

1 General problem

1. Cylindrical container with radius $R = 12$ cm.
2. Total fields are a superposition of background flow and wave disturbances: Velocity field $\mathbf{v} = \mathbf{U} + \mathbf{u}$. Pressure $p_{\text{total}} = P + p$. Surface elevation $z = \zeta + \eta$.
3. The steady background laminar azimuthal flow has velocity $\mathbf{U} = U(r, z)\mathbf{i}_\theta$, pressure $P(r, z)$ and surface displacement $\zeta(r)$. We place the origin of the cylindrical coordinate system in the middle of the container.
4. The steady background flow has up to 20 RPM, the angular velocity is up to $\Omega = 2\pi \cdot 20 \text{ min}^{-1} = 2\pi \cdot 20/60 \text{ s}^{-1} = 2.09 \text{ s}^{-1}$. We identify the small parameters $\Omega\sqrt{R/g} = 0.23$ and $\Omega/\omega_c = 0.23/14.8 = 0.016$ provided $\omega_c^2 = gk_c$.
5. The steady background flow provokes displacement of steady background surface up to $z = \zeta = \Omega^2 R^2/(2g)$. We identify the small parameter $\zeta/R = \Omega^2 R/g = 0.053$.
6. Experimentally observed waves due to drop impact have characteristic wavenumber approximately $k_c R = 14.8$, $(k_c R)^{-1} = 0.068$, meaning the cylindrical container can support more than 4 characteristic wavelengths, thus it may be reasonable to consider group modulation. Their characteristic surface elevation is a , and steepness $k_c a \ll 1$.
7. We do not consider unsteady or turbulent background flow. We do also not consider background flow without free surface. It is true that a laminar background flow can be unsteady due to instabilities not associated with a free surface. This is the topic of Drazin & Reid chapter 3, sections 14–19. These instabilities limit the strength of the laminar background flow that we can achieve in the laboratory. We insist that our experiment should have a laminar background flow.

quantity	typical value	scaling
$k_c a$	0.1	ϵ
$(k_c R)^{-1}$	0.068	ϵ^2
a/R	0.0068	ϵ^3
Ω/ω_c	0.016	ϵ^2
$\Omega\sqrt{R/g}$	0.23	ϵ
$\zeta/R \sim \Omega^2 R/g$	0.053	ϵ^2

2 Inviscid equations in coordinate free form

It is an advantage to use coordinate free representation as much as possible in order to avoid getting lost in the details of any particular coordinate system.

At this stage, limit to only one natural coordinate direction, $\mathbf{i}_z$, determined by the acceleration of gravity $\mathbf{g} = -g\mathbf{i}_z$. Thus we let the z -axis point upwards.

Here we let the total velocity be $\mathbf{v}$, the total pressure be p , and the total free surface elevation be $z = \eta$ relative to the vertical direction as indicated by gravity. There are three constants in the following: The acceleration of gravity g , the water density ρ , and the constant air pressure p_a .

The continuity equation

$$\nabla \cdot \mathbf{v} = 0 \quad (1)$$

The Euler equation

$$\frac{D\mathbf{v}}{dt} \equiv \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p + \mathbf{g} \quad (2)$$

where

$$\frac{D}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \quad (3)$$

If we rewrite the pressure $p = p_a - \rho g z + \tilde{p}$ where p_a is the constant air pressure, the Euler equation becomes

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla \tilde{p} \quad (4)$$

Taking the curl of the Euler equation, we get the vorticity equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \mathbf{u} = \mathbf{0} \quad (5)$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ is the vorticity.

Rewriting the advective term $\mathbf{v} \cdot \nabla \mathbf{v} = \nabla(\frac{v^2}{2}) - \mathbf{v} \times \boldsymbol{\omega}$, the Euler equation can be rewritten

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{v^2}{2} + \frac{\tilde{p}}{\rho} \right) = \mathbf{v} \times \boldsymbol{\omega} \quad (6)$$

The kinematic bottom condition for surface wave disturbances

$$|\mathbf{v}| \rightarrow 0 \quad \text{as } z \rightarrow -\infty \quad (7)$$

The kinematic surface condition

$$\frac{D\eta}{dt} \equiv \frac{\partial \eta}{\partial t} + \mathbf{v} \cdot \nabla \eta = v_z \quad \text{at } z = \eta \quad (8)$$

The dynamic surface condition

$$p - p_a - \gamma \nabla \cdot \mathbf{n} = 0 \quad \text{at } z = \eta \quad (9)$$

where p_a is the constant air pressure above the water surface γ is the coefficient of surface tension, and $\mathbf{n}$ is a unit normal vector to the free surface pointing out from water into air. This can be re-expressed

$$\tilde{p} - \rho g \eta - \gamma \nabla \cdot \mathbf{n} = 0 \quad \text{at } z = \eta \quad (10)$$

We have

$$\nabla \cdot \mathbf{n} = - \frac{\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} + \frac{\partial^2 \eta}{\partial x^2} \left(\frac{\partial \eta}{\partial y} \right)^2 + \frac{\partial^2 \eta}{\partial y^2} \left(\frac{\partial \eta}{\partial x} \right)^2 - 2 \frac{\partial \eta}{\partial x} \frac{\partial \eta}{\partial y} \frac{\partial^2 \eta}{\partial x \partial y}}{\left(1 + \left(\frac{\partial \eta}{\partial x} \right)^2 + \left(\frac{\partial \eta}{\partial y} \right)^2 \right)^{\frac{3}{2}}}. \quad (11)$$

2.1 Taylor-expand the surface boundary conditions

Assuming that η and $\mathbf{v}$ are small, we can Taylor expand the surface boundary conditions around $z = 0$.

The Taylor-expanded kinematic surface condition is

$$\frac{\partial \eta}{\partial t} + \mathbf{v} \cdot \nabla \eta + \eta \nabla \frac{\partial \mathbf{v}}{\partial z} \cdot \nabla \eta + \dots = v_z + \eta \frac{\partial v_z}{\partial z} + \frac{1}{2} \eta^2 \frac{\partial^2 v_z}{\partial z^2} + \dots \quad \text{at } z = 0 \quad (12)$$

The dynamic surface condition

$$\tilde{p} + \eta \frac{\partial \tilde{p}}{\partial z} + \frac{1}{2} \eta^2 \frac{\partial^2 \tilde{p}}{\partial z^2} + \dots = \rho g \eta + \gamma \nabla \cdot \mathbf{n} \quad \text{at } z = 0 \quad (13)$$

2.2 Horizontal boundary conditions?

If the phenomena to be discussed are of an extent comparable to the container, then horizontal boundary conditions must be imposed, and wave numbers will be discrete.

If the phenomena to be discussed are of an extent much smaller than the container, then no horizontal boundary need to be imposed, and wave numbers should be continuous.

We will assume that the background flow should obey the horizontal boundary conditions $\mathbf{U} \cdot \mathbf{i}_r = 0$ at $r = R$, and $\mathbf{U} = \mathbf{0}$ at $r = 0$ and all of $\mathbf{U}$ and P and ζ are continuously periodic over the azimuthal interval $0 \leq \theta \leq 2\pi$.

2.3 Helmholtz decomposition

For a general flow we can set $\mathbf{v} = \nabla\phi + \nabla \times \mathbf{A}$. The vorticity of the velocity field is then $\boldsymbol{\omega} = \nabla \times \mathbf{v} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$. With Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ we get the simpler relationship $\boldsymbol{\omega} = -\nabla^2 \mathbf{A}$. Most likely the vector potential can be described by a stream function, and we can set $\mathbf{A} = \mathbf{i}_z \psi$.

It is probably not correct to assume that the wave disturbances will be exclusively described by ϕ because the wave field will likely be (weakly) rotational due to the influence of the background flow.

2.4 Irrotational time-dependent flow

For irrotational flow, $\boldsymbol{\omega} = \mathbf{0}$, we set $\mathbf{v} = \nabla\phi$. The continuity equation yields the Laplace equation $\nabla^2\phi = 0$. The Euler equation can be integrated to give the Euler pressure equation

$$p = p_a - \rho \left(\frac{\partial\phi}{\partial t} + gz + \frac{(\nabla\phi)^2}{2} \right) \quad (14)$$

where p_a can in principle be any constant, but we select it to be equal to the air pressure above the water surface. This allows an alternative expression for the dynamic surface condition

$$\frac{\partial\phi}{\partial t} + g\eta - \frac{\gamma}{\rho} \nabla^2 \eta + \frac{(\nabla\phi)^2}{2} = 0 \quad \text{at } z = \eta \quad (15)$$

2.5 Rotational steady flow

Assume velocity in the azimuthal direction, $\mathbf{U} = U(r)\mathbf{i}_\theta$, pressure $P(r, z)$ and surface elevation $z = \zeta(r)$. In particular we assume U does not depend on z . The vorticity is $\nabla \times \mathbf{U} = \mathbf{i}_z \frac{1}{r} \frac{\partial}{\partial r}(rU(r))$. The continuity equation (18) is satisfied, $\nabla \cdot \mathbf{U} = 0$. The azimuthal component of the Euler equation (20) is identically satisfied. The two remaining components of the Euler equation (19) and (21) are

$$-\frac{U^2}{r} = -\frac{1}{\rho} \frac{\partial P}{\partial r}$$

and

$$0 = -\frac{1}{\rho} \frac{\partial P}{\partial z} - g$$

which have the solution for the pressure

$$P(r, z) = \rho \int_0^r \frac{U^2(\xi)}{\xi} d\xi - \rho g z + c \quad (16)$$

where c is a constant.

The dynamic surface condition combined with the condition $\zeta(0) = 0$ gives

$$\zeta(r) = \int_0^r \frac{U^2(\xi)}{g\xi} d\xi \quad (17)$$

(Here the surface tension was un-intentionally left out...)

All other boundary conditions are already satisfied and do not need to be considered.

For the special case of rigid body rotation with angular velocity $\mathbf{\Omega} = \Omega \mathbf{i}_z$, we have $\mathbf{U} = \mathbf{\Omega} \times \mathbf{r} = \Omega r \mathbf{i}_\theta$. In this case the vorticity is $\boldsymbol{\omega} = \nabla \times \mathbf{U} = 2\mathbf{\Omega}$. The vorticity equation is identical to zero since all three terms are zero. We get $\zeta(r) = \frac{\Omega^2 r^2}{2g}$.

3 Inviscid time-dependent equations in cylindrical coordinates

This seems to be unnecessary since the waves are assumed to be much shorter than the container size. However, we have:

Represent the velocity $\mathbf{v} = v_r \mathbf{i}_r + v_\theta \mathbf{i}_\theta + v_z \mathbf{i}_z$, the pressure p , and the free surface elevation $z = \eta$, in cylindrical coordinates $\{r, \theta, z\}$.

The continuity equation

$$\frac{1}{r} \frac{\partial}{\partial r} (r v_r) + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \quad (18)$$

The Euler equations

$$\frac{Dv_r}{dt} - \frac{v_\theta^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} \quad (19)$$

$$\frac{Dv_\theta}{dt} + \frac{v_r v_\theta}{r} = -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} \quad (20)$$

$$\frac{Dv_z}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g \quad (21)$$

where

$$\frac{D}{dt} = \frac{\partial}{\partial t} + v_r \frac{\partial}{\partial r} + \frac{v_\theta}{r} \frac{\partial}{\partial \theta} + v_z \frac{\partial}{\partial z} \quad (22)$$

The kinematic bottom condition for surface wave disturbances

$$|\mathbf{v}| \rightarrow 0 \quad \text{as } z \rightarrow -\infty \quad (23)$$

The kinematic and dynamic surface and bottom conditions are as before...

Finally, the boundary conditions $v_r = 0$ at $r = R$, $\mathbf{v} = \mathbf{0}$ at $r = 0$, and all quantities $\mathbf{v}$, p and η continuously periodic around the entire azimuthal angle θ .

4 Brilliant perturbation analysis

Let us assume that

1. Wave steepness small, $u \sim \epsilon$.
2. Background circulation small, $U \sim \epsilon$.
3. From previous analysis we anticipate background surface deflection very small, $\zeta \sim \epsilon^2$, so we can Taylor expand around a horizontal equilibrium surface. However, this does not imply that the background surface deflection does not play a role in the forthcoming perturbation analysis.
4. Triad interaction can be neglected for the time being. It will be a concern when we account for surface tension, and possibly also when we account for Doppler shift or interaction with background current?
5. Leading order wave disturbance is potential, $\mathbf{u} \approx \nabla\phi$.
6. Take analysis to second order ϵ^2 .

The perturbation expansions therefore start out like

$$\mathbf{v} = \epsilon\mathbf{v}_1 + \epsilon^2\mathbf{v}_2 + \mathcal{O}(\epsilon^3) \quad (24)$$

$$\eta = \epsilon\eta_1 + \epsilon^2\eta_2 + \mathcal{O}(\epsilon^3) \quad (25)$$

$$\tilde{p} = \epsilon\tilde{p}_1 + \epsilon^2\tilde{p}_2 + \mathcal{O}(\epsilon^3) \quad (26)$$

5 Order $\mathcal{O}(\epsilon)$

Decompose the velocity into an irrotational and rotational part

$$\mathbf{v}_1 = \nabla\phi_1 + \mathbf{U}_1 = \nabla\phi_1 + \Omega r\mathbf{i}_\theta \quad (27)$$

where $\mathbf{U}_1 = \Omega r\mathbf{i}_\theta$ is the background rotational flow of a rigid body.

The vorticity is $\boldsymbol{\omega}_1 = \nabla \times \mathbf{U}_1 = 2\Omega\mathbf{i}_z$ which is constant. Therefore the vorticity equation (5) is already identically satisfied.

All terms in the Euler equation (6) have gradient form

$$\nabla \left(\frac{\partial \phi_1}{\partial t} + \frac{\tilde{p}_1}{\rho} \right) = \mathbf{0} \quad (28)$$

from which we get the pressure

$$\tilde{p}_1 = -\rho \frac{\partial \phi_1}{\partial t} + f_1(t) \quad (29)$$

where $f_1(t)$ is an arbitrary function of time which can be incorporated into ϕ_1 .

The surface kinematic condition (12) becomes

$$\frac{\partial \eta_1}{\partial t} - \frac{\partial \phi_1}{\partial z} = 0 \quad \text{at } z = 0 \quad (30)$$

The surface dynamic condition (13) becomes

$$\tilde{p}_1 = \rho g \eta_1 - \gamma \nabla^2 \eta_1 \quad \text{at } z = 0 \quad (31)$$

or

$$\frac{\partial \phi_1}{\partial t} + g \eta_1 - \frac{\gamma}{\rho} \nabla^2 \eta_1 = 0 \quad \text{at } z = 0 \quad (32)$$

We now have the choice to solve the remaining problem in Cartesian or cylindrical coordinates.

In Cartesian coordinates the continuity equation becomes

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} + \frac{\partial^2 \phi_1}{\partial z^2} = 0 \quad (33)$$

and in polar coordinates it becomes

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi_1}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi_1}{\partial \theta^2} + \frac{\partial^2 \phi_1}{\partial z^2} = 0 \quad (34)$$

For Cartesian coordinates assume separable solution

$$\begin{pmatrix} \eta_1 \\ \phi_1 \end{pmatrix} = \begin{pmatrix} \hat{\eta}_1 \\ \hat{\phi}_1(z) \end{pmatrix} e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \quad (35)$$

where $\mathbf{x} = x\mathbf{i}_x + y\mathbf{i}_y$ and $\mathbf{k} = k_x\mathbf{i}_x + k_y\mathbf{i}_y$.

The continuity equation and the bottom condition gives the solution

$$\hat{\phi}_1(z) = \hat{\Phi} e^{kz} \quad (36)$$

where $k = |\mathbf{k}| = \sqrt{k_x^2 + k_y^2}$.

The Cartesian coordinates are a natural choice if the waves are short compared to the size of the container, and then it is not relevant to impose horizontal boundary conditions.

For polar coordinates it is tempting to use the above result of exponential attenuation as hindsight. Thus we assume separable solution

$$\begin{pmatrix} \eta_1 \\ \phi_1 \end{pmatrix} = \begin{pmatrix} \check{\eta}(r) \\ \check{\phi}(r)e^{kz} \end{pmatrix} e^{i(\nu\theta - \omega t)} \quad (37)$$

Obviously, the parameter ν must be restricted to integer values, $\nu \in \mathbb{Z}$, for continuously periodic solutions in the azimuthal direction. Substituted into the continuity condition, this requires $\check{\phi}(r)$ to satisfy the Bessel equation

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{d\check{\phi}}{dr} \right) - \frac{\nu^2}{r^2} \check{\phi} + k^2 \check{\phi} = 0 \quad (38)$$

and together with the boundary condition that $\check{\phi}$ is nonsingular at $r = 0$ this may constrain the permissible solutions.

In the case that it is relevant to impose a wall boundary condition, $\frac{\partial \check{\phi}}{\partial r} = 0$ at $r = R$, the wavenumber k may be limited to certain discrete values.

In either case, Cartesian or polar, horizontal boundary conditions or not, we get a linear system of the following kind

$$\begin{pmatrix} -i\omega & -k \\ g + \frac{\gamma}{\rho} k^2 & -i\omega \end{pmatrix} \begin{pmatrix} \hat{\eta} \\ \hat{\Phi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (39)$$

or with the substitution $\hat{\Phi} \rightarrow \check{\phi}$ and $\hat{\eta} \rightarrow \check{\eta}$, and the criterion for nontrivial solution is in both cases the dispersion relation

$$\omega^2 = gk + \frac{\gamma}{\rho} k^3 \quad (40)$$

Notice that k acts as an eigenvalue for the vertical structure, and as a wavenumber magnitude in the horizontal direction!

6 Order $\mathcal{O}(\epsilon^2)$

Decompose the velocity $\mathbf{v}_2 = \nabla \phi_2 + \mathbf{U}_2$ where the second term is allowed to have nonzero vorticity.

Introduce slow time $t_1 = \epsilon t$ and space $\mathbf{r}_1 = \epsilon \mathbf{r}$ dependence of all wave-related quantities.

The vorticity is $\boldsymbol{\omega}_2 = \nabla \times \mathbf{U}_2$. The vorticity equation (5) becomes

$$\frac{\partial \boldsymbol{\omega}_2}{\partial t} + \mathbf{u}_1 \cdot \nabla \boldsymbol{\omega}_1 - \boldsymbol{\omega}_1 \cdot \nabla \mathbf{u}_1 = \mathbf{0} \quad (41)$$

Recall that $\boldsymbol{\omega}_1 = 2\Omega \mathbf{i}_z$ is a constant, so $\nabla \boldsymbol{\omega}_1$ is zero. We have $\boldsymbol{\omega}_1 \cdot \nabla = 2\Omega \frac{\partial}{\partial z}$, so $\boldsymbol{\omega}_1 \cdot \nabla \mathbf{u}_1 = 2\Omega \nabla \frac{\partial \phi_1}{\partial z}$. We must solve

$$\frac{\partial \boldsymbol{\omega}_2}{\partial t} = 2\Omega \nabla \frac{\partial \phi_1}{\partial z} = 2\Omega k (\mathbf{i}_k + k \mathbf{i}_z) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t) + kz} \quad (42)$$

Assuming the time response of $\boldsymbol{\omega}_2$ is $e^{-i\omega t}$ we must solve

$$\nabla \times \mathbf{U}_2 = \frac{2i\Omega k}{\omega} (\mathbf{i}_k + k \mathbf{i}_z) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t) + kz} \quad (43)$$

A possible solution is

$$\mathbf{U}_2 = -\frac{2\Omega}{\omega} (\mathbf{k} \times \mathbf{i}_z) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t) + kz} \quad (44)$$

The Euler equation (6) becomes

$$\nabla \left(\frac{\partial \phi_2}{\partial t} + \frac{v_1^2}{2} + \frac{\tilde{p}_2}{\rho} \right) + \nabla_1 \left(\frac{\partial \phi_1}{\partial t} + \frac{\tilde{p}_1}{\rho} \right) + \nabla \frac{\partial \phi_1}{\partial t_1} = -\frac{\partial \mathbf{U}_2}{\partial t} + \nabla \phi_1 \times \boldsymbol{\omega}_1 + \mathbf{U}_1 \times \boldsymbol{\omega}_1 \quad (45)$$

however, the second parenthesis has been declared to be a function of t only, and that t -behavior has been incorporated into ϕ_1 , so the slow gradient of the second parenthesis evaluates to zero. We can therefore express this in all-gradient form

$$\nabla \left(\frac{\partial \phi_2}{\partial t} + \frac{v_1^2}{2} + \frac{\tilde{p}_2}{\rho} - \Omega^2 r^2 + \frac{\partial \phi_1}{\partial t_1} \right) = \mathbf{0} \quad (46)$$

Recall the expression for $\mathbf{v}_1 = \nabla \phi_1 + \mathbf{U}_1$ giving

$$v_1^2 = 2\Omega r \mathbf{i}_\theta \cdot \nabla \phi_1 + \Omega^2 r^2 \quad (47)$$

We thus get the pressure

$$\tilde{p}_2 = -\rho \left(\frac{\partial \phi_2}{\partial t} + \Omega r \mathbf{i}_\theta \cdot \nabla \phi_1 - \frac{\Omega^2 r^2}{2} + \frac{\partial \phi_1}{\partial t_1} \right) + f_2(t) \quad (48)$$

where $f_2(t)$ is an arbitrary function of time which can be incorporated into ϕ_2 .

The surface kinematic condition

$$\frac{\partial \eta_2}{\partial t} - \frac{\partial \phi_2}{\partial z} = -\nabla \phi_1 \cdot \nabla \eta_1 - \Omega r \mathbf{i}_\theta \cdot \nabla \eta_1 + \eta_1 \frac{\partial^2 \phi_1}{\partial z^2} - \frac{\partial \eta_1}{\partial t_1} \quad \text{at } z = 0 \quad (49)$$

The surface dynamic condition

$$\tilde{p}_2 + \eta_1 \frac{\partial \tilde{p}_1}{\partial z} = \rho g \eta_2 - \gamma \nabla^2 \eta_2 - 2\gamma \nabla \cdot \nabla_1 \eta_1 \quad \text{at } z = \eta \quad (50)$$

or

$$\frac{\partial \phi_2}{\partial t} + g \eta_2 - \frac{\gamma}{\rho} \nabla^2 \eta_2 = -\Omega r \mathbf{i}_\theta \cdot \nabla \phi_1 + \frac{\Omega^2 r^2}{2} \eta_1 \frac{\partial^2 \phi_1}{\partial t \partial z} - \frac{\partial \phi_1}{\partial t_1} + 2\gamma \nabla \cdot \nabla_1 \eta_1 \quad \text{at } z = 0 \quad (51)$$

At this point we introduce the decomposition of the second order surface displacement

$$\eta_2 = \zeta(r) + \tilde{\eta}_2 \quad (52)$$

where the stationary part satisfies

$$\frac{\Omega^2 r^2}{2} = g \zeta - \frac{\gamma}{\rho} \nabla^2 \zeta = g \zeta - \frac{\gamma}{\rho} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \zeta}{\partial r} \right) \quad (53)$$

and we are satisfied by including only the particular solution

$$\zeta = \frac{\Omega^2 r^2}{2g} + \frac{2\gamma \Omega^2}{\rho g^2} \quad (54)$$

We are now left with the reduced surface conditions

$$\frac{\partial \tilde{\eta}_2}{\partial t} - \frac{\partial \phi_2}{\partial z} = -\nabla \phi_1 \cdot \nabla \eta_1 - \Omega r \mathbf{i}_\theta \cdot \nabla \eta_1 + \eta_1 \frac{\partial^2 \phi_1}{\partial z^2} - \frac{\partial \eta_1}{\partial t_1} \quad \text{at } z = 0 \quad (55)$$

and

$$\frac{\partial \phi_2}{\partial t} + g \tilde{\eta}_2 - \frac{\gamma}{\rho} \nabla^2 \tilde{\eta}_2 = -\Omega r \mathbf{i}_\theta \cdot \nabla \phi_1 - \eta_1 \frac{\partial^2 \phi_1}{\partial t \partial z} - \frac{\partial \phi_1}{\partial t_1} + 2\gamma \nabla \cdot \nabla_1 \eta_1 \quad \text{at } z = 0 \quad (56)$$

Clearly, the second-order problem is forced by zeroth, first and second harmonic forcing.

$$\begin{pmatrix} \tilde{\eta}_2 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \hat{\eta}_{2,0} \\ \hat{\phi}_{2,0}(z) \end{pmatrix} + \begin{pmatrix} \hat{\eta}_{2,1} \\ \hat{\phi}_{2,1}(z) \end{pmatrix} e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} + \begin{pmatrix} \hat{\eta}_{2,2} \\ \hat{\phi}_{2,2}(z) \end{pmatrix} e^{2i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \quad (57)$$

We are (so far) only interested in the first harmonic.

$$-i\omega \hat{\eta}_{2,1} - \frac{\partial \hat{\phi}_{2,1}}{\partial z} = -\Omega r \mathbf{i}_\theta \cdot \nabla \hat{\eta}_1 - \frac{\partial \hat{\eta}_1}{\partial t_1} \quad \text{at } z = 0 \quad (58)$$

and

$$-i\omega\hat{\phi}_{2,1} + g\hat{\eta}_{2,1} + \frac{\gamma}{\rho}k^2\hat{\eta}_{2,1} = -\Omega r\mathbf{i}_\theta \cdot \nabla\hat{\phi}_1 - \frac{\partial\hat{\phi}_1}{\partial t_1} + 2i\gamma\mathbf{k} \cdot \nabla_1\hat{\eta}_1 \quad \text{at } z = 0 \quad (59)$$

In addition we have

$$\frac{\partial^2\hat{\phi}_{2,1}}{\partial z^2} - k^2\hat{\phi}_{2,1} = 0 \quad \text{for } z < 0 \quad (60)$$

and the bottom condition.

At this point we need to impose a solvability condition, which should yield something like

$$\frac{\partial\hat{\eta}_{1,1}}{\partial t_1} + \mathbf{c}_g \cdot \nabla_1\hat{\eta}_{1,1} + i\Omega r\mathbf{i}_\theta \cdot \mathbf{k}\hat{\eta}_{1,1} = 0 \quad (61)$$

where the last term represents the Doppler shift. The interesting questions are:

- Can we merge this amplitude equatin with the dispersion relation in order to incorporate the Doppler shift into the dispersion relation?
- Can this result be presented graphically as an adjustment to the red curve already presented?
- Can we experience reflection points?