac{\partial \eta_2}{\partial t} - \frac{\partial \phi_2}{\partial z} = -\frac{\partial \phi_1}{\partial x}\frac{\partial \eta_1}{\partial x} + \eta_1\frac{\partial^2 \phi_1}{\partial z^2} \qquad \text{at } z = 0 \tag{33}$$

$$\frac{\partial \phi_2}{\partial t} + g\eta_2 = -\eta_1\frac{\partial^2 \phi_1}{\partial z \partial t} - \frac{1}{2}\left[\left(\frac{\partial \phi_1}{\partial x}\right)^2 + \left(\frac{\partial \phi_1}{\partial z}\right)^2\right] \qquad \text{at } z = 0 \tag{34}$$

$$\frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial z^2} = 0 \qquad \text{for } -\infty < z < 0 \tag{35}$$

$$\frac{\partial \phi_2}{\partial z} = 0 \qquad \text{at } z \to -\infty \tag{36}$$

The inhomogeneous terms on the right-hand sides are all quadratic products of the first-order terms, which are first-harmonic terms, therefore we understand that the forcing has rapid phase oscillations either of the kind $\mathrm{e}^{0\mathrm{i}\chi}$ or $\mathrm{e}^{\pm 2\mathrm{i}\chi}$. We may therefore anticipate that the second-order solution is a superposition of zeroth- and second-harmonic terms

$$\eta_2 = \hat{\eta}_{2,0} + \frac{1}{2}\left(\hat{\eta}_{2,2}\mathrm{e}^{2\mathrm{i}\chi} + \mathrm{c.c.}\right) \tag{37}$$

$$\phi_2 = \hat{\phi}_{2,0} + \frac{1}{2}\left(\hat{\phi}_{2,2}\mathrm{e}^{2\mathrm{i}\chi} + \mathrm{c.c.}\right) \tag{38}$$

Second-order zeroth-harmonic problem

$$\frac{\partial \hat{\phi}_{2,0}}{\partial z} = 0 \qquad \text{at } z = 0 \tag{39}$$

$$\hat{\eta}_{2,0} = 0 \qquad \text{at } z = 0 \tag{40}$$

$$\frac{\partial^2 \hat{\phi}_{2,0}}{\partial z^2} = 0 \qquad \text{for } -\infty < z < 0 \tag{41}$$

$$\frac{\partial \hat{\phi}_{2,0}}{\partial z} = 0 \qquad \text{at } z \to -\infty \tag{42}$$

We see that solution is

$$\hat{\eta}_{2,0} = 0 \qquad \text{and} \qquad \hat{\phi}_{2,0} = \Phi(x, t, t_2) \tag{43}$$

thus allowing an arbitrary horizontal current.

Second-order second-harmonic problem

$$-2\mathrm{i}\omega\hat{\eta}_{2,2} - \frac{\partial \hat{\phi}_{2,2}}{\partial z} = -\frac{\mathrm{i}gk^2}{\omega}A^2 \qquad \text{at } z = 0 \tag{44}$$

$$-2\mathrm{i}\omega\hat{\phi}_{2,2} + g\hat{\eta}_{2,2} = \frac{gk}{2}A^2 \qquad \text{at } z = 0 \tag{45}$$

$$\frac{\partial^2 \hat{\phi}_{2,2}}{\partial z^2} - 4k^2\hat{\phi}_{2,2} = 0 \qquad \text{for } -\infty < z < 0 \tag{46}$$

$$\frac{\partial \hat{\phi}_{2,2}}{\partial z} = 0 \qquad \text{at } z \to -\infty \tag{47}$$

We thus find the solution

$$\hat{\eta}_{2,2} = \frac{k}{2}A^2 \qquad \text{and} \qquad \hat{\phi}_{2,2} = 0. \tag{48}$$

It is rather remarkable that the second-order second-harmonic velocity potential vanishes!

Third-order problem at order $\mathcal{O}(\epsilon^2)$

$$\begin{aligned}\frac{\partial \eta_3}{\partial t} - \frac{\partial \phi_3}{\partial z} &= -\frac{\partial \eta_1}{\partial t_2} - \frac{\partial \phi_1}{\partial x}\frac{\partial \eta_2}{\partial x} - \frac{\partial \phi_2}{\partial x}\frac{\partial \eta_1}{\partial x} - \eta_1 \frac{\partial^2 \phi_1}{\partial x \partial z}\frac{\partial \eta_1}{\partial x} \\ &\quad + \eta_1 \frac{\partial^2 \phi_2}{\partial z^2} + \eta_2 \frac{\partial^2 \phi_1}{\partial z^2} + \frac{1}{2}\eta_1^2 \frac{\partial^3 \phi_1}{\partial z^3} \qquad \text{at } z = 0\end{aligned} \tag{49}$$

$$\begin{aligned}\frac{\partial \phi_3}{\partial t} + g\eta_3 &= -\frac{\partial \phi_1}{\partial t_2} - \eta_1 \frac{\partial^2 \phi_2}{\partial z \partial t} - \eta_2 \frac{\partial^2 \phi_1}{\partial z \partial t} - \frac{1}{2}\eta_1^2 \frac{\partial^3 \phi_1}{\partial z^2 \partial t} - \frac{\partial \phi_1}{\partial x}\frac{\partial \phi_2}{\partial x} - \frac{\partial \phi_1}{\partial z}\frac{\partial \phi_2}{\partial z} \\ &\quad - \eta_1 \left[\frac{\partial \phi_1}{\partial x}\frac{\partial^2 \phi_1}{\partial x \partial z} + \frac{\partial \phi_1}{\partial z}\frac{\partial^2 \phi_1}{\partial z^2} \right] \qquad \text{at } z = 0\end{aligned} \tag{50}$$

$$\frac{\partial^2 \phi_3}{\partial x^2} + \frac{\partial^2 \phi_3}{\partial z^2} = 0 \qquad \text{for } -\infty < z < 0 \tag{51}$$

$$\frac{\partial \phi_3}{\partial z} = 0 \qquad \text{at } z \to -\infty \tag{52}$$

The inhomogeneous terms on the right-hand sides are all cubic products of the first-order terms, or products between first- and second-order terms, in such a way that we understand the forcing has rapid phase oscillations either of the kind $\mathrm{e}^{\pm \mathrm{i}\chi}$ or $\mathrm{e}^{\pm 3\mathrm{i}\chi}$. We may therefore anticipate that the third-order solution is a superposition of first- and third-harmonic terms

$$\eta_3 = \frac{1}{2}\left(\hat{\eta}_{3,1}\mathrm{e}^{\mathrm{i}\chi} + \hat{\eta}_{3,3}\mathrm{e}^{3\mathrm{i}\chi} + \text{c.c.} \right) \tag{53}$$

$$\phi_3 = \frac{1}{2}\left(\hat{\phi}_{3,1}\mathrm{e}^{\mathrm{i}\chi} + \hat{\phi}_{3,3}\mathrm{e}^{3\mathrm{i}\chi} + \text{c.c.} \right) \tag{54}$$

Third-order first-harmonic problem

$$-\mathrm{i}\omega\hat{\eta}_{3,1} - \frac{\partial \hat{\phi}_{3,1}}{\partial z} = -\frac{\partial A}{\partial t_2} - \frac{5\mathrm{i}gk^3}{8\omega}|A|^2 A \equiv F \qquad \text{at } z = 0 \tag{55}$$

$$-\mathrm{i}\omega\hat{\phi}_{3,1} + g\hat{\eta}_{3,1} = \frac{\mathrm{i}g}{\omega}\frac{\partial A}{\partial t_2} - \frac{3gk^2}{8}|A|^2 A \equiv G \qquad \text{at } z = 0 \tag{56}$$

$$\frac{\partial^2 \hat{\phi}_{3,1}}{\partial z^2} - k^2 \hat{\phi}_{3,1} = 0 \qquad \text{for } -\infty < z < 0 \tag{57}$$

$$\frac{\partial \hat{\phi}_{3,1}}{\partial z} = 0 \qquad \text{at } z \to -\infty \tag{58}$$

This problem is singular, since we already insisted that the linear dispersion relation should be satisfied. The Fredholm alternative then tells us that the inhomogeneous forcing needs to satisfy a constraint. We have previously derived that constraint by application of Green's theorem. Alternatively, we can recognize that as $\hat{\phi}_{3,1}$ must be proportional to e^{kz}, the two surface equations can be combined to $-\mathrm{i}\omega F + kG = 0$, which leads to

$$\frac{\partial A}{\partial t_2} + \frac{\mathrm{i}\omega k^2}{2}|A|^2 A = 0 \tag{59}$$

which has solution

$$A = a\mathrm{e}^{\mathrm{i}\left(\theta - \frac{1}{2}\omega k^2 a^2 t_2\right)} \tag{60}$$

where a and θ are two real constants.

Third-order third-harmonic problem

(This is included just for amusement)

$$-3\mathrm{i}\omega\hat{\eta}_{3,3} - \frac{\partial \hat{\phi}_{3,3}}{\partial z} = -\frac{9\mathrm{i}gk^3}{8\omega}A^3 \qquad \text{at } z = 0 \tag{61}$$

$$-3\mathrm{i}\omega\hat{\phi}_{3,3} + g\hat{\eta}_{3,3} = \frac{3gk^2}{8}A^3 \qquad \text{at } z = 0 \tag{62}$$

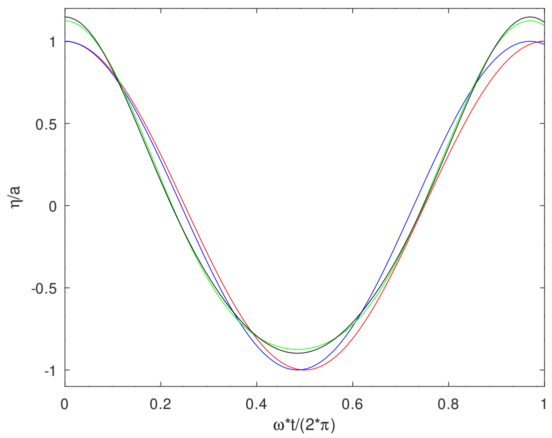
$$\frac{\partial^2 \hat{\phi}_{3,3}}{\partial z^2} - k^2 \hat{\phi}_{3,3} = 0 \qquad \text{for } -\infty < z < 0 \tag{63}$$

$$\frac{\partial \hat{\phi}_{3,3}}{\partial z} = 0 \qquad \text{at } z \to -\infty \tag{64}$$

We thus find the solution

$$\hat{\eta}_{3,3} = \frac{3k^2}{8}A^3 \qquad \text{and} \qquad \hat{\phi}_{3,3} = 0. \tag{65}$$

It is rather remarkable that the third-order third-harmonic velocity potential vanishes!



Figur 1: Stokes wave with steepness $\epsilon = 0.25$: red, linear wave without frequency modification; blue, first-order wave with frequency modification; green, second-order wave; black, third-order wave.

Finally, we summarize what we have found, and with no loss of generality we may set the solutions to the first-harmonic third-order problem and the zeroth-harmonic second-order problem both equal to zero, we then get

$$\eta = a\cos\psi + \frac{1}{2}ka^2\cos 2\psi + \frac{3}{8}k^2a^3\cos 3\psi \tag{66}$$

$$\phi = \frac{ga}{\omega}\mathrm{e}^{kz}\sin\psi \tag{67}$$

where

$$\psi = kx - \omega\left(1 + \frac{1}{2}(ka)^2\right)t + \theta \tag{68}$$

where a is the linear amplitude and θ is a constant phase.

A crest occurs for $\psi = 2\pi n$, for which $\eta = a + \frac{1}{2}ka^2 + \frac{3}{8}k^2a^3$. A trough occurs for $\psi = \pi + 2\pi n$, for which $\eta = -a + \frac{1}{2}ka^2 - \frac{3}{8}k^2a^3$. The wave height, the vertical difference between a crest and a trough, is $H = 2a + \frac{3}{4}k^2a^3$. The second-order wave height is equal to the linear wave height, while the third-order wave height is slightly larger.

Figure 1 shows the first-, second- and third-order profiles of a Stokes. It is seen that nonlinearity causes the crest to be higher and thinner, the trough to be shallower and wider, and the wave period to be reduced. Stokes waves appear to "point upward". The nonlinear correction to the phase speed and the group velocity is that both are faster than for linear waves.

It is remarkable that the velocity potential is unaffected by the nonlinear corrections at the second and third order. This is only true for Stokes waves on infinite depth.