

Experimental evidence of transversal modulation and frequency down-shift of uniform waves in a long tank

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Stokes waves are unstable — modulational instability — Benjamin & Feir (1967)

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The disintegration of wave trains on deep water

Part 1. Theory

By T. BROOKE BENJAMIN AND J. E. FEIR†

Department of Applied Mathematics and Theoretical Physics,
University of Cambridge

(Received 21 June 1966)

The phenomenon in question arises when a periodic progressive wave train with fundamental frequency ω is formed on deep water—say by radiation from an oscillating paddle—and there are also present residual wave motions at adjacent side-band frequencies $\omega(1 \pm \delta)$, such as would be generated if the movement of the paddle suffered a slight modulation at low frequency. In consequence of coupling through the non-linear boundary conditions at the free surface, energy is then transferred from the primary motion to the side bands at a rate that, as will be shown herein, can increase exponentially as the interaction proceeds. The result is that the wave train becomes highly irregular far from its origin, even when the departures from periodicity are scarcely detectable at the start.

In this paper a theoretical investigation is made into the stability of periodic wave trains to small disturbances in the form of a pair of side-band modes, and Part 2 which will follow is an account of some experimental observations in accord with the present predictions. The main conclusion of the theory is that infinitesimal disturbances of the type considered will undergo unbounded magnification if

$$0 < \delta < (\sqrt{2}/2)ka,$$

where k and a are the fundamental wave-number and amplitude of the perturbed wave train. The asymptotic rate of growth is a maximum for $\delta = ka$.

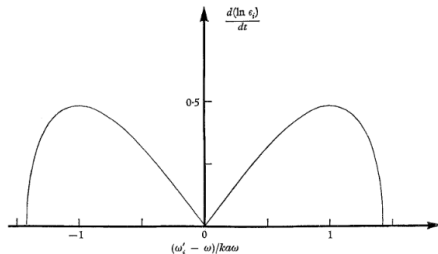


FIGURE 1. The asymptotic growth rate of the side-band amplitudes as a function of frequency.

The Fermi–Pasta–Ulam (FPU) recurrence — 1955

A long chain of masses linked by springs that obey Hooke's law (a linear interaction), but with a weak nonlinear term.

One day the computer wasn't stopped as intended, and the calculation was left running. They found, to their great surprise, that nearly all of the energy returned to the initial mode, and the original state was almost perfectly recovered.

The energy did not drift toward the equipartition predicted by statistical physics but periodically returned to the original mode.

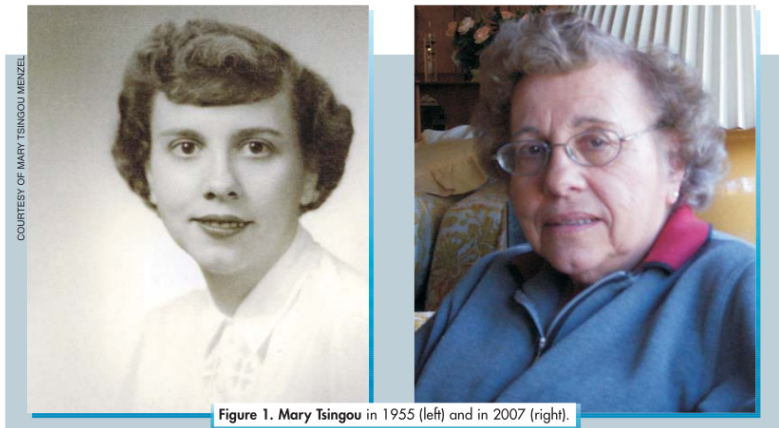
The first page of the report states the following:

Report written by Fermi, Pasta, and Ulam.

Work done by Fermi, Pasta, Ulam, and Tsingou.

Story told according to Thierry Dauxois, Physics Today (2008)

Now called the Fermi–Pasta–Ulam–Tsingou recurrence



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January 2008 Physics Today 55

Example of Fermi–Pasta–Ulam–Tsingou recurrence (1965)

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266. - *Studies of non Linear Problems*

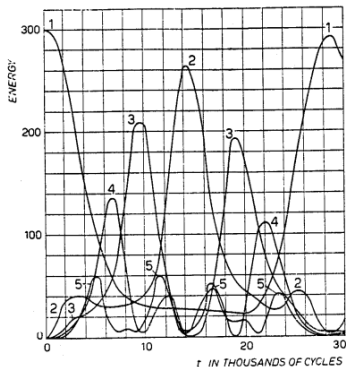


Fig. 1. - The quantity plotted is the energy (kinetic plus potential in each of the first five modes). The units for energy are arbitrary. $N = 32$; $\alpha = 1/4$; $8t^2 = 1/8$. The initial form of the string was a single sine wave. The higher modes never exceeded in energy 20 of our units. About 30,000 computation cycles were calculated.

Look for Fermi–Pasta–Ulam–Tsingou recurrence in water waves — Lake, Yuen, Rungaldier & Ferguson (1977) — they found recurrence with downshift, different from the Fermi–Pasta–Ulam–Tsingou recurrence

J. Fluid Mech. (1977), vol. 83, part 1, pp. 49–74
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Nonlinear deep-water waves: theory and experiment. Part 2. Evolution of a continuous wave train

By BRUCE M. LAKE, HENRY C. YUEN,
HARALD RUNGALDIER AND WARREN E. FERGUSON†

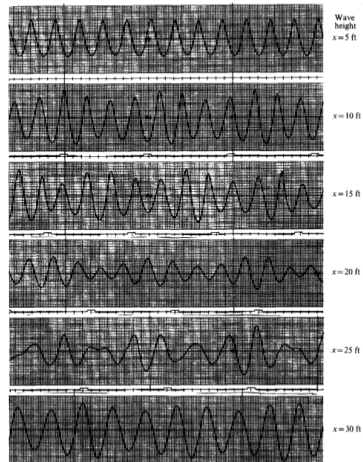
Fluid Mechanics Department, TRW/DSSG, One Space Park,
Redondo Beach, California 90278

(Received 13 December 1976 and in revised form 16 May 1977)

Results of an experimental investigation of the evolution of a nonlinear wave train on deep water are reported. The initial stage of evolution is found to be characterized by exponential growth of a modulational instability, as was first discovered by Benjamin & Feir. At later stages of evolution it is found that the instability does not lead to wave-train disintegration or loss of coherence. Instead, the modulation periodically increases and decreases, and the wave train exhibits the Fermi–Pasta–Ulam recurrence phenomenon. Results of an earlier study of nonlinear wave packets by Yuen & Lake, in which solutions of the nonlinear Schrödinger equation were shown to provide quantitatively correct descriptions of the properties of nonlinear wave packets, are applied to describe the experimentally observed wave-train phenomena. A comparison between the laboratory data and numerical solutions of the nonlinear Schrödinger equation for the long-time evolution of nonlinear wave trains is given.

Nonlinear deep-water waves

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One-dimensional conservative nonlinear models predict perfect Fermi–Pasta–Ulam–Tsingou recurrence

For example:

- NLS = Nonlinear Schrödinger equation
- MNLS = Modified Nonlinear Schrödinger equation
= Dysthe equation
- etc.

Master thesis of Karsten Trulsen 1987–1989

supervised by Kristian B. Dysthe

With the wave modulations described by the Dysthe equation,
can dissipation, induced by wave breaking, as a consequence of
wave modulation, produce a permanent downshift?

Yes, it can!

FREQUENCY DOWN-SHIFT THROUGH SELF MODULATION AND BREAKING

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ABSTRACT. We simulate the development of a moderately steep wavetrain, using evolution equations correct to fourth order in the wave steepness (see Dysthe (1979) and Lo and Mei (1985)), with an added term simulating effects of wave breaking. It is found that the breaking damps the developing sidebands selectively, such that the most unstable lower sideband comes out of the modulation-breaking process being the dominant one. The reason for this, in our opinion, is the tendency towards spatial localization of the part of the wavetrain contributing to the upper sidebands. This “high-frequency” part of the signal seems to concentrate around the steepest portions of the wavetrain, where breaking occurs. This is found both in our simulations and in experimental records (Melville 1983).

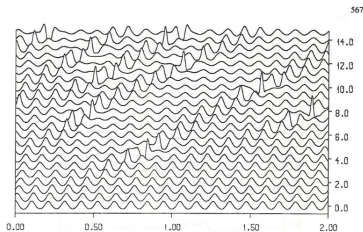


Figure 4: Wave profiles corresponding to the perturbed carrier wave.

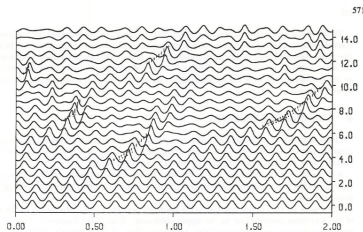


Figure 9: Wave profiles corresponding to the perturbed carrier wave, with breaking.

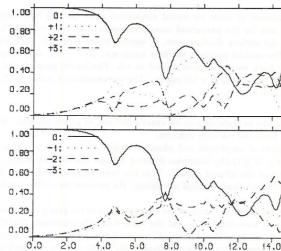


Figure 5: Spectral development for the three upper and lower sidebands, corresponding to the perturbed carrier wave.

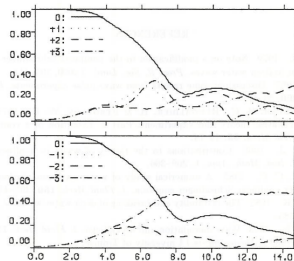


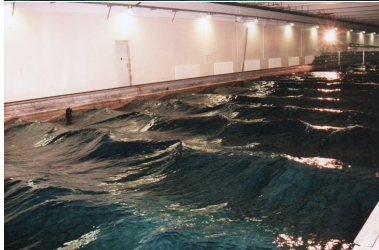
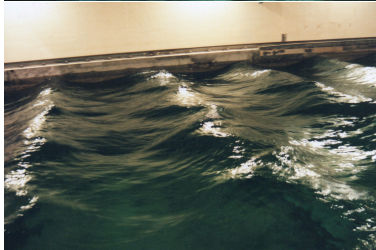
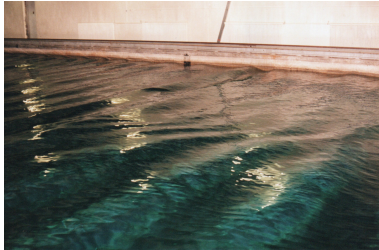
Figure 10: Spectral development for the three upper and lower sidebands, corresponding to the perturbed carrier wave, with breaking.

Very many publications follow this up ...

The one-dimensional downshift is not sensitive to the fine details of the heuristic damping / non-conservative mechanism.

(Here I should have included a long list of references. . .)

How to make a wave tank long :
Employ high-frequencies / short-wavelengths
but then the modulation is not long-crested

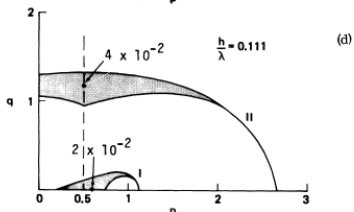
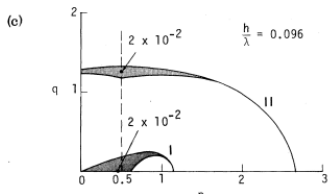
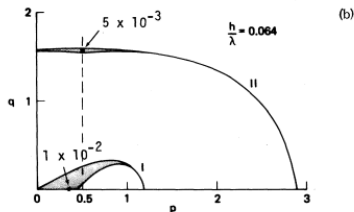
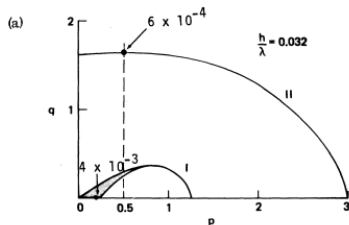


Hierarchy of modulational instabilities — collinear and transversal — type-I and type-II — McLean et al. (1981)

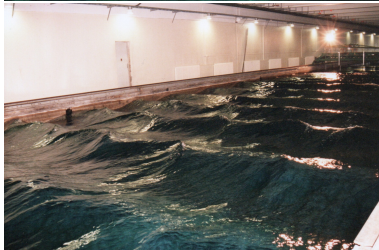
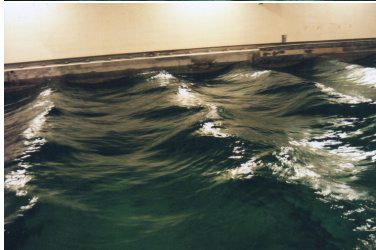
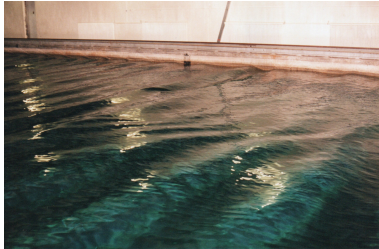
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30 MARCH 1981



We can easily distinguish type-I and type-II transversal instability in experiments by visual inspection



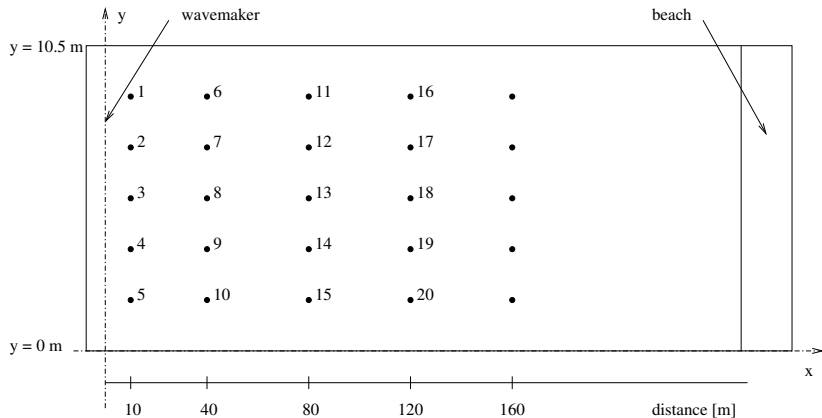
Have all the “one-dimensional” experiments fooled us to believe that the downshift is a dissipative process?

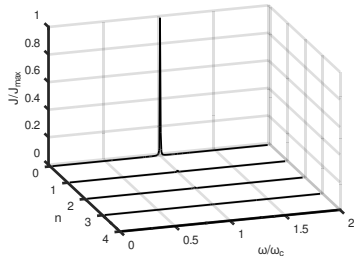
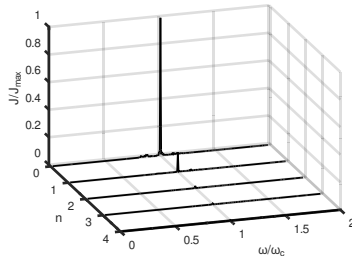
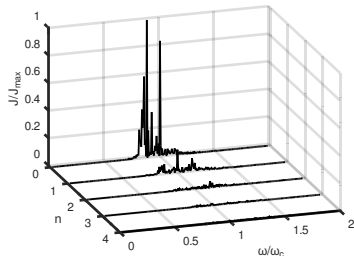
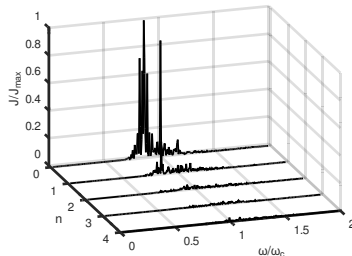
Maybe the “one-dimensional” experiments were actually “two-dimensional”, employing short wavelengths in order to make the wave tanks long?

Is dissipation less important than previously believed due to energy “hiding” in transversal modes?

We need new experiments with better resolution of transversal modes!

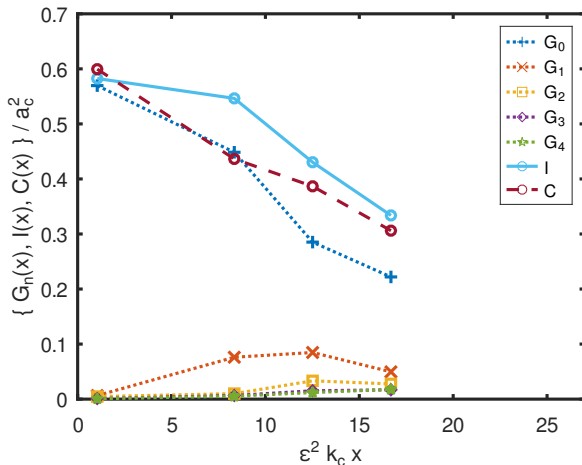
Design of new experiments at MARINTEK (now renamed SINTEF Ocean)



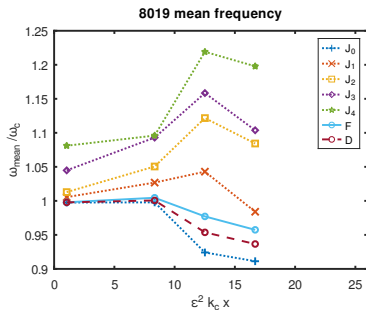
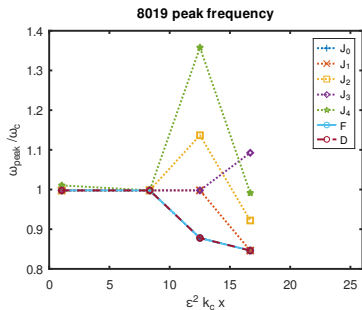
8019, $\epsilon^2 k_c x_1 = 1.04$ 8019, $\epsilon^2 k_c x_2 = 8.34$ 8019, $\epsilon^2 k_c x_3 = 12.5$ 8019, $\epsilon^2 k_c x_4 = 16.7$ 

Frequency power spectrum $J_n(x, \omega)$ for each transversal mode number n at each station x , normalized peak value 1 at each station x .

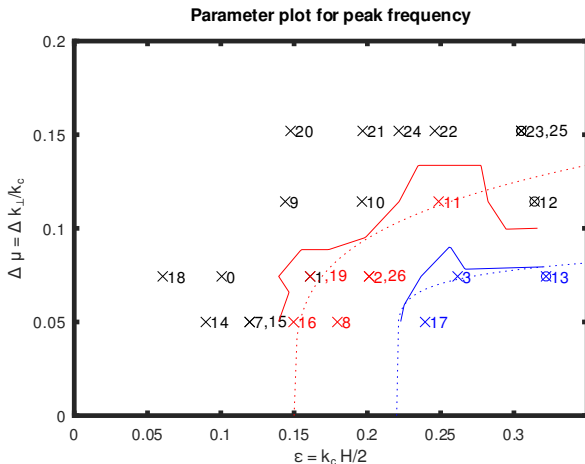
8019 mean power



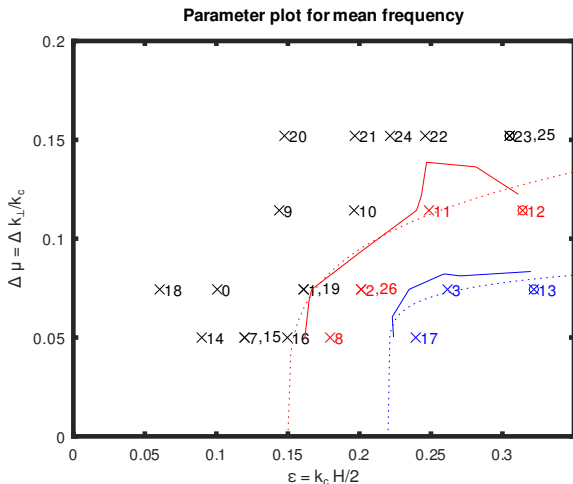
Mean power along the tank for each transversal mode $G_n(x)$, the total mean power $I(x)$, and the mean power along the centerline of the tank $C(x)$.



Peak and mean frequencies along the tank, $\omega_{\text{peak},n}(x)$ and $\omega_{\text{mean},n}(x)$ for each transversal mode $J_n(x, \omega)$, $\omega_{\text{peak}}(x)$ and $\omega_{\text{mean}}(x)$ for the combined measurements across the tank $F(x, \omega)$, $\omega_{\text{peak},c}(x)$ and $\omega_{\text{mean},c}(x)$ for the centerline of the tank $D(x, \omega)$.



Parametric plot for peak frequency downshift. Runs without, $\times$, and with, $\otimes$, clear visual observation of type II horseshoe patterns. Run numbers reduced by 8000. Colors: black, no downshift $\Delta\omega_{\text{peak}}/\omega_c > 0.9$; red, small downshift $0.9 \geq \Delta\omega_{\text{peak}}/\omega_c > 0.76$; blue, clear downshift $\Delta\omega_{\text{peak}}/\omega_c \leq 0.76$.



Parametric plot for mean frequency downshift. Runs without, $\times$, and with, $\otimes$, clear visual observation of type II horseshoe patterns.

Run numbers reduced by 8000. Colors: black, no downshift

$\Delta\omega_{\text{mean}}/\omega_c > 0.95$; red, small downshift

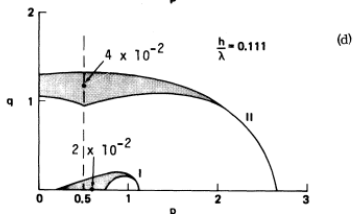
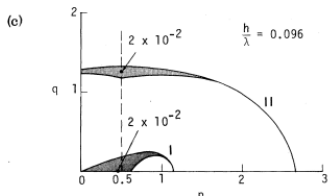
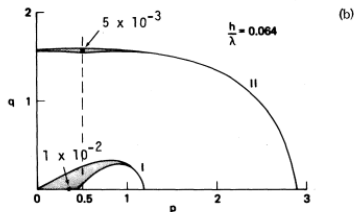
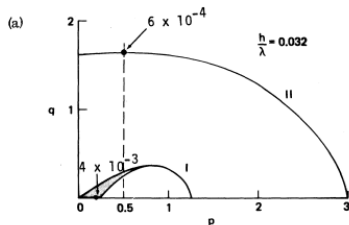
$0.95 \geq \Delta\omega_{\text{mean}}/\omega_c > 0.89$; blue, clear downshift

Downshift occurs when transversal type-I instabilities are enabled — McLean et al. (1981)

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Conclusion

- Recurrence with downshift only when transversal type-I instabilities are enabled
- Type-II instabilities do not correlate with downshift
- Insufficient measurement resolution across the tank can result in overestimating dissipation and incorrect estimation of the amount of downshifting, because energy and frequency content can be “hidden” in transversal modes.
- Downshift due to dissipation is possible, but may pertain to an idealized world that does not exist?
- Downshift without dissipation has not been observed in physical experiments?
- A criterion for downshift appears to be that the wave tank should accomodate crest lengths of at least 10 wavelengths.