Experimental evidence of transversal modulation and frequency downshift of uniform waves in a long tank

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Abstract

Experiments in a long tank show that transversal Benjamin–Feir instability of Stokes waves can lead to a significant energy transfer into oscillations across the tank. We observe frequency downshift in the long term evolution of Stokes waves essentially only when significant interactions with unstable narrow-banded transversal modes are involved. With insufficient resolution of measurements across the tank, the energy transferred to transversal modes could be misinterpreted as dissipation. Our experiments suggest that the frequency downshift depends as much on narrow-banded unstable transversal modulations of type I as it does on damping or wave breaking. Broad-banded unstable modulations of type II and horseshoe modulation patterns do not imply downshift.

1 Introduction

Uniform long-crested water surface waves of permanent form on deep water, commonly known as Stokes waves (Stokes, 1847), are unstable to sideband perturbations. The instability was first discovered for narrow band collinear perturbations (Lighthill, 1965; Benjamin & Feir, 1967; Benjamin, 1967) and is commonly known as the Benjamin–Feir instability. The collinear narrow band instability vanishes for finite depth $kh < 1.363$ (Benjamin, 1967; Hasimoto & Ono, 1972). Here $k = 2\pi/\lambda$ is the wavenumber, λ is the wavelength and h is the depth. The instability also extends to transversal perturbations which are unstable for both greater and smaller depths (Benney & Roskes, 1969).

In the course of long-term evolution, following the initial instability and after a stage of heavy modulations, there may be recurrence back to a uniform wave train. There is laboratory evidence that the recurrence can either be back to the original Stokes wave frequency or it can be accompanied by a downshift to a lower frequency (Lake *et al.*, 1977). The downshift has been observed together with wave breaking and with the growth of a continuous background spectrum (Melville, 1982; Tulin & Waseda, 1999).

Simple model equations predict the Fermi–Pasta–Ulam recurrence in which the initial uniform wave train recurs (Yuen & Ferguson, 1978*b*; Janssen, 1981). Although a temporal downshift is predicted by the non-dissipative Dysthe equation (Dysthe, 1979; Janssen, 1983; Lo & Mei, 1985), a permanent recurrence with downshift has

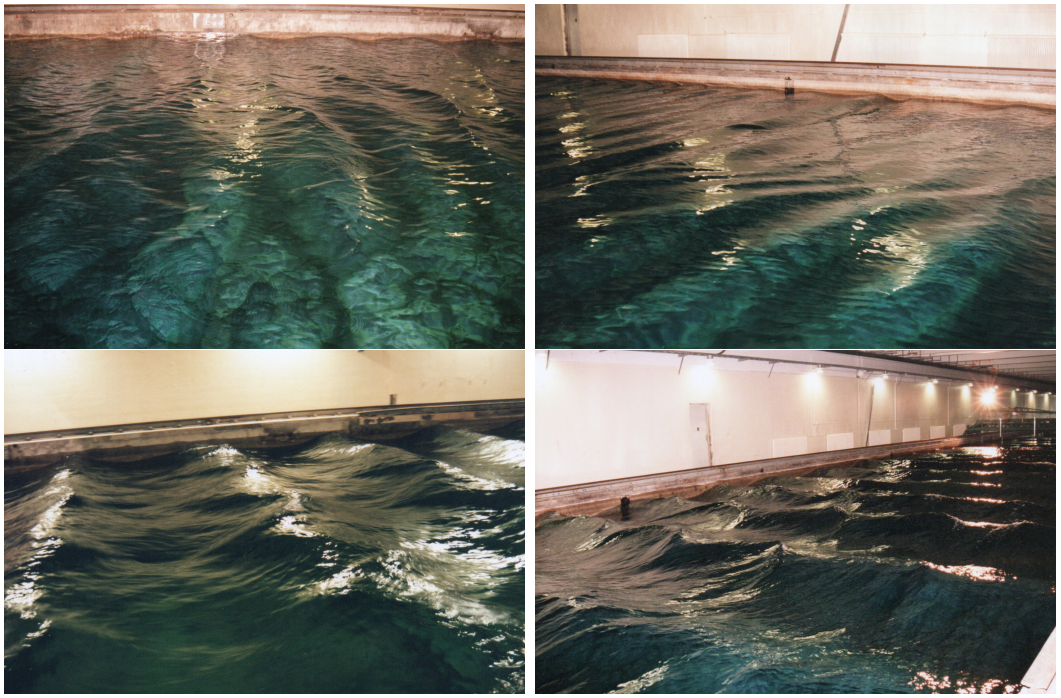


Figure 1: Upper left: Transversal modulation (TM) of type I in Run 8008. Upper right: TM of type I in Run 8015. Lower left and right: Horseshoe patterns typical of TM of type II in Run 8012.

not been captured by non-dissipative long-crested model equations. A permanent downshift has been captured by long-crested model equations equipped with various types of dissipation, including ad-hoc models for wave breaking (Trulsen & Dysthe, 1990), models for viscous damping without breaking (Uchiyama & Kawahara, 1994; Kato & Oikawa, 1995; Skandrani *et al.*, 1996; Segur *et al.*, 2005; Islas & Schober, 2011*a*,*b*), combinations of wind forcing and damping (Hara & Mei, 1991; Trulsen & Dysthe, 1992; Schober & Strawn, 2015), and models for energy and momentum loss due to breaking (Tulin, 1996; Tulin & Waseda, 1999). A review of long-crested downshift was given by Carter *et al.* (2019).

Stokes waves are unstable to infinitesimal perturbations due to both four-wave and five-wave nonlinear resonance. In McLean *et al.* (1981) these are denoted instabilities of type I and II, respectively. The Benjamin–Feir instability is of narrow band type I. These instabilities are nicely summarized in figure 1 of McLean *et al.* (1981) from which it is observed that type I is mainly collinear and narrow band while type II is mainly transversal and wide band. Type II instability of Stokes waves gives rise to characteristic horseshoe patterns of transversal length equal to the longitudinal wavelength (Su *et al.*, 1982; Su, 1982). Transversal modulations of narrow band type I and broad band type II are easily distinguished by visual inspection of the wave field, as can be appreciated in photos from our experiments in figure 1.

The emphasis of this paper is on weakly nonlinear and narrow-banded transversal instability of type I. For sufficiently narrow wave tanks there is numerical evidence that non-dissipative transversal modulations can lead to recurrence without down-

shift (Yuen & Ferguson, 1978*a*; Lo & Mei, 1987). For slightly wider wave tanks there is numerical evidence for a permanent non-dissipative downshift to transversal modes (Trulsen & Dysthe, 1997). Trulsen *et al.* (1999) provided experimental evidence that the downshift can be accompanied by substantial transfer of energy from long-crested modes to transversal modes, which can easily be misinterpreted as dissipation in case of insufficient measurements in the transversal direction. Accurate numerical simulations of short-crested waves by Fuhrman *et al.* (2006) further support the conclusions of Trulsen & Dysthe (1997).

Trulsen & Dysthe (1997) and Trulsen *et al.* (1999) anticipated that the evolution with transversal modes depends on a width parameter $\mu = k_\perp/k_c$ where $k_\perp$ is a permissable transversal wavenumber vector component, while $k_c = 2\pi/\lambda$ is the characteristic wavenumber and λ is the characteristic wavelength of the original Stokes wave. In a wave tank of width b the permissable transversal wavenumber components will be discretized in multiples of $\Delta k_\perp = \pi/b$ and the width parameter will be discretized in multiples of $\Delta\mu = \Delta k_\perp/k_c = \lambda/(2b)$. On deep water this evaluates to

$$\Delta\mu = \frac{\Delta k_\perp}{k_c} = \frac{\lambda}{2b} = \frac{gT^2}{4\pi b} \tag{1}$$

where T is the characteristic wave period and g is the acceleration of gravity. Transversal type I instability can be excited in experiments when $\Delta\mu$ is small enough to be inside the unstable domain. Visual inspection of figure 1 in McLean *et al.* (1981) suggests that the transversal instability of type I is contained within $\mu < 0.36$. From the same figure we also see that unstable perturbations of type II are essentially limited to $\mu > 1$ for the small steepnesses $\epsilon = a_c k_c < 0.33$ employed in our experiments. Here we define the characteristic amplitude to be $a_c = \sqrt{2}\sigma$, where σ is the standard deviation of the surface elevation, thus for uniform waves a_c is the amplitude and $H = 2a_c$ is the wave height.

The single experiment reported by Trulsen *et al.* (1999) had $\Delta\mu = 0.074$, with unseeded Stokes waves generated by the wave maker, and with transversal measurements limited to only one propagation distance along the tank, where only two probes were located across the tank. The original experiments of Lake *et al.* (1977) had $\mu = 0.066$ (see Trulsen & Dysthe, 1997) with only one probe across the tank. Toffoli *et al.* (2013) reported experiments with $\Delta\mu$ between 0.011 and 0.031, with seeded Stokes waves with transversal modulations generated by the wave maker, unfortunately without transversal measurements. Pinho & Babanin (2015) reported experiments with $\Delta\mu$ between 0.0094 and 0.026, with unseeded Stokes waves generated by the wave maker, and with transversal measurements limited to only one propagation distance along the tank where nine probes were located across the tank. Numerical simulations with $\mu = 0.098$ and with seeded perturbations of Stokes waves reported by Fernandez *et al.* (2014) showed substantial growth of oblique sidebands.

There is clearly a lack of experiments resolving the evolution of several transveral modes of Stokes waves propagating along a tank of finite width. We here report such experiments, with $\Delta\mu$ between 0.05 and 0.15, with unseeded Stokes waves generated by the wave maker, and with transversal measurements using five probes across the tank at four distances along the tank. This gives adequate resolution for unstable mode dynamics within the domain of type I instability, while it does not resolve

dynamics of type II instability. The presence of both type I and II modulations were however observed visually. These tests are run in a 260 m long and 10.5 m wide tank, which is long enough to study the problem at a reasonable scale.

This paper deals with the evolution of uniform waves in a long tank for the limiting regime that the tank is just wide enough for unstable transversal modulations to modify the long range evolution compared to a narrow tank. We discover that the limiting width corresponds to $\Delta\mu \approx 0.1$, thus the tank should accomodate crest lengths of at least 10 wavelengths. In this paper we do not consider the opposite limit of wide basins with $\Delta\mu \ll 0.1$, or irregular waves for which the modulational instability is suppressed (Alber, 1978) such as Onorato *et al.* (2009); Toffoli *et al.* (2010); Cherneva & Guedes Soares (2011) who employed irregular waves with small $\Delta\mu = 0.016$.

It is a curious coincidence, not necessarily related, that Gramstad & Trulsen (2007) concluded the Benjamin–Feir index (BFI) is a good indicator of increased freak wave activity only when the wave field is long-crested with characteristic crest length at least 10 characteristic wavelengths.

2 Experimental setup

The experiments were carried out during the summer of 1998 in the towing tank at MARINTEK (later renamed SINTEF Ocean). The tank is 260 m long and $b = 10.5$ m wide. The depth of the tank is $h = 10$ m for the first 80 m closest to the wave maker, and $h = 5$ m elsewhere, with a step-type bathymetric connection. The experimental setup, using 20 probes arranged with five across the tank at four distances from the wavemaker, is shown in figure 2. The x-axis runs along the tank measuring the distance from the wavemaker. The y-axis is across with $y = 0$ at one side. The probes are at transversal positions $y_l = lb/6$ for $l = 1, 2, 3, 4, 5$, between the lateral tank walls at $y = 0$ and $y = b$. The wave tank is sufficiently long that waves reflected from the damping beach do not reach back within the time window used for analysis.

The experiments consisted of generating long series of un-seeded uniform long-crested waves with given period T and wave height H. Unstable perturbations grew naturally and the resulting modulations were recorded by the 20 probes along and across the tank. A total of 27 experimental runs were carried out, of which four were repeated runs. Linear harmonic signals were used as input to the wavemaker. This means that free "parasitic" overharmonic components are also present after some additional time. However, for the present study of modulations and downshift we do not consider that these effects disturb our findings.

Table 1 gives a summary of the run numbers and the corresponding nominal period T and nominal wave height H. The nominal wave height is supposed to be the peak-to-trough height of the generated uniform wave. We define the characteristic amplitude to be half the nominal wave height $a_c = H/2$, and we define the characteristic frequency to be given by the nominal period $\omega_c = 2\pi/T$. We let the characteristic wavenumber be given by the linear dispersion relation $\omega_c^2 = gk_c \tanh k_c h$, where $g = 9.81\,\mathrm{m/s^2}$ is the acceleration of gravity and h is the depth. For all but

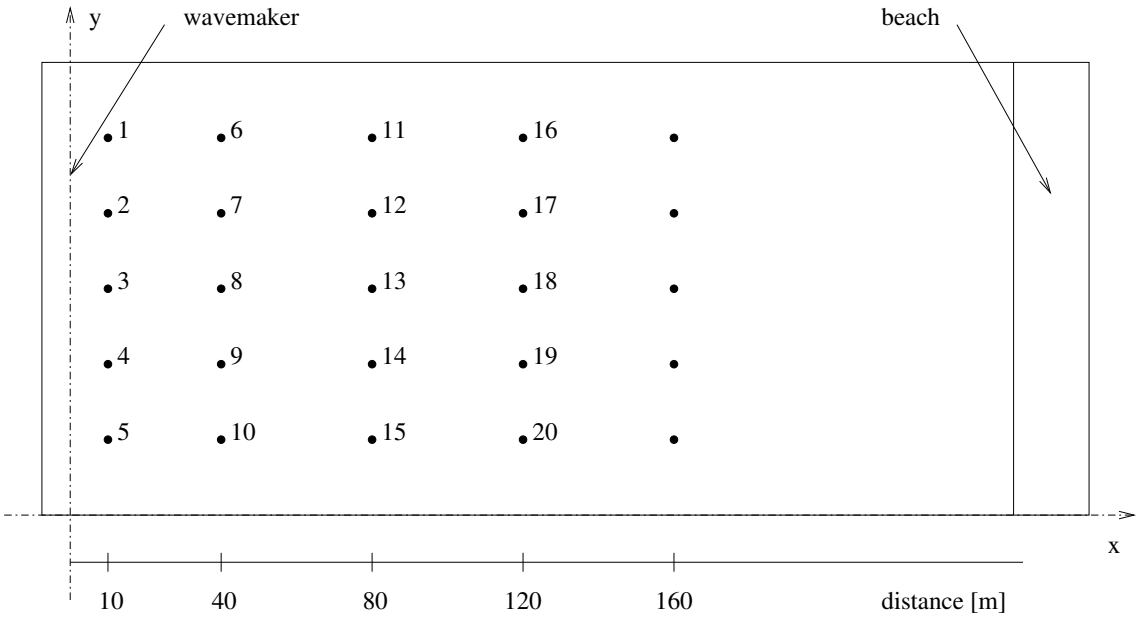


Figure 2: Experimental setup with array of 20 probes. For runs 8000–8017 probes were located at $x_1 = 10\,\mathrm{m}$, $x_2 = 40\,\mathrm{m}$, $x_3 = 80\,\mathrm{m}$ and $x_4 = 120\,\mathrm{m}$. For runs 8018–8026 the probes and probe numbers at $x = 40\,\mathrm{m}$, $80\,\mathrm{m}$ and $120\,\mathrm{m}$ were moved to $80\,\mathrm{m}$, $120\,\mathrm{m}$ and $160\,\mathrm{m}$, respectively.

runs 8004–8006 we can safely use the deep-water dispersion relation $\omega_c^2 = gk_c$. For this analysis we do not employ nonlinear corrections to the linear dispersion relation, leaving the nonlinear corrections to be taken care of by nonlinear evolution equations (e.g. Trulsen & Dysthe, 1997). In the table we show the resulting values for the dimensionless depth $k_c h$ and steepness $\epsilon = a_c k_c$ for the shallower depth $h = 5\,\mathrm{m}$. The transversal parameter $\Delta\mu$ from equation (1) is also shown. The largest fetch $\epsilon^2 k_c x_4$ is an important parameter telling us how long the tank is in terms of nonlinear modulational evolution. For ease of comparison we employ the same definition of fetch as in Trulsen & Dysthe (1997) inspired by the effect of cubic nonlinearities in the nonlinear Schrödinger equation. The last four columns of the table are explained in the next section.

Runs 4–6 are test cases with large transversal parameter $\Delta\mu$ and small fetch $\epsilon^2 k_c x_4$, not interesting and therefore not included in the following discussion.

3 Data analysis

The x-axis is along the tank, the y-axis is across, and the z-axis is vertical pointing up. The surface elevation is $z = \eta$ with quiescent surface $z = 0$. The velocity field is $\boldsymbol{v} = \nabla\phi$ according to an assumption of potential flow.

At the lateral vertical walls there is no flow through the walls $v_y = 0$. In the case of gravity waves and potential flow, the normal derivative of the surface elevation vanishes at the walls, $\frac{\partial \eta}{\partial y} = 0$. This can be established from the exact dynamic

Run	T [s]	H [m]	$k_c h$	ϵ	$\Delta\mu$	$\epsilon^2 k_c x_4$	$\frac{\omega_{\text{peak}}(x_4)}{\omega_c}$	$\frac{\omega_{\text{peak},c}(x_4)}{\omega_c}$	$\frac{\omega_{\text{mean}}(x_4)}{\omega_c}$	$\frac{\omega_{\text{mean},c}(x_4)}{\omega_c}$
8000	1	0.05	20	0.1	0.074	4.9	1	1	1	1
8001	1	0.08	20	0.16	0.074	13	1	0.85	0.99	0.97
8002	1	0.1	20	0.2	0.074	20	0.81	0.81	0.92	0.91
8003	1	0.13	20	0.26	0.074	33	0.73	0.75	0.88	0.87
8004	2.5	0.1	3.2	0.032	0.46	0.08	0.98	0.98	0.98	0.98
8005	2	0.1	5	0.05	0.3	0.31	0.98	0.98	0.98	0.98
8006	3	0.1	2.2	0.022	0.67	0.027	1	1	1	1
8007	0.82	0.04	30	0.12	0.05	10	1	1	1	0.99
8008	0.82	0.06	30	0.18	0.05	23	0.82	0.83	0.92	0.92
8009	1.24	0.11	13	0.14	0.11	6.5	1	1	1	1
8010	1.24	0.15	13	0.2	0.11	12	1	1	1	1
8011	1.24	0.19	13	0.25	0.11	19	0.8	0.8	0.94	0.92
8012	1.24	0.24	13	0.31	0.11	31	1	1	0.95	0.91
8013	1	0.16	20	0.32	0.074	50	0.73	0.73	0.87	0.86
8014	0.82	0.03	30	0.09	0.05	5.8	1	1	1	1
8015	0.82	0.04	30	0.12	0.05	10	1	0.89	1	1
8016	0.82	0.05	30	0.15	0.05	16	0.85	0.85	0.97	0.95
8017	0.82	0.08	30	0.24	0.05	41	0.74	0.74	0.88	0.87
8018	1	0.03	20	0.06	0.074	2.3	1	1	1	1
8019	1	0.08	20	0.16	0.074	17	0.85	0.85	0.96	0.94
8020	1.43	0.15	9.8	0.15	0.15	6.9	1	1	1	1
8021	1.43	0.2	9.8	0.2	0.15	12	1	1	1	1
8022	1.43	0.25	9.8	0.25	0.15	19	1	1	0.96	0.94
8023	1.43	0.31	9.8	0.31	0.15	29	1	1	0.96	0.95
8024	1.43	0.225	9.8	0.22	0.15	15	1	1	1	0.99
8025	1.43	0.31	9.8	0.31	0.15	29	1	1	0.96	0.96
8026	1	0.1	20	0.2	0.074	26	0.8	0.8	0.9	0.9

Table 1: First column run number. Next six columns nominal parameter values: period, wave height, dimensionless depth, steepness, width parameter, largest fetch. Final four columns downshift at largest fetch: peak across tank, peak at centerline, mean across tank, mean at centerline

surface condition

$$\frac{\partial \phi}{\partial t} + g\eta + \frac{1}{2}(\nabla\phi)^2 = 0 \quad \text{at } z = \eta$$

Upon taking the y-derivative and evaluating at the lateral wall we get the desired result.

In the following we outline a Fourier analysis in the transversal (y) and time (t) domains, with no Fourier decomposition in the longitudinal (x) direction.

Having established that $\frac{\partial \eta}{\partial y} = 0$ at the lateral walls, we can represent the transversal behavior of the surface elevation by an infinite cosine series

$$\eta(x, y, t) = \sum_{n=0}^{\infty} \tilde{\eta}_n(x, t) \cos(\frac{\pi n y}{b}). \tag{2}$$

We proceed as in Trulsen *et al.* (1999), but now truncating equation (2) to the first five terms, insisting on exact reproduction at the collocation points

$$\eta(x, y_l, t) = \sum_{n=0}^{4} \check{\eta}_n(x, t) \cos(\frac{\pi n y_l}{b}) \qquad \text{for} \quad l = 1, 2, 3, 4, 5. \tag{3}$$

With $y_l/b = l/6$ the coefficients are obtained by the transform

$$\begin{pmatrix} \check{\eta}_0 \\ \check{\eta}_1 \\ \check{\eta}_2 \\ \check{\eta}_3 \\ \check{\eta}_4 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 2 & 0 & 2 & 0 & 2 \\ 2\sqrt{3} & 0 & 0 & 0 & -2\sqrt{3} \\ 3 & -3 & 0 & -3 & 3 \\ \sqrt{3} & -3 & 0 & 3 & -\sqrt{3} \\ 1 & -3 & 4 & -3 & 1 \end{pmatrix} \begin{pmatrix} \eta(y_1) \\ \eta(y_2) \\ \eta(y_3) \\ \eta(y_4) \\ \eta(y_5) \end{pmatrix}$$

In the following we use the truncated equation (3) as the basis for interpolation across the tank

$$\eta(x, y, t) = \sum_{n=0}^{4} \check{\eta}_n(x, t) \cos(\frac{\pi n y}{b}) \tag{4}$$

We take the temporal Fourier transform over a time interval τ starting at an offset time depending on the position $\tau_0(x)$

$$\check{\eta}_n(x, t) = \sum_j \hat{\eta}_{n,j}(x) \mathrm{e}^{-\mathrm{i}\omega_j t} \qquad \text{for} \quad \tau_0(x) \le t < \tau_0(x) + \tau \tag{5}$$

where $\omega_j = 2\pi j/\tau$. The importance of employing the offset starting time $\tau_0(x)$ is to recognize that energy is transported with the group velocity. Between two positions x_1 and x_2 we have $x_1 - x_2 = c_g(\tau_0(x_1) - \tau_0(x_2))$ where c_g is the group velocity corresponding to the nominal wave period T.

With the advantage of the orthogonality of the interpolating functions, we can now compute the mean power in the combined time and transversal domain

$$I(x) = \frac{1}{\tau} \int_{\tau_0(x)}^{\tau_0(x)+\tau} \frac{1}{b} \int_0^b \eta^2(x, y, t) \, \mathrm{d}y\mathrm{d}t = \sum_j \sum_{n=0}^{4} \nu_n |\hat{\eta}_{n,j}(x)|^2 \tag{6}$$

where we have introduced the coefficient ν_n such that $\nu_0 = 1$ and $\nu_n = 1/2$ for $n > 0$.

Introducing the power of each mode

$$J_n(x, \omega_j) = \nu_n |\hat{\eta}_{n,j}(x)|^2$$

the transversally integrated power

$$F(x, \omega_j) = \sum_{n=0}^{4} J_n(x, \omega_j)$$

and the temporally integrated power

$$G_n(x) = \sum_j J_n(x, \omega_j)$$

we have that the total mean power is

$$I(x) = \sum_{n=0}^{4} G_n(x) = \sum_j F(x, \omega_j) = \sum_j \sum_{n=0}^{4} J_n(x, \omega_j)$$

Sometimes experiments are carried out with only one probe across the tank for each longitudinal postion x, typically such probes are located along the centerline of the tank. In our case this corresponds to transversal position $y_3 = b/2$ and we define $\zeta(x,t) \equiv \eta(x, y_3, t) = \check{\eta}_0(x,t) - \check{\eta}_2(x,t) + \check{\eta}_4(x,t)$ where we have employed the interpolation formula (4). Again we take the temporal Fourier transform over a time interval τ starting at an offset time depending on position $\tau_0(x)$

$$\zeta(x,t) \equiv \eta(x, y_3, t) = \sum_j \hat{\zeta}_j(x) \mathrm{e}^{-\mathrm{i}\omega_j t} \qquad \text{for} \quad \tau_0(x) \le t < \tau_0(x) + \tau \tag{7}$$

from which we identify the mean power in the time domain

$$C(x) = \frac{1}{\tau} \int_{\tau_0(x)}^{\tau_0(x)+\tau} \eta^2(x, y_3, t)\, \mathrm{d}t = \sum_j |\hat{\zeta}_j(x)|^2 = \sum_j D(x, \omega_j) \tag{8}$$

where $D(x, \omega_j) = |\hat{\zeta}_j(x)|^2$.

We have computed three different kinds of peak frequencies of the power spectrum: The peak frequency of each transversal mode $\omega_{\mathrm{peak},n}(x)$ is the frequency ω_j that maximizes $J_n(x, \omega_j)$ for each $n \in \{0, 1, 2, 3, 4\}$. The peak frequency for the combined measurements across the tank $\omega_{\mathrm{peak}}(x)$ is the frequency ω_j that maximizes $F(x, \omega_j)$. The peak frequency at the centerline of the tank $\omega_{\mathrm{peak},c}(x)$ is the frequency ω_j that maximizes $D(x, \omega_j)$.

For all of the peak frequencies, we limit the search to a frequency range strictly within the first harmonic domain $\delta < \omega_{\mathrm{peak}}/\omega_c < 2 - \delta$ where δ is a small number.

Similarly, we have computed three different kinds of mean frequencies of the power spectrum: The mean frequency of each transversal mode is

$$\omega_{\mathrm{mean},n}(x) = \sum_j |\omega_j| J_n(x, \omega_j) / G_n(x)$$

The mean frequency for the combined measurements across the tank is

$$\omega_{\text{mean}}(x) = \sum_j |\omega_j| F(x, \omega_j)/I(x)$$

The mean frequency at the centerline of the tank is

$$\omega_{\text{mean},c}(x) = \sum_j |\omega_j| D(x, \omega_j)/C(x)$$

The evolution of these quantities along the tank is expected to be dominated by cubic nonlinear interactions, e.g. described by the nonlinear Schrödinger equation, therefore we have chosen to normalize the evolution x-axis by the square of the nominal steepness ϵ (Trulsen & Dysthe, 1997).

4 Results

Here we give detailed results for run 8019. Detailed results for all runs can be found in the Supplementary Material published online alongside this paper.

In order to help the interpretation of the following figures we offer Table 2 with a summary of the quantities that are being presented.

Figure 3 shows the frequency power spectrum $J_n(x, \omega)$ for each transversal mode number n at each station x, normalized such that the peak value is 1 at each station x. Two different views are shown, to the left spectral curves as a function of ω for each value of n, to the right intensity plots in the (ω, n)-plane. Notice that ω increases to the right in the left column while it increases downward in the right column. We observe significant downshift of the peak frequency, and a certain amount of energy transfer to transversal modes $n > 0$. The first transversal mode $n = 1$ has a significant downshifted peak, while all modes have significant energy leakage to higher frequencies. All figures are limited to the first harmonic frequency domain $0 < \omega < 2\omega_c$.

Figure 4 shows the mean power of each transversal mode $G_n(x)$, the total mean power $I(x)$, and the mean power along the centerline of the tank $C(x)$. If we limit consideration to only the centerline, or only the longcrested part of the wave $n = 0$, then we may overestimate the amount of dissipation. A significant portion of the energy is transferred to transversal modes $n > 0$.

Quantity	Explanation
$J_n(x, \omega_j)$	power of each transversal mode n and frequency ω_j
$F(x, \omega_j)$	transversally integrated power of each frequency ω_j
$G_n(x)$	temporally integrated power of each transversal mode n
$I(x)$	temporally and transversally integrated power
$D(x, \omega_j)$	power at centerline of each frequency ω_j
$C(x)$	temporally integrated power at centerline

Table 2: Summary of quantities presented.

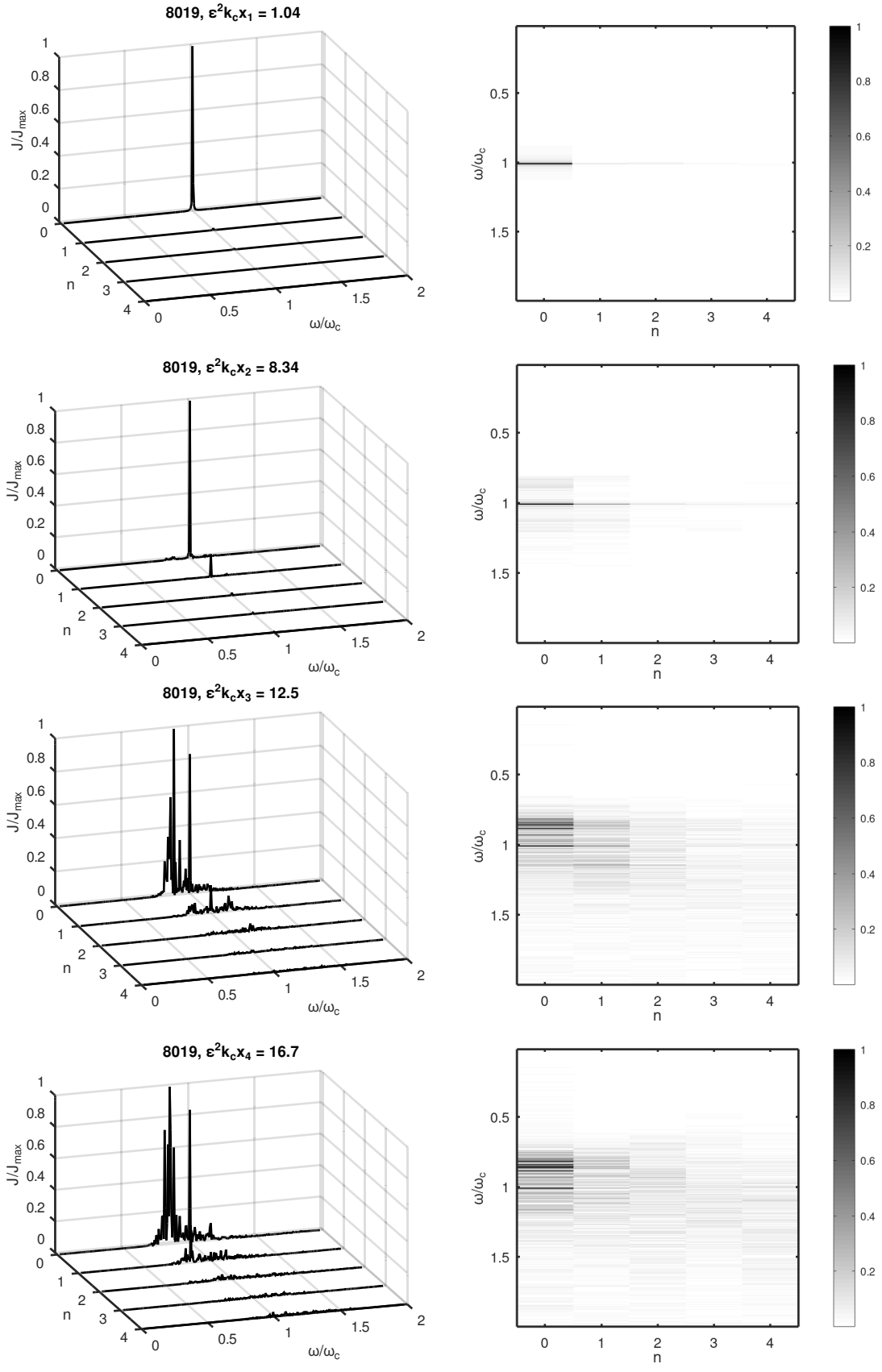


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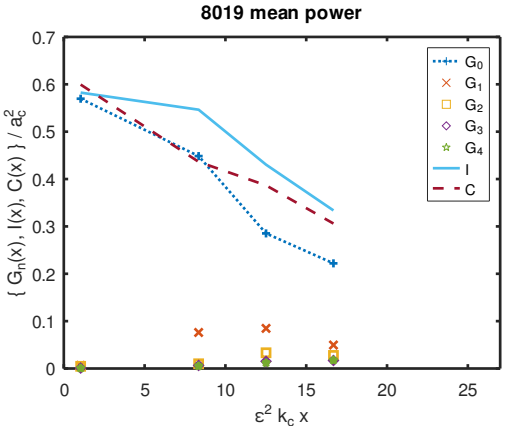


Figure 4: Mean power along the tank for each transversal mode $G_n(x)$, the total mean power $I(x)$, and the mean power along the centerline of the tank $C(x)$.

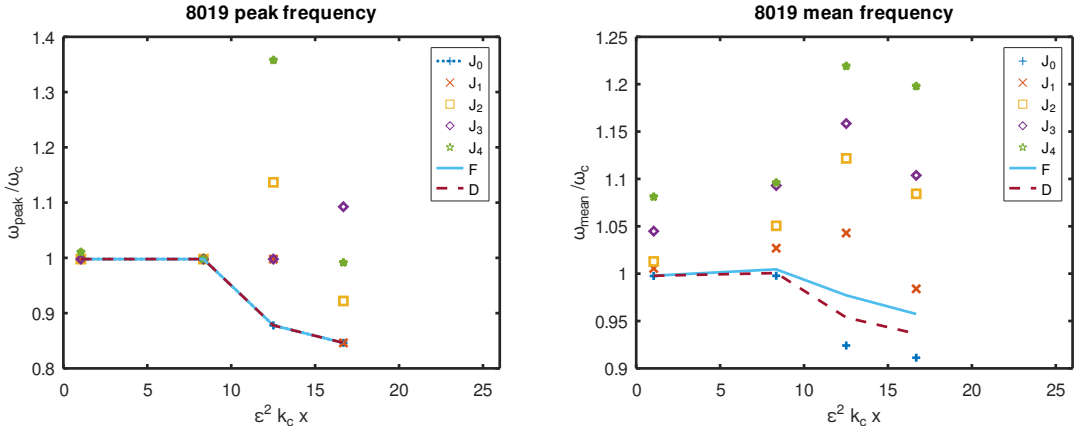


Figure 5: Peak and mean frequencies along the tank, $\omega_{\text{peak},n}(x)$ and $\omega_{\text{mean},n}(x)$ for each transversal mode $J_n(x,\omega)$, $\omega_{\text{peak}}(x)$ and $\omega_{\text{mean}}(x)$ for the combined measurements across the tank $F(x,\omega)$, $\omega_{\text{peak},c}(x)$ and $\omega_{\text{mean},c}(x)$ for the centerline of the tank $D(x,\omega)$.

Figure 5 shows the peak and mean frequencies. To the left we see the peak frequency $\omega_{\text{peak},n}(x)$ for each transversal mode $J_n(x,\omega)$, the peak frequency $\omega_{\text{peak}}(x)$ for the combined measurements across the tank $F(x,\omega)$, and the peak frequency $\omega_{\text{peak},c}(x)$ at the centerline of the tank $D(x,\omega)$. To the right we see the mean frequency $\omega_{\text{mean},n}(x)$ for each transversal mode $J_n(x,\omega)$, the mean frequency $\omega_{\text{mean}}(x)$ for the combined measurements across the tank $F(x,\omega)$, and the mean frequency $\omega_{\text{mean},c}(x)$ at the centerline of the tank $D(x,\omega)$. Several of the transversal modes $n > 0$ may actually experience significant upshift, but they do not carry significant amount of energy, therefor there is a clear tendency of downshift for the wave system as a whole and along the centerline.

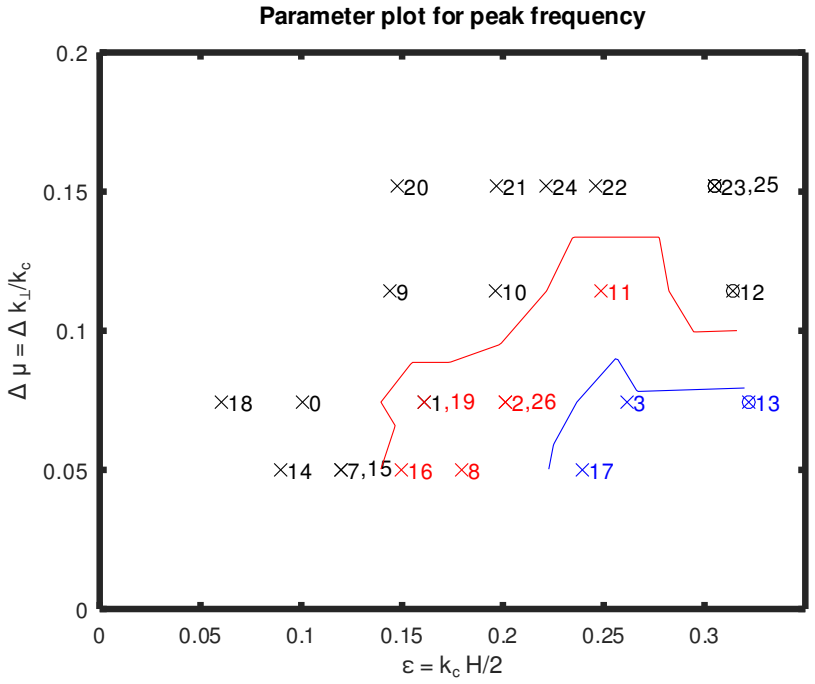


Figure 6: Parametric plot for peak frequency downshift. Runs without, $\times$, and with, $\otimes$, clear visual observation of type II horseshoe patterns. Run numbers reduced by 8000. Colors: black, no downshift $\Delta\omega_{\text{peak}}/\omega_c > 0.9$; red, small downshift $0.9 \geq \Delta\omega_{\text{peak}}/\omega_c > 0.76$; blue, clear downshift $\Delta\omega_{\text{peak}}/\omega_c \leq 0.76$.

5 Discussion

The results can be summarized in parametric $(\epsilon, \Delta\mu)$-plots in order to distinguish which parametric regimes that give downshift. Figure 6 shows results for the downshift of the peak frequency $\omega_{\text{peak}}(x)$, while figure 7 shows results for the downshift of the mean frequency $\omega_{\text{mean}}(x)$, in both cases for the combined measurements across the tank $F(x, \omega)$. Each run corresponds to a cross, $\times$, or a cross in a circle, $\otimes$.

Colors indicate the amount of downshift at the last probe x_4. For the duplicate runs with identical parameter values $(\epsilon, \Delta\mu)$, only the highest run numbers with the largest value of x_4 were employed in order to compute the contour levels. Runs with the largest amount of downshift are shown in blue. Runs with smaller amounts of downshift are shown in red. Runs without downshift or with little downshift are shown in black. The chosen contour levels are different for the peak frequency in figure 6 and for the mean frequency in figure 7, and are given in the figure captions, because the peak frequency downshift is larger than the mean frequency downshift.

Run numbers 2 and 26 both correspond to the single experiment reported in Trulsen *et al.* (1999). We confirm that this choice of $(\epsilon, \Delta\mu)$ indeed produces a downshift, as previously reported. The fact that run number 10, with equal steepness ϵ and greater $\Delta\mu$, does not produce a downshift suggests that transversal modulations are essential for this downshift to occur. Consulting with the detailed results for these three runs, in the Supplementary Material published online alongside this paper, we indeed see much less transfer of energy to transversal modes for run 10.

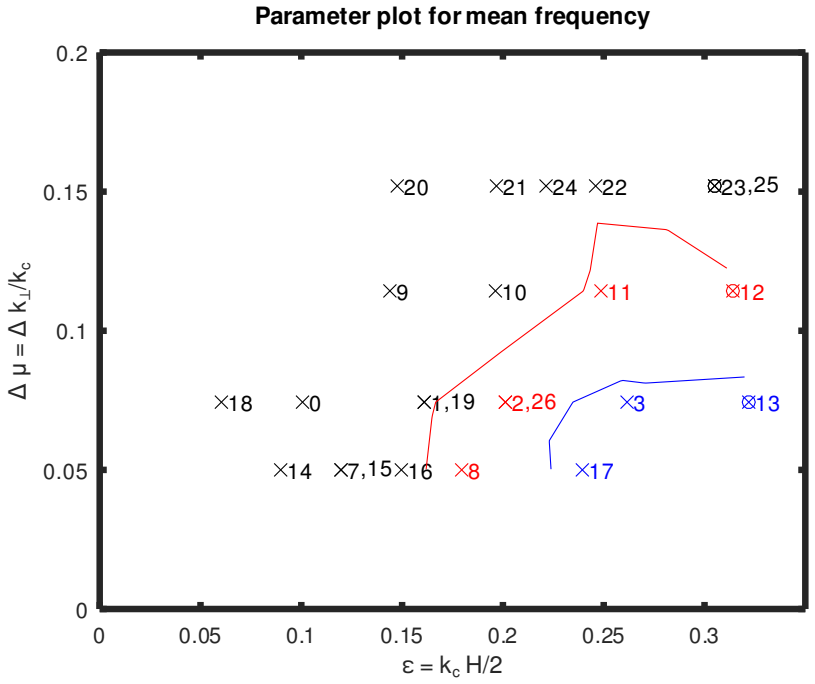


Figure 7: Parametric plot for mean frequency downshift. Runs without, $\times$, and with, $\otimes$, clear visual observation of type II horseshoe patterns. Run numbers reduced by 8000. Colors: black, no downshift $\Delta\omega_{\text{mean}}/\omega_c > 0.95$; red, small downshift $0.95 \geq \Delta\omega_{\text{mean}}/\omega_c > 0.89$; blue, clear downshift $\Delta\omega_{\text{mean}}/\omega_c \leq 0.89$.

Run 11 has downshift while run 10 does not. They have the same value of $\Delta\mu$, but un 11 has greater steepness ϵ than run 10. This is likely associated with the greater amount of energy loss in the steeper run 11, possibly giving support to previous findings that damping associated with heavy modulations can give downshift for long-crested waves (e.g. Trulsen & Dysthe, 1990)

Experimental runs with clear visual observation of type II horseshoe patterns are indicated by circles with crosses $\otimes$, this happens for the largest values of steepness $\epsilon > 0.3$. The results clearly show that the downshift does not depend on type II modulational instability or the horseshoe modulation pattern. It is interesting to note that Run 12 does not have a clear downshift of the peak frequency, while Run 11 does, suggesting that the downshift depends on type I rather than type II modulational instability.

The downshift clearly depends on the combination of the tank being sufficiently wide and the wave being sufficiently steep. The thresholds are approximately $\Delta\mu \approx 0.1$ and $\epsilon \approx 0.15$. Downshift can happen in wide wave tanks $\Delta\mu < 0.1$ for steepness $\epsilon > 0.15$. For narrower wave tanks $0.1 < \Delta\mu < 0.12$, downshift can happen for larger steepness $\epsilon \approx 0.25$, but it is not clear that an even greater steepness $\epsilon > 0.3$ will guarantee downshift for these values of $\Delta\mu$. For even narrower wave tanks $\Delta\mu > 0.12$ it is not clear that a downshift can happen regardless of the steepness.

Those runs with the largest steepness denoted by circle and cross, $\otimes$, correspond to clear observations of type II broadband horseshoe patterns along the wave crests. Not all of the cases with horseshoe patterns did show downshift, only those with

sufficiently small $\Delta\mu$ showed downshift. We therefore do not expect that type II modulations play any role in the frequency downshift.

The detailed results in the Supplementary Material published online alongside this paper clearly show that if measurements were limited to only the centerline along the tank, with only one probe across the tank, the conclusions would have been different from those discussed above. This could have resulted in an overestimation of both the energy dissipation and the frequency downshift. This is because energy and frequency content can be "hidden" in the transversal modes.

The most significant limitation of these experiments is the maximum normalized evolution distance $\epsilon^2 k_c x_4$, possibly concealing downshift at greater distances for some of the runs.

6 Conclusions

We have presented experimental evidence for the numerical prediction of Trulsen & Dysthe (1997), confirming and extending the results of Trulsen *et al.* (1999), that transversal narrow-banded modulations can play an essential role in the evolution and possible frequency downshift of Stokes waves. We observe clear downshift only in those cases that allow significant interactions with unstable narrow-banded transversal modes, corresponding to transversal type I instability. We do not necessarily observe downshift with modulations of unstable wide-banded transversal modes, corresponding to type II instability. These experiments have significant damping, regardless of displaying downshift or not, therefore damping by itself cannot be the origin of the downshift. However, we find evidence that damping associated with heavy modulations can play a role for the downshift.

A criterion for downshift appears to be that the wave tank should accomodate crest lengths of at least 10 wavelengths.

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Supplementary Material published online alongside the paper

In section A of the Supplementary Material we give a detailed account for all runs, similar to the detailed account for run 8019 given in section 4 of the paper.

In section B of the Supplementary Material we give photos taken during some of the runs.

First we reproduce table 1 of the paper for a summary of all the runs.

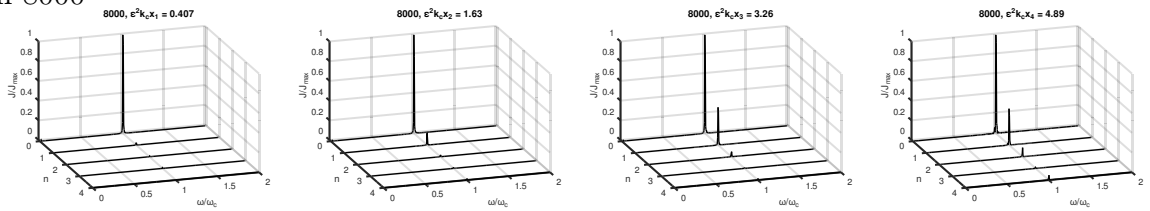
Run	T [s]	H [m]	$k_c h$	ϵ	$\Delta\mu$	$\epsilon^2 k_c x_4$	$\frac{\omega_{\text{peak}}(x_4)}{\omega_c}$	$\frac{\omega_{\text{peak},c}(x_4)}{\omega_c}$	$\frac{\omega_{\text{mean}}(x_4)}{\omega_c}$	$\frac{\omega_{\text{mean},c}(x_4)}{\omega_c}$
8000	1	0.05	20	0.1	0.074	4.9	1	1	1	1
8001	1	0.08	20	0.16	0.074	13	1	0.85	0.99	0.97
8002	1	0.1	20	0.2	0.074	20	0.81	0.81	0.92	0.91
8003	1	0.13	20	0.26	0.074	33	0.73	0.75	0.88	0.87
8004	2.5	0.1	3.2	0.032	0.46	0.08	0.98	0.98	0.98	0.98
8005	2	0.1	5	0.05	0.3	0.31	0.98	0.98	0.98	0.98
8006	3	0.1	2.2	0.022	0.67	0.027	1	1	1	1
8007	0.82	0.04	30	0.12	0.05	10	1	1	1	0.99
8008	0.82	0.06	30	0.18	0.05	23	0.82	0.83	0.92	0.92
8009	1.24	0.11	13	0.14	0.11	6.5	1	1	1	1
8010	1.24	0.15	13	0.2	0.11	12	1	1	1	1
8011	1.24	0.19	13	0.25	0.11	19	0.8	0.8	0.94	0.92
8012	1.24	0.24	13	0.31	0.11	31	1	1	0.95	0.91
8013	1	0.16	20	0.32	0.074	50	0.73	0.73	0.87	0.86
8014	0.82	0.03	30	0.09	0.05	5.8	1	1	1	1
8015	0.82	0.04	30	0.12	0.05	10	1	0.89	1	1
8016	0.82	0.05	30	0.15	0.05	16	0.85	0.85	0.97	0.95
8017	0.82	0.08	30	0.24	0.05	41	0.74	0.74	0.88	0.87
8018	1	0.03	20	0.06	0.074	2.3	1	1	1	1
8019	1	0.08	20	0.16	0.074	17	0.85	0.85	0.96	0.94
8020	1.43	0.15	9.8	0.15	0.15	6.9	1	1	1	1
8021	1.43	0.2	9.8	0.2	0.15	12	1	1	1	1
8022	1.43	0.25	9.8	0.25	0.15	19	1	1	0.96	0.94
8023	1.43	0.31	9.8	0.31	0.15	29	1	1	0.96	0.95
8024	1.43	0.225	9.8	0.22	0.15	15	1	1	1	0.99
8025	1.43	0.31	9.8	0.31	0.15	29	1	1	0.96	0.96
8026	1	0.1	20	0.2	0.074	26	0.8	0.8	0.9	0.9

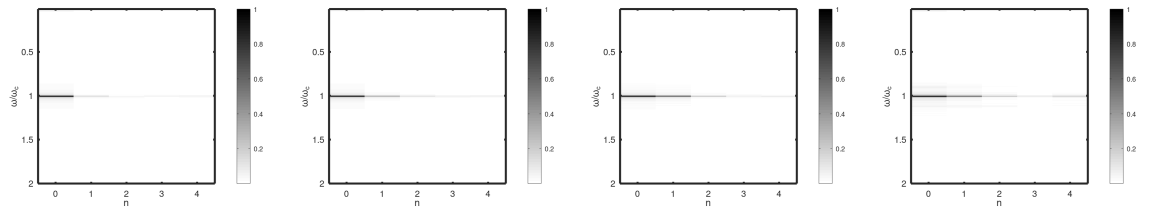
We also reproduce Table 2 of the paper for a summary of all the quantities that are being presented in the following figures.

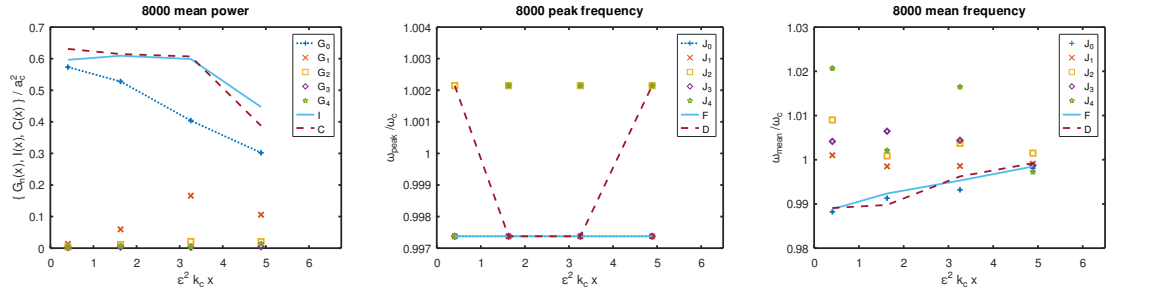
Quantity	Explanation
$J_n(x, \omega_j)$	power of each transversal mode n and frequency ω_j
$F(x, \omega_j)$	transversally integrated power of each frequency ω_j
$G_n(x)$	temporally integrated power of each transversal mode n
$I(x)$	temporally and transversally integrated power
$D(x, \omega_j)$	power at centerline of each frequency ω_j
$C(x)$	temporally integrated power at centerline

A Detailed analysis for each run

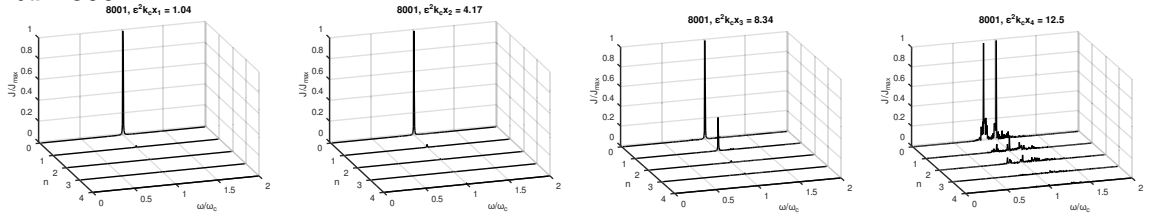
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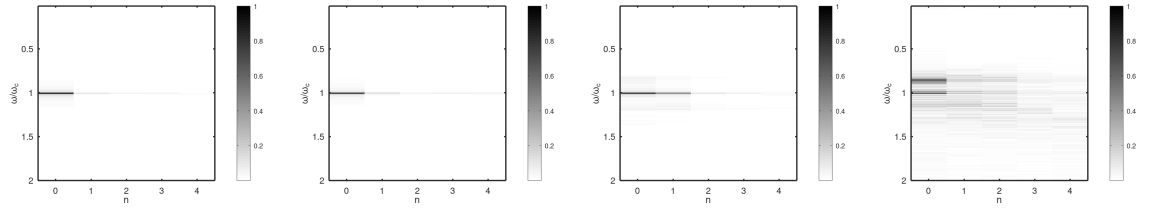


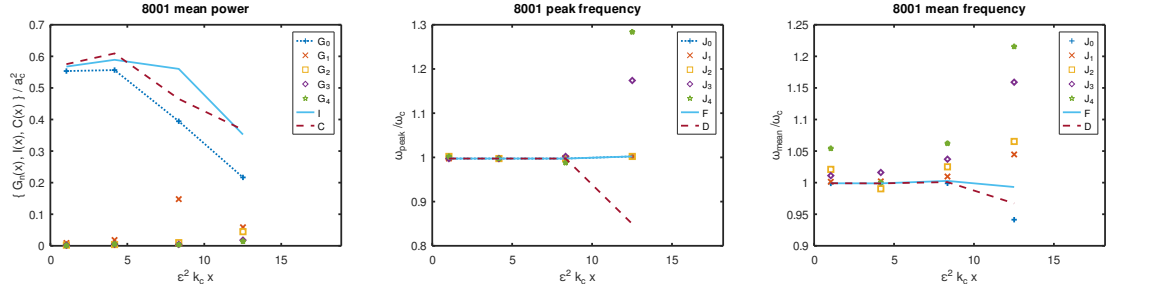




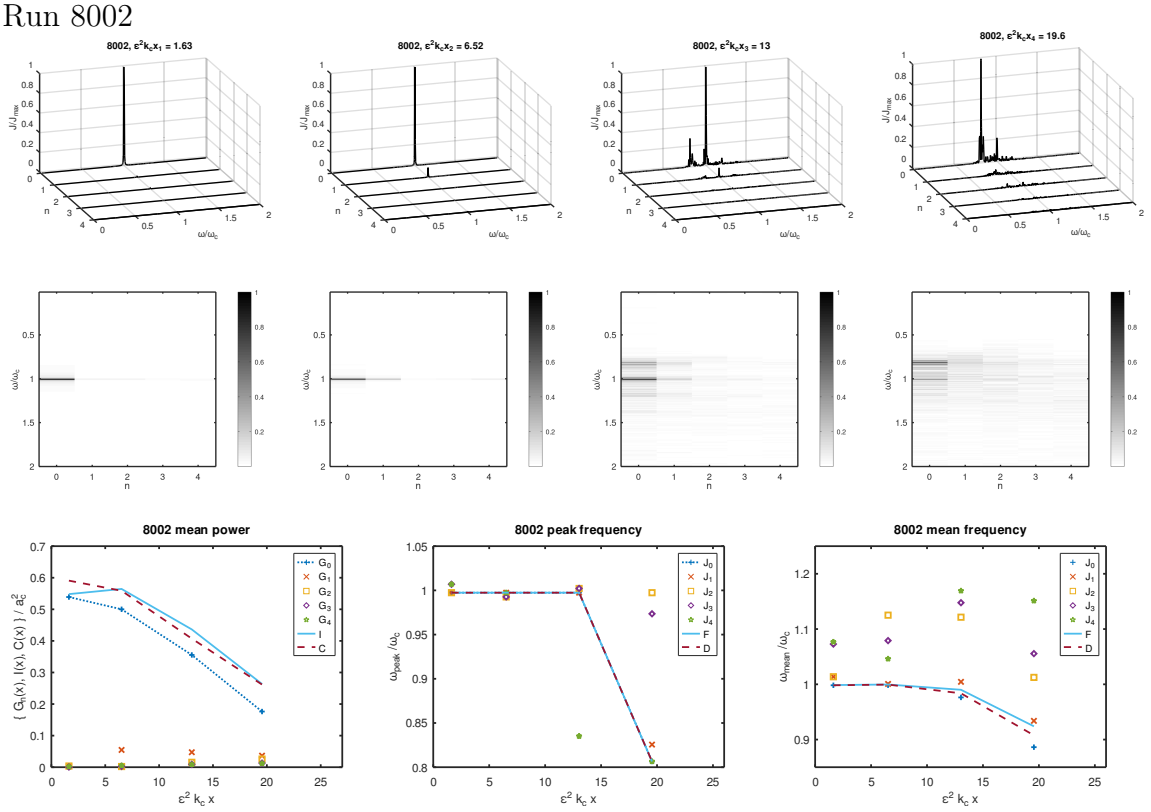
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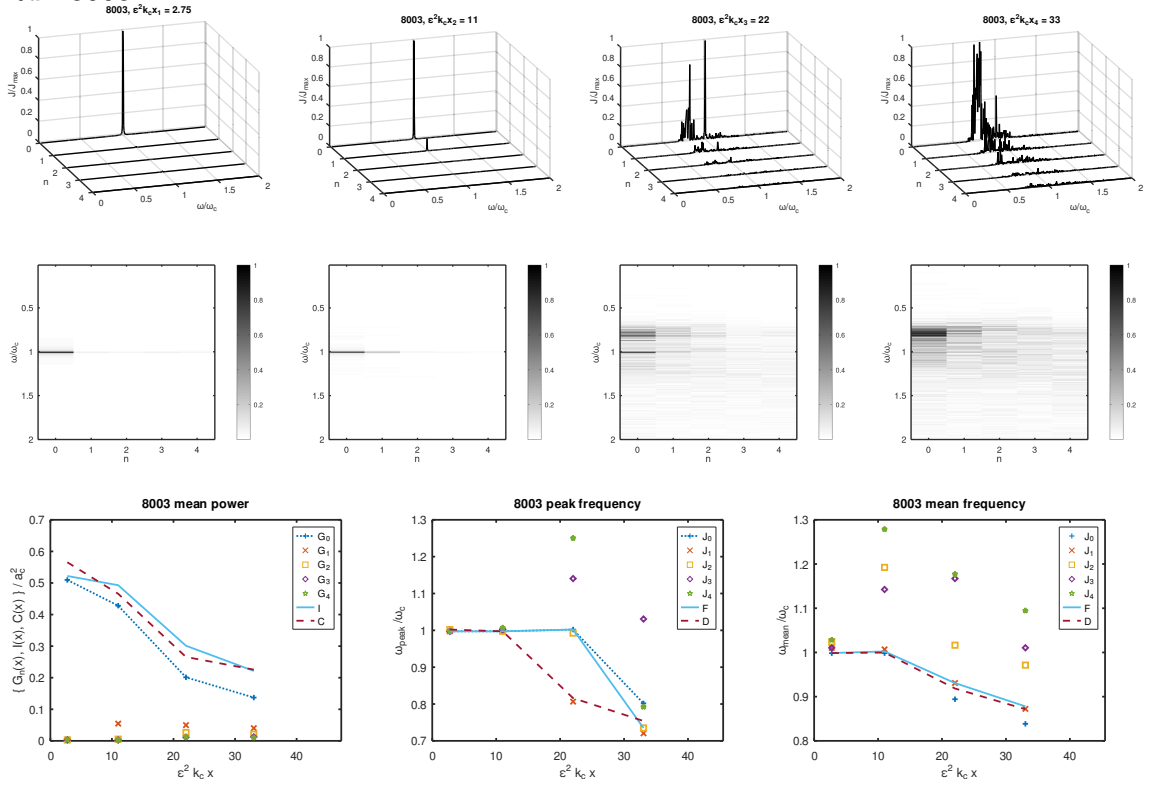




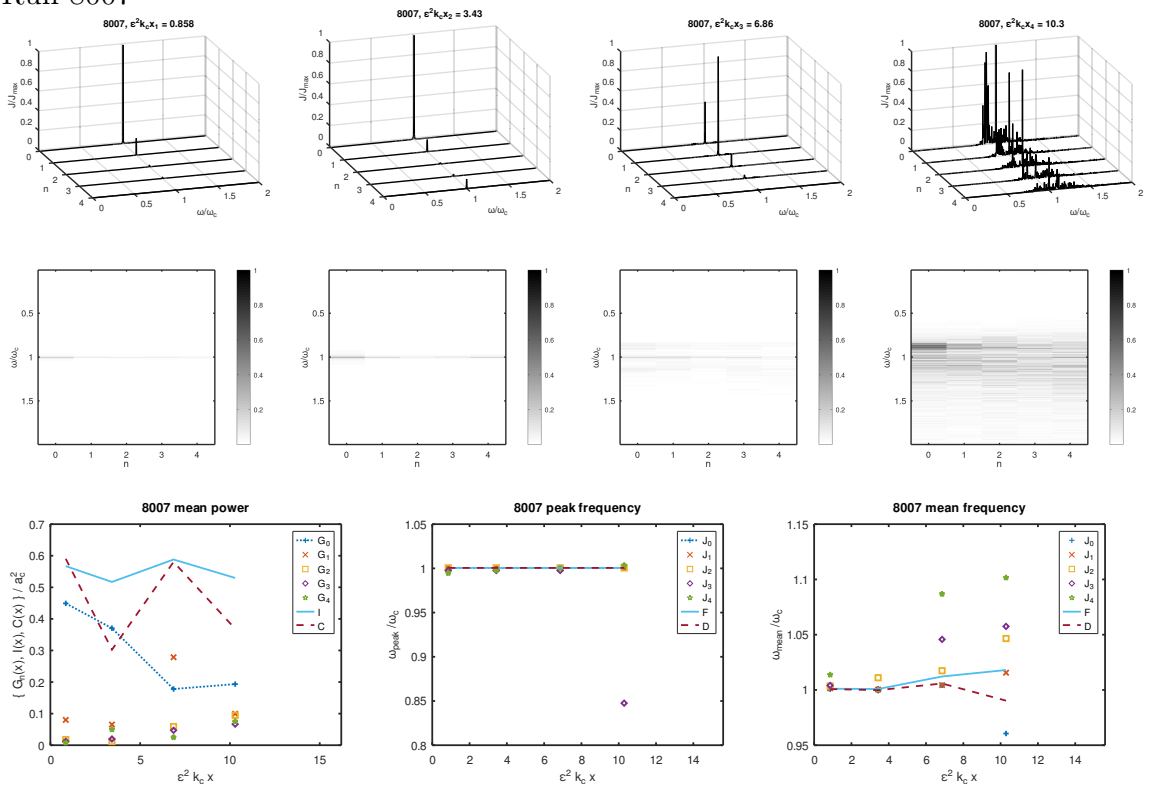
Run 8002



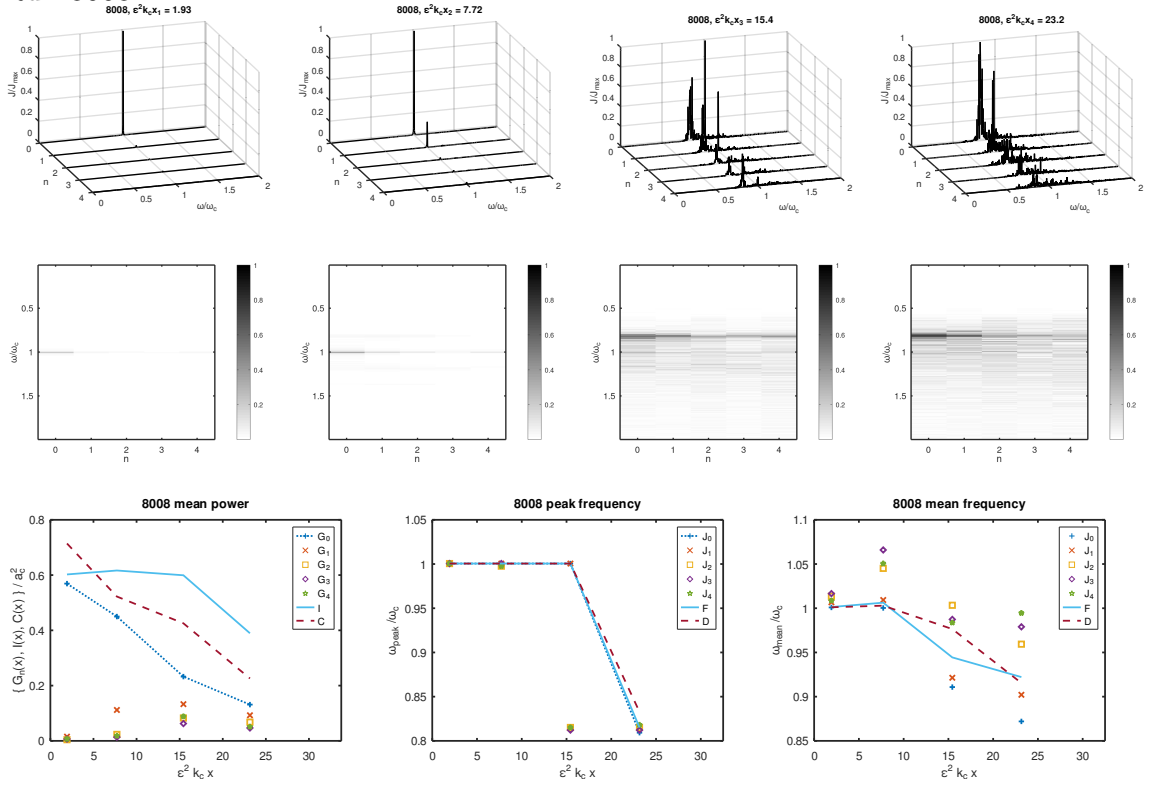
Run 8003



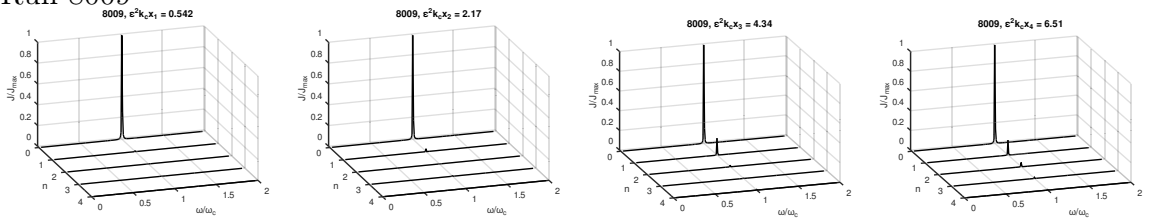
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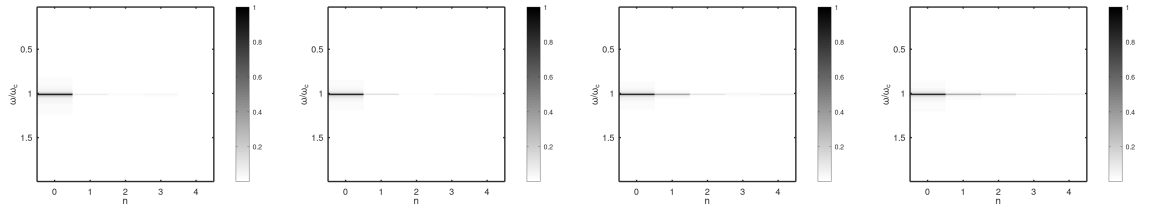


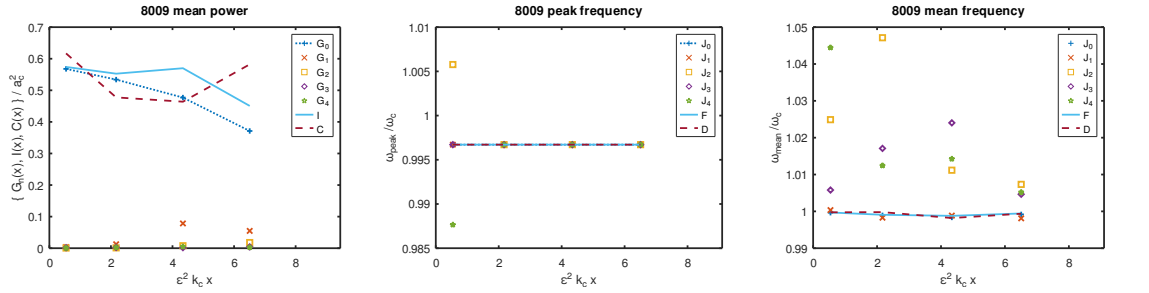
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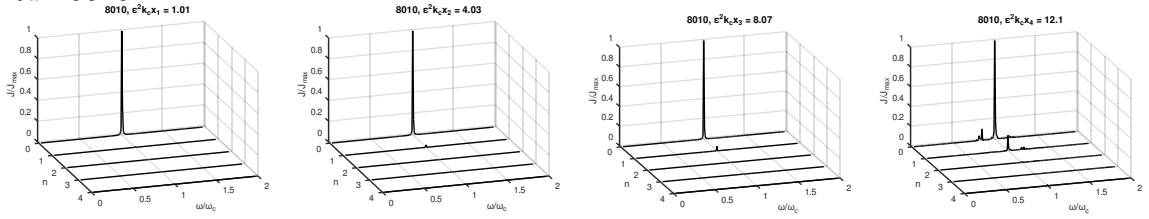
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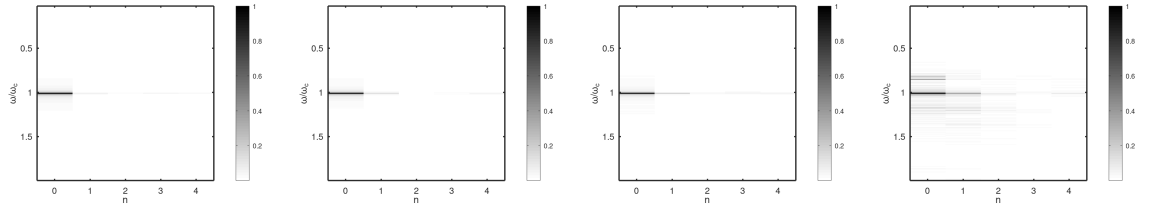


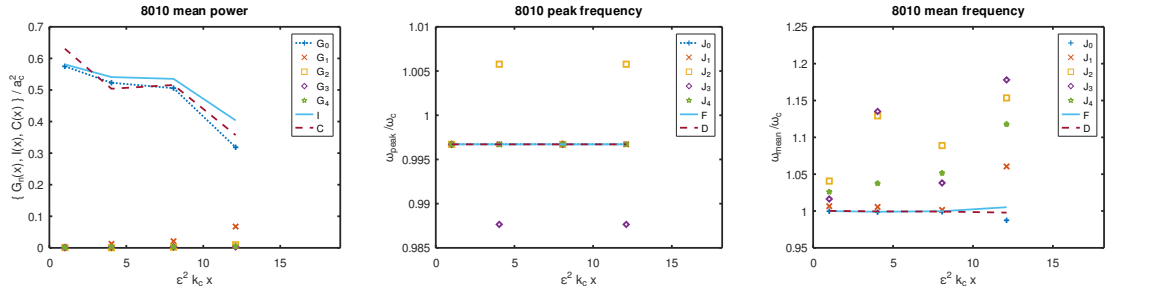




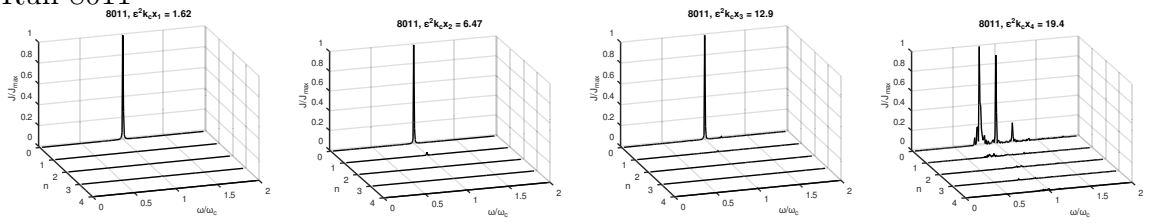
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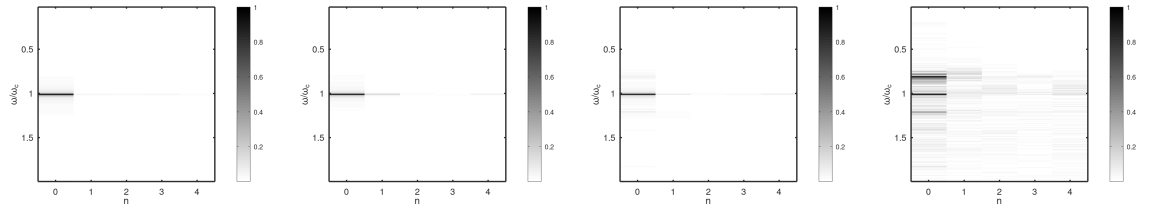


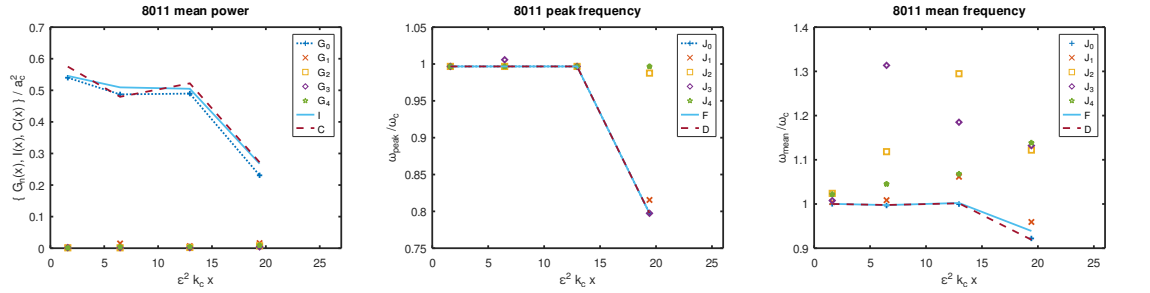




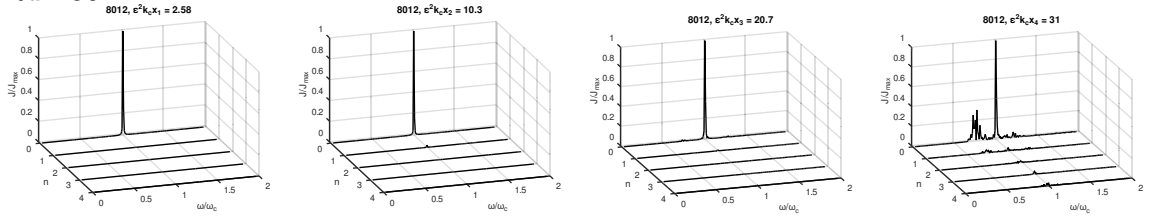
Run 8011

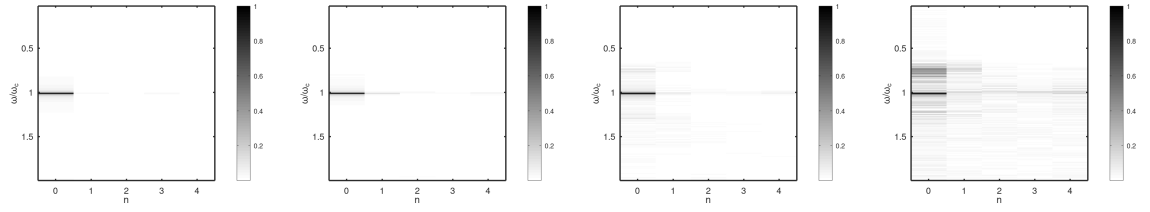


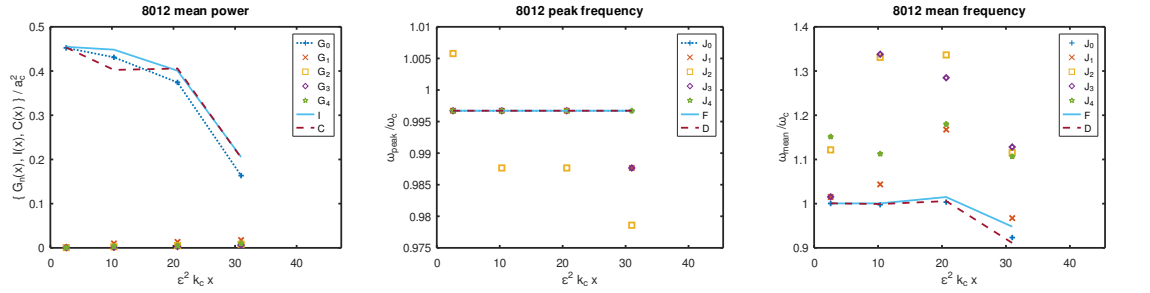




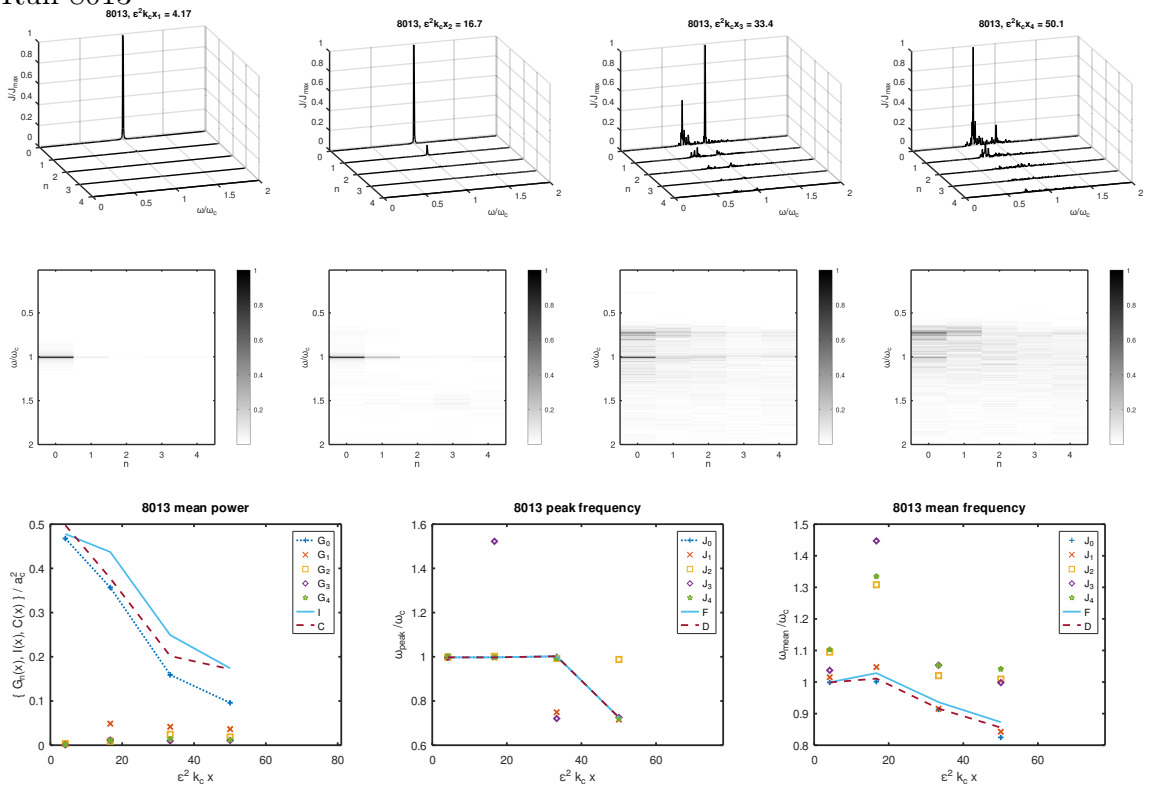
Run 8012



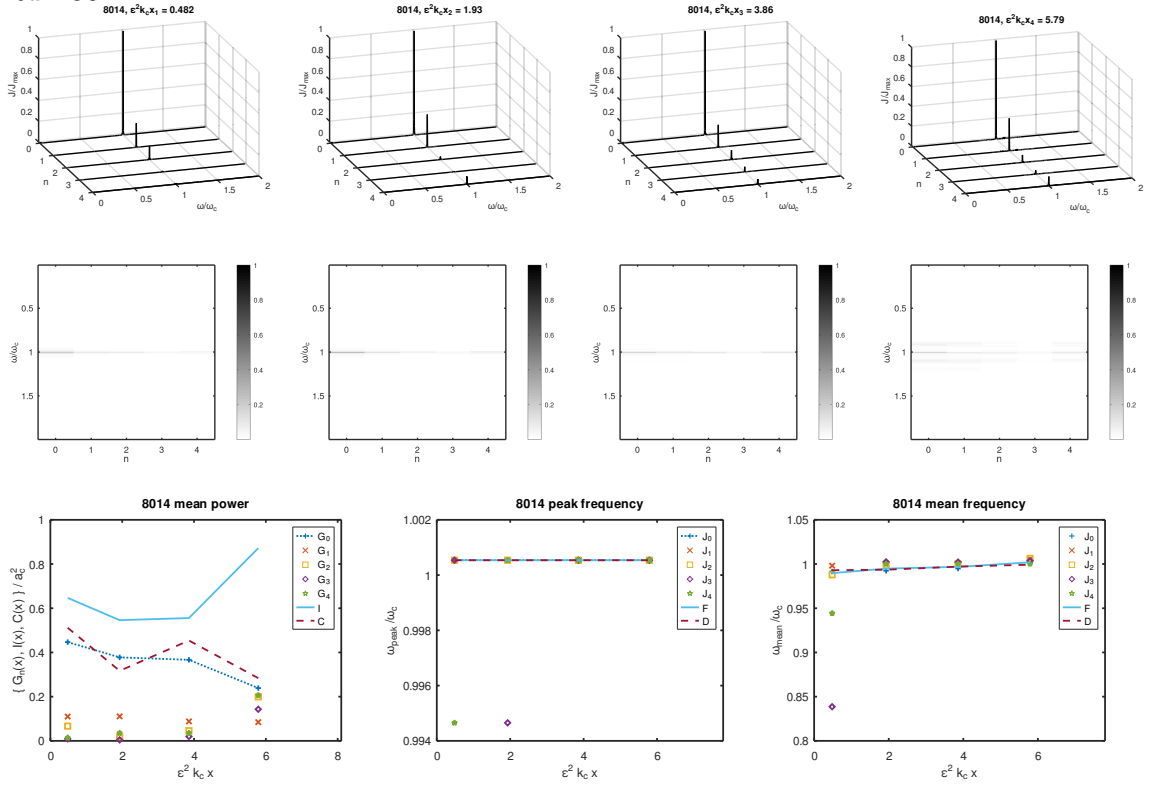




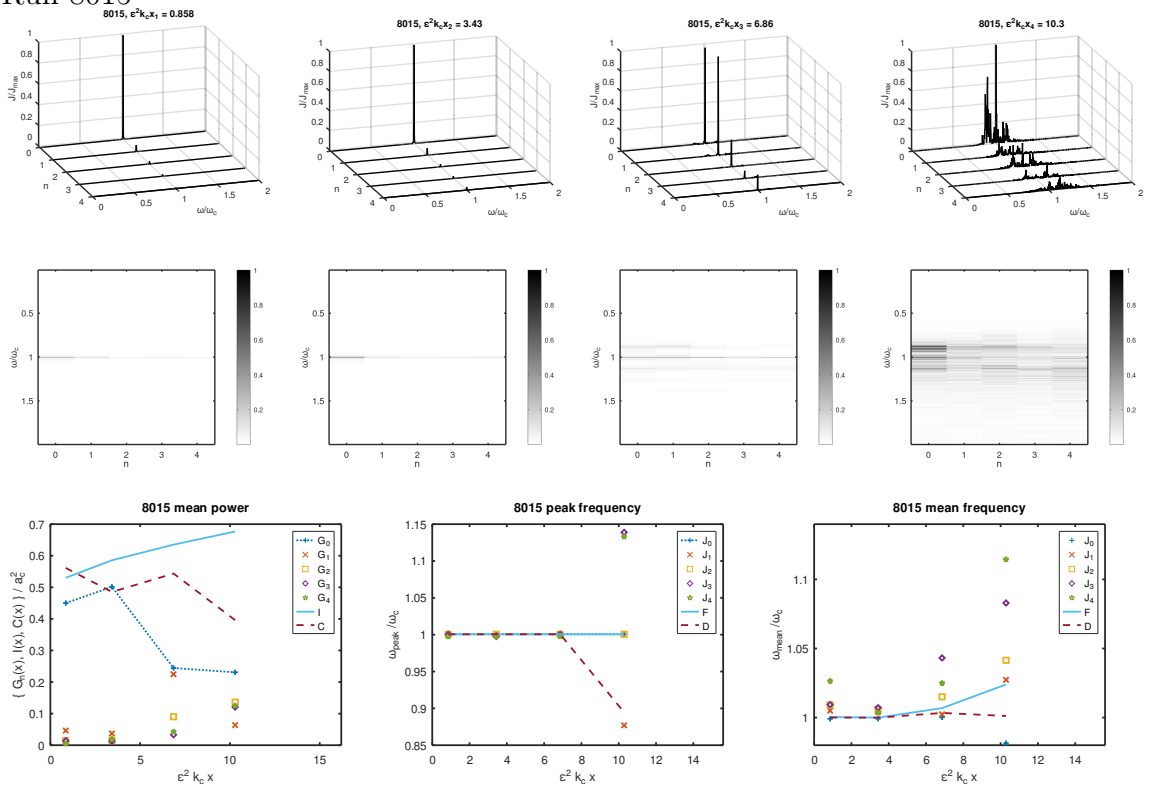
Run 8013



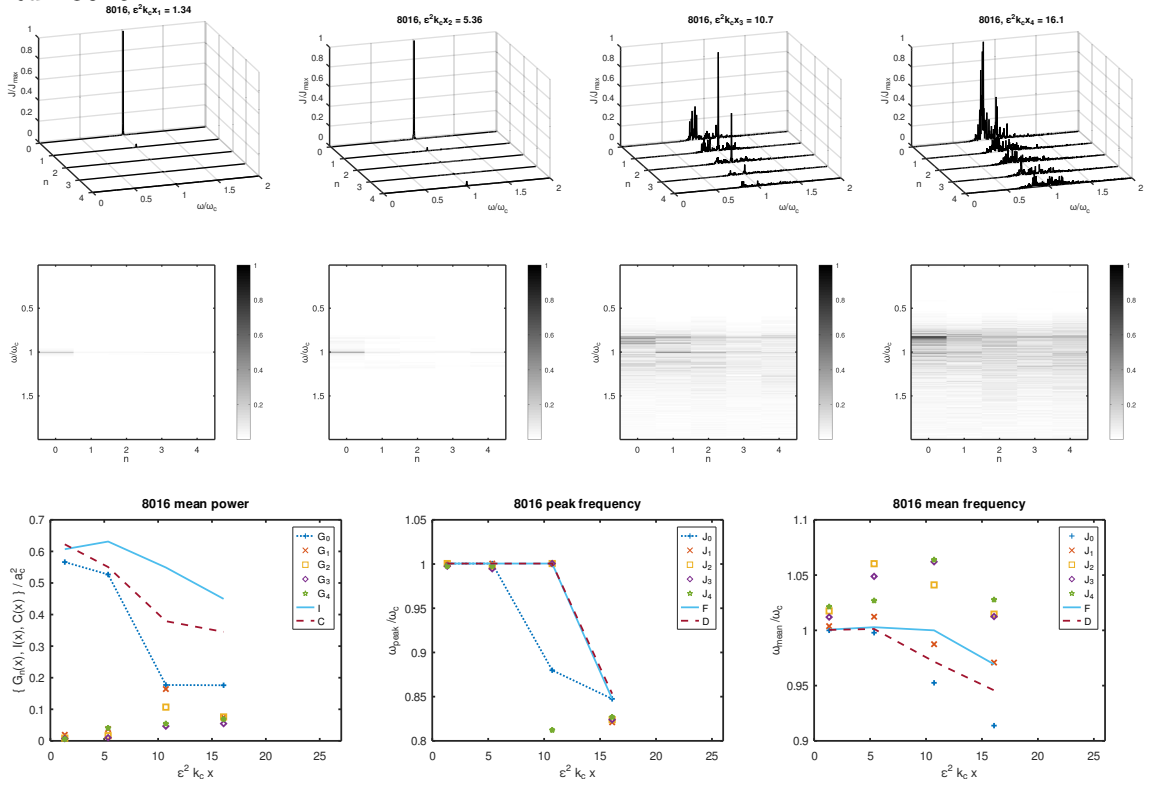
Run 8014



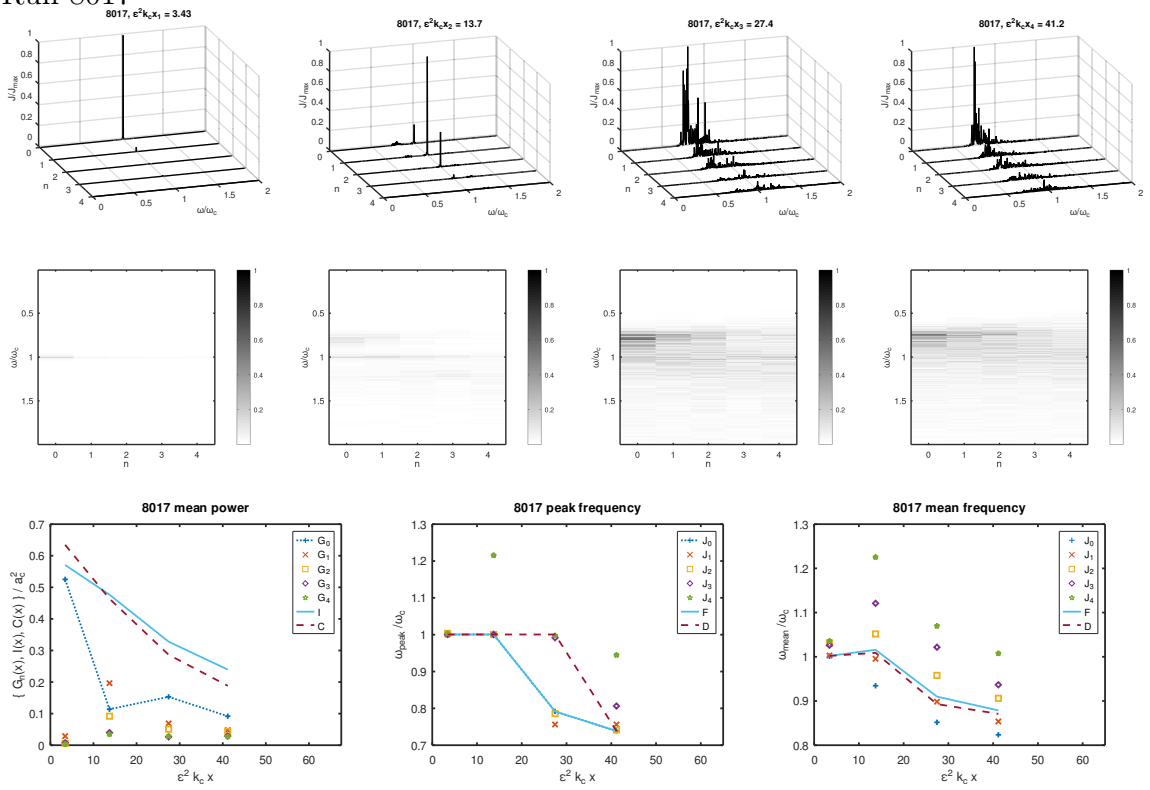
Run 8015



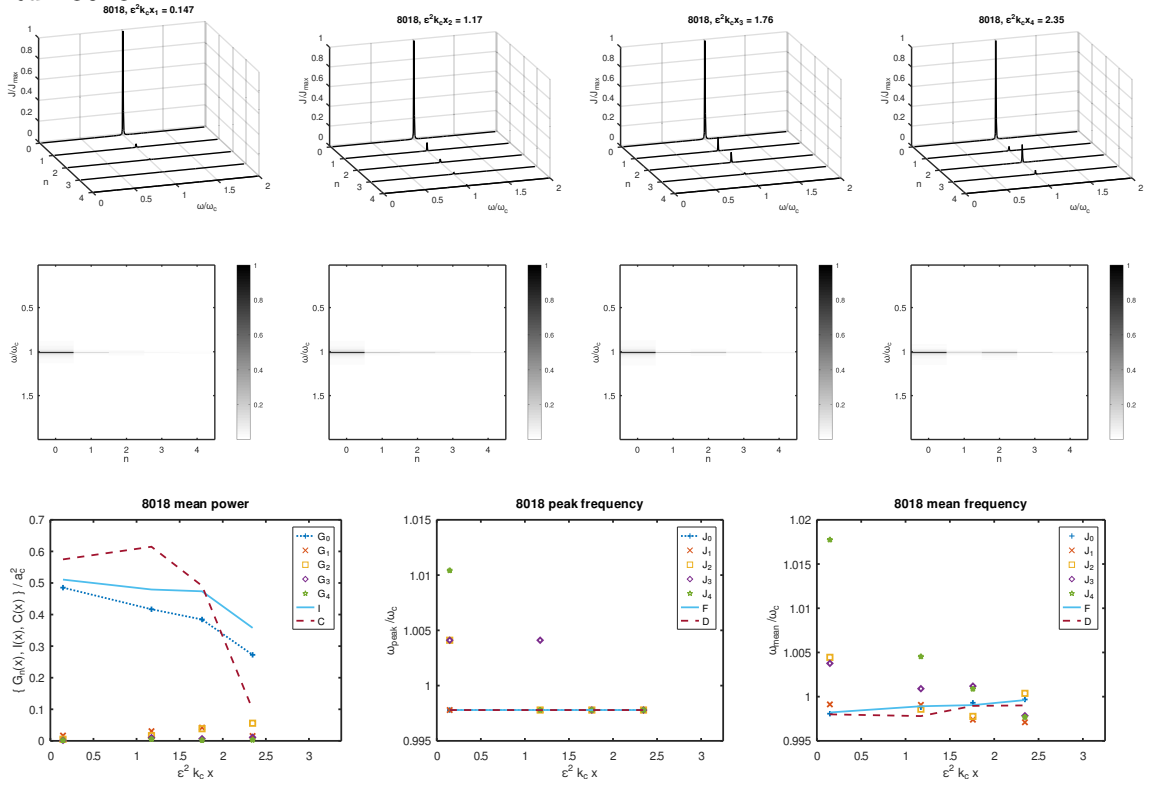
Run 8016



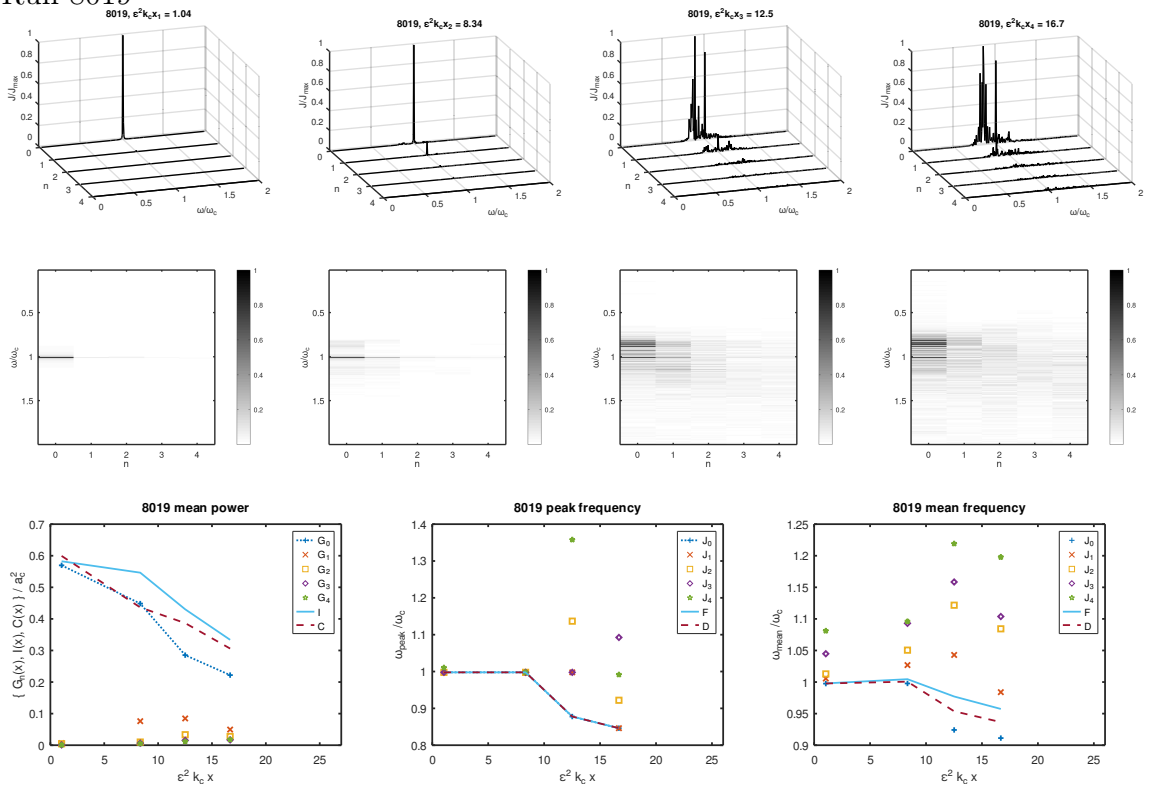
Run 8017



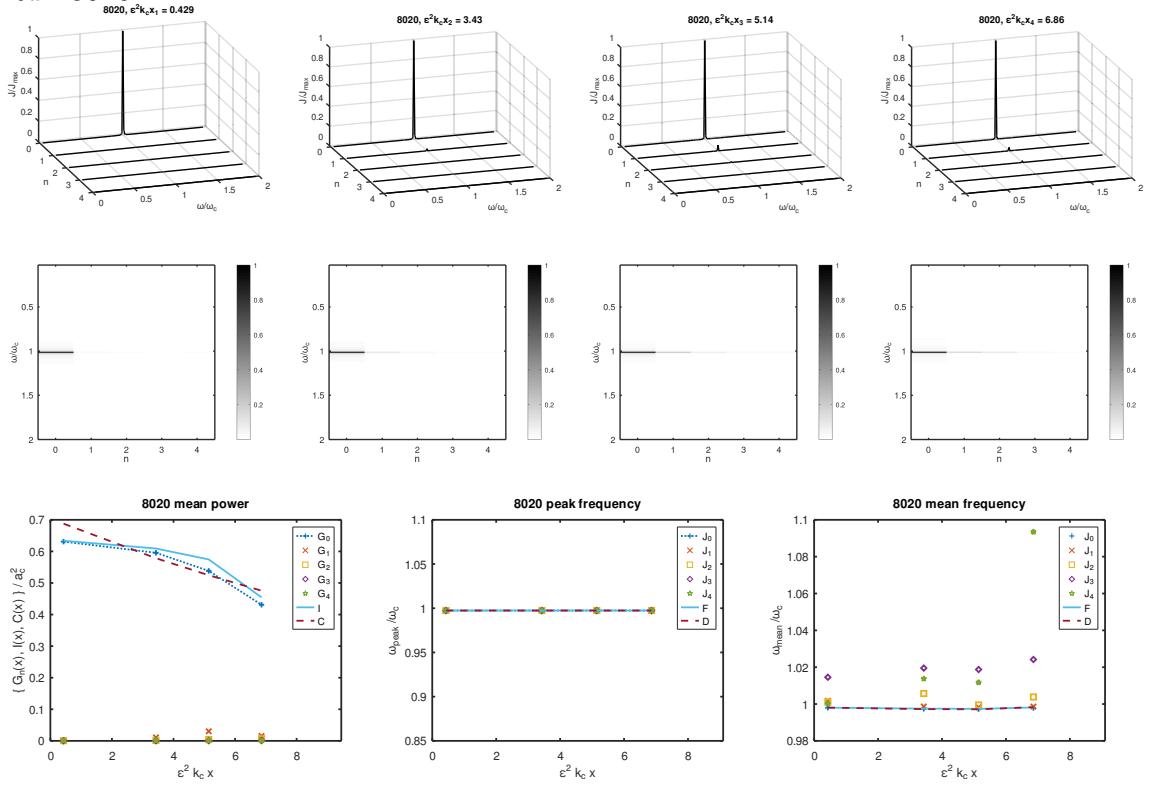
Run 8018



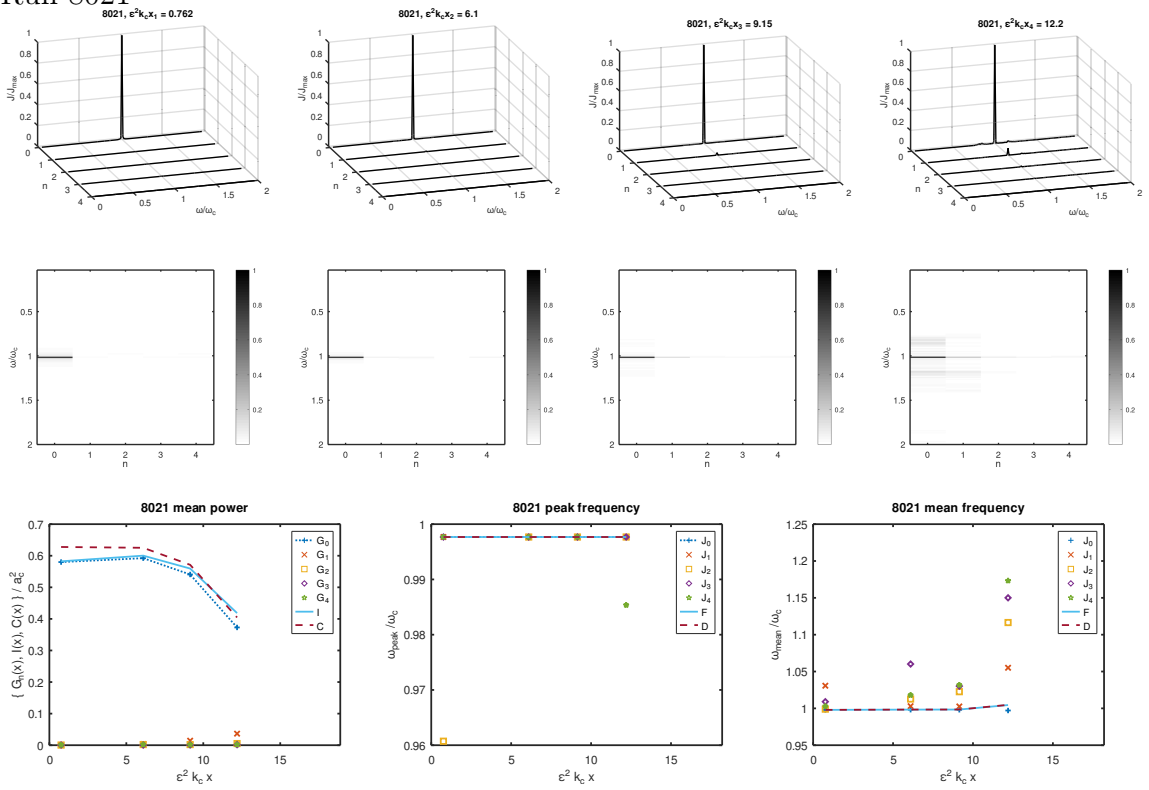
Run 8019



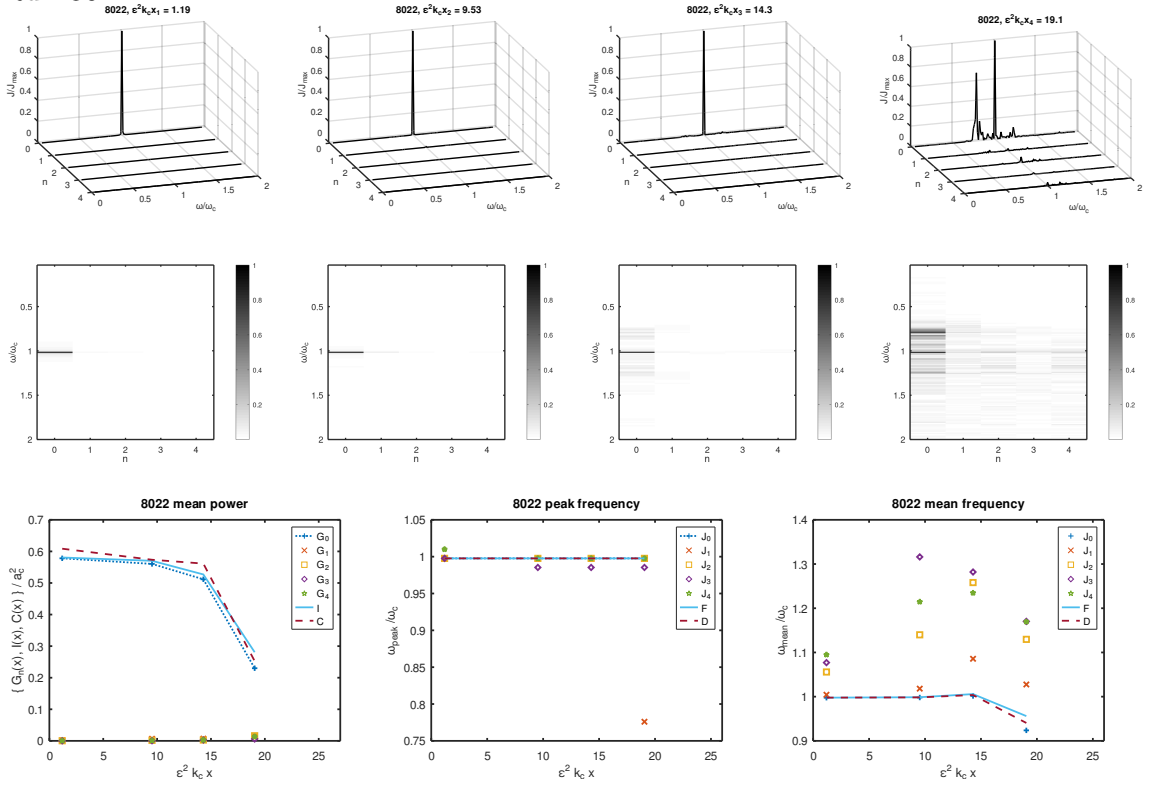
Run 8020



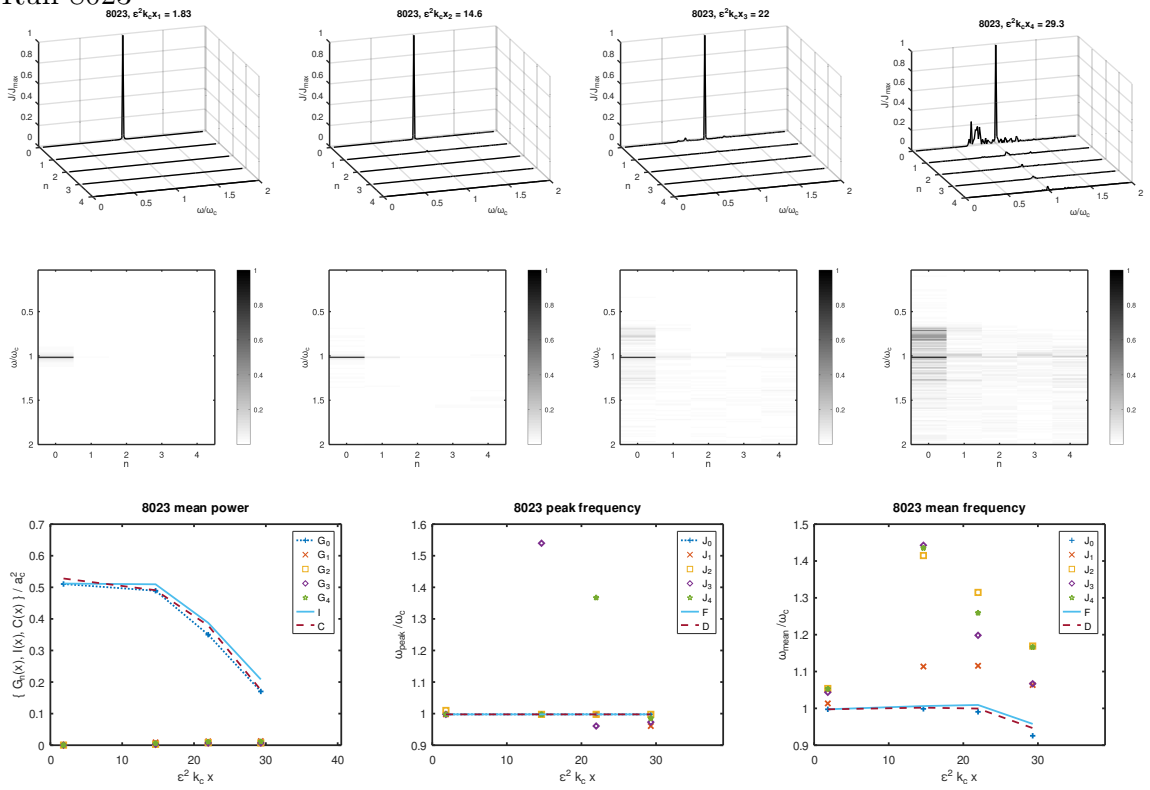
Run 8021



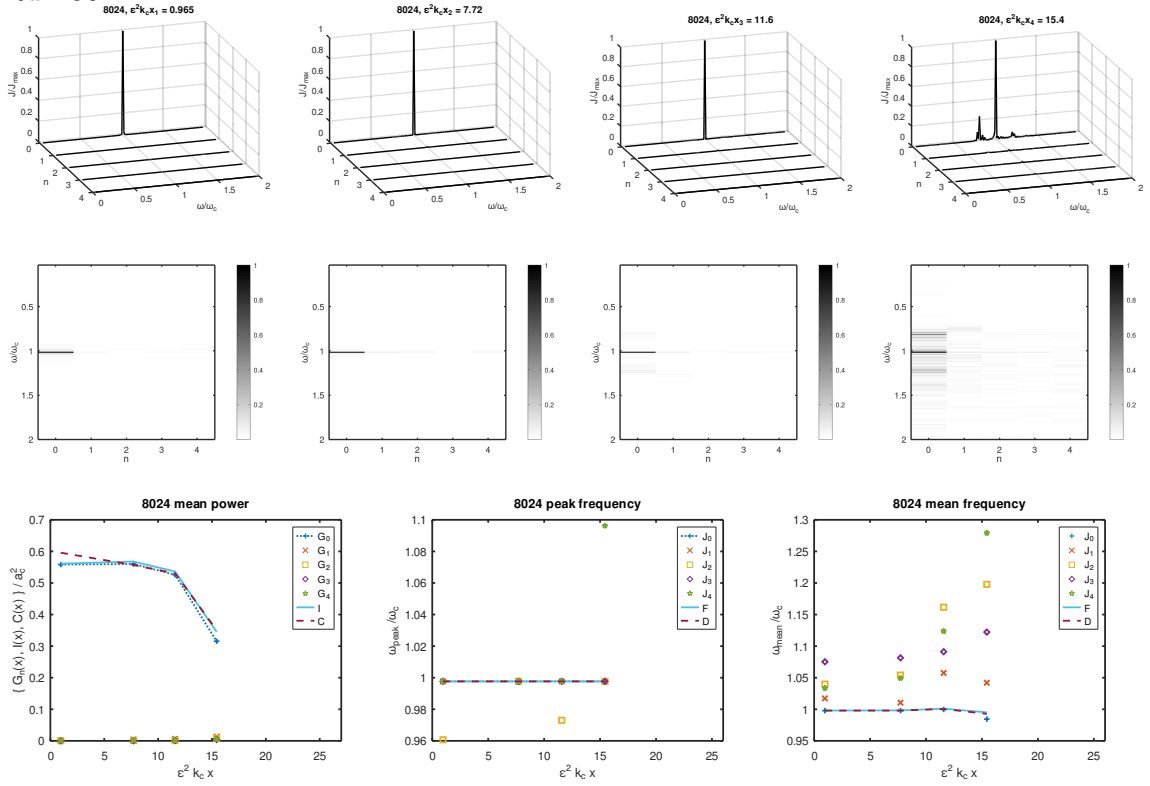
Run 8022



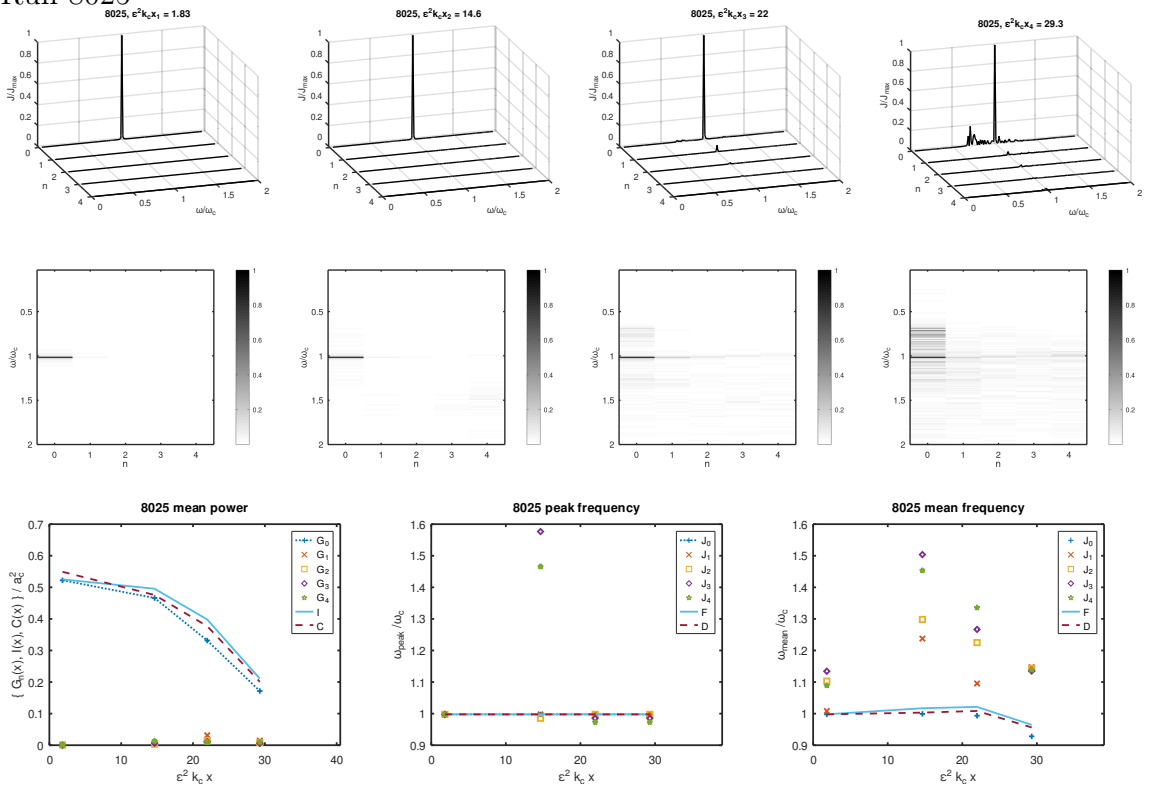
Run 8023



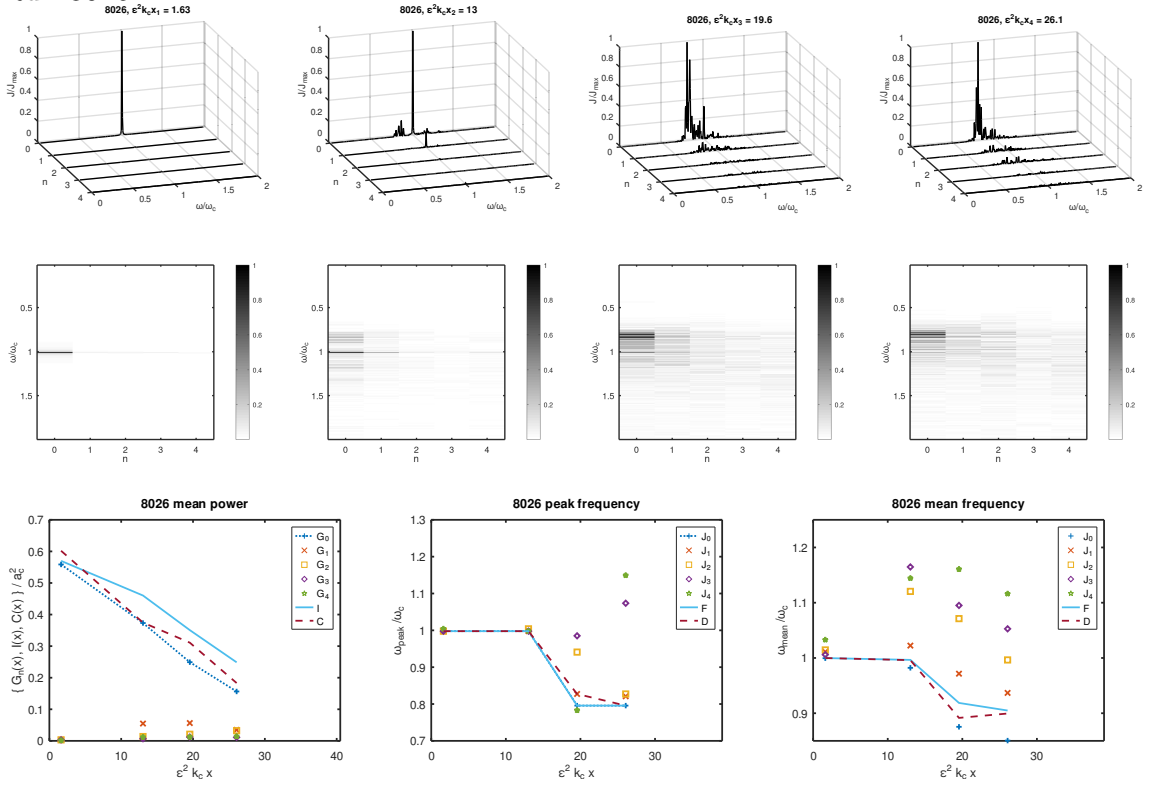
Run 8024



Run 8025



Run 8026



B Pictures from various runs

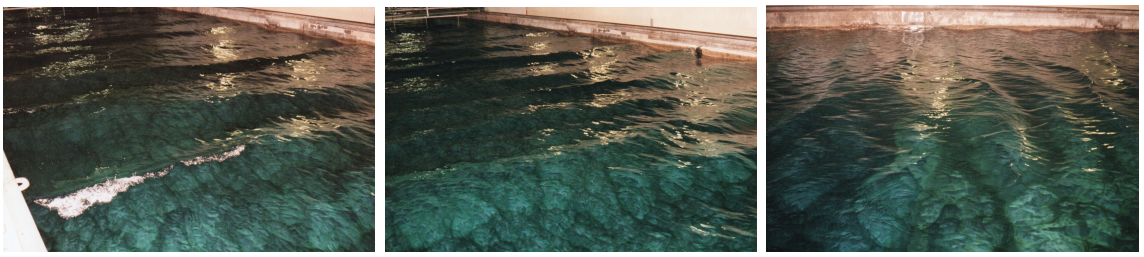


Figure 8: Pictures from run 8008.

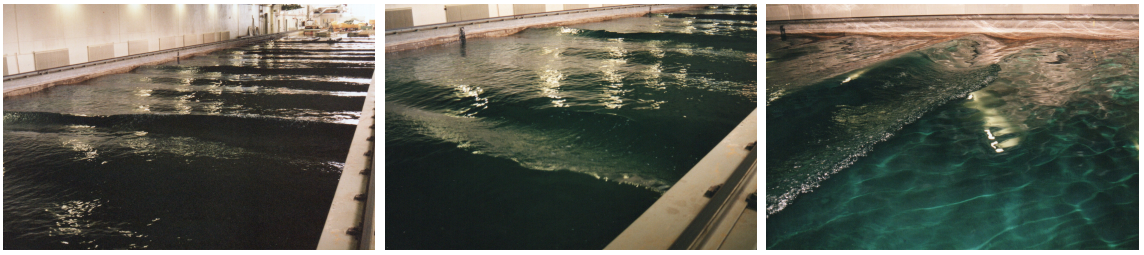


Figure 9: Pictures from run 8009.



Figure 10: Pictures from run 8010.



Figure 11: Pictures from run 8011.

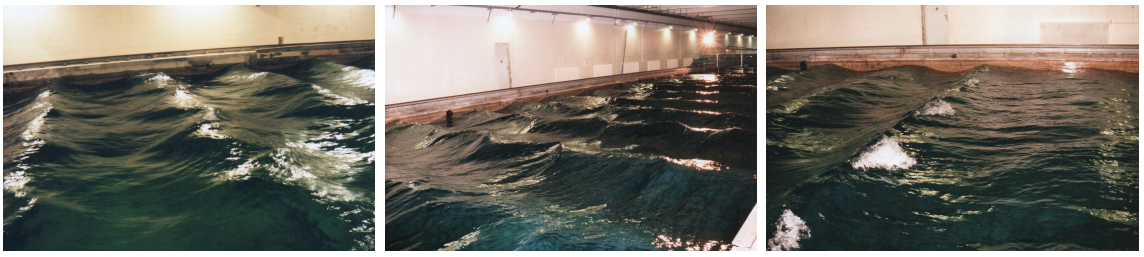


Figure 12: Pictures from run 8012.

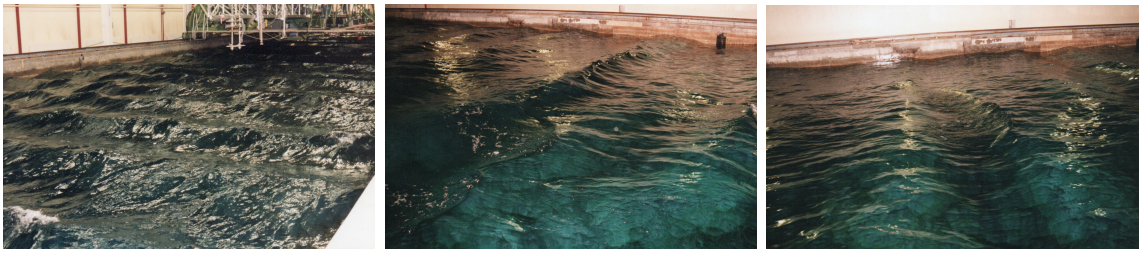


Figure 13: Pictures from run 8013.



Figure 14: Pictures from run 8014.

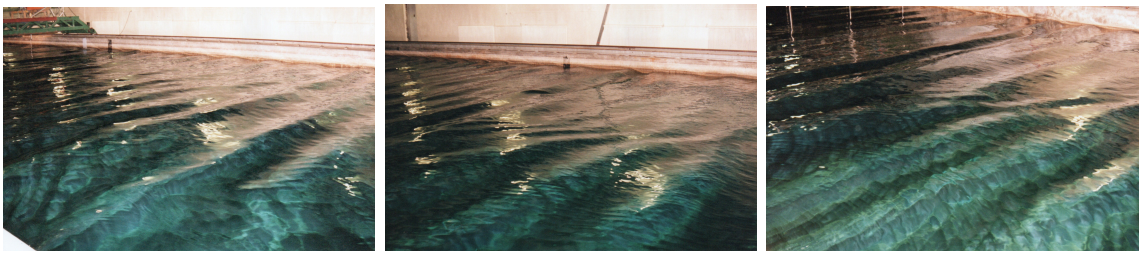


Figure 15: Pictures from run 8015.

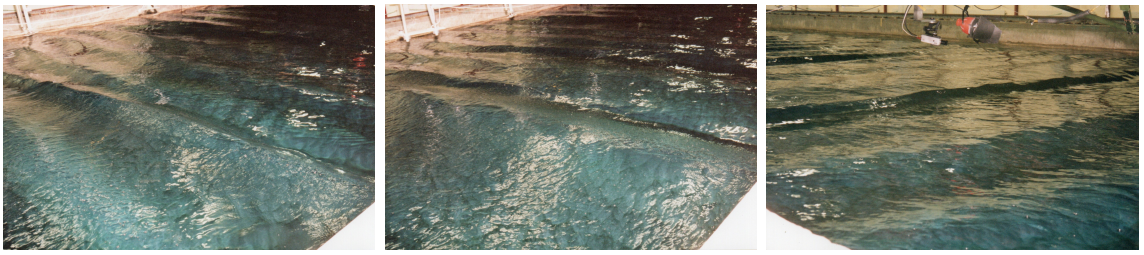


Figure 16: Pictures from run 8016.

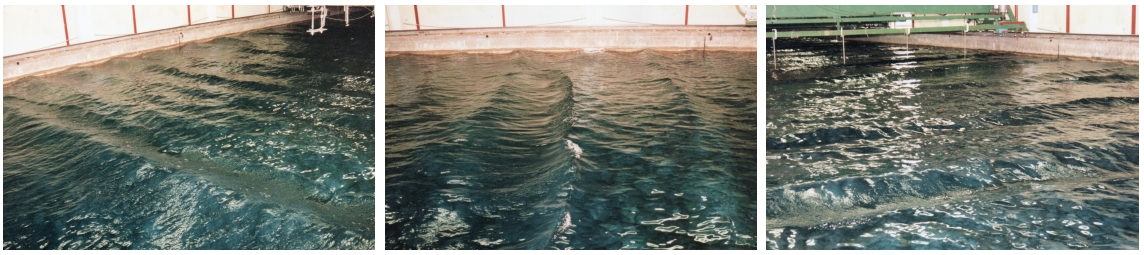


Figure 17: Pictures from run 8017.

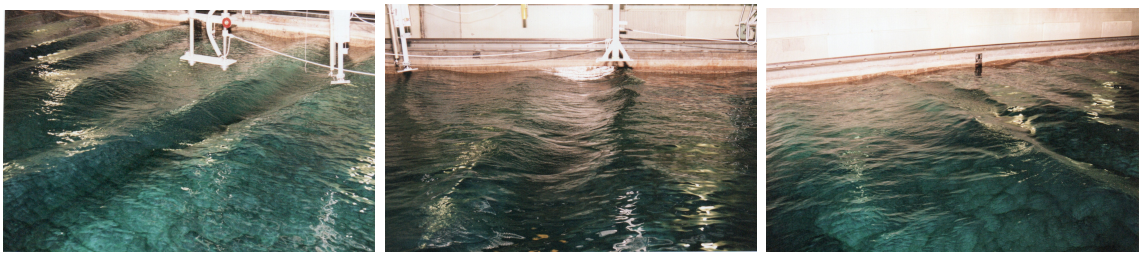


Figure 18: Pictures from run 8019.



Figure 19: Pictures from run 8025.