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THE EVOLUTION OF AN EVOLUTION EQUATION

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Introduction

Surface gravity waves are nonlinear and dispersive. Consider a wave field that is narrow-banded with a central wave vector $\mathbf{k}$ and a typical bandwidth Δk . Let a be a characteristic amplitude and assume that the wave field is only weakly nonlinear. We then have two small dimensionless parameters: the relative bandwidth $\mu = \Delta k / |\mathbf{k}|$ and the wave steepness $\epsilon = a |\mathbf{k}|$ which are measures of the importance of dispersion and nonlinearity respectively.

Studying such a system, it is well known that perturbation expansions in small parameters like μ and ϵ may produce model equations that are more tractable than the exact equations. They may even be better suited for numerical simulation in the sense that they may permit an expanded computational domain.

For the above mentioned system a perturbation development of the basic equations using μ and ϵ produces a hierarchy of model equations depending on the relative ordering between the two parameters. It is pointed out that at each step in this hierarchy some new physics is taken into account. Numerically we compare the predictions of some of the model equations and some experimental data.

Basic equations

We start by assuming that our fluid is ideal, homogeneous, incompressible and irrotational. Consequently the continuity equation is simply

$$\nabla^2 \varphi = 0 \tag{1}$$

where φ is the velocity potential. We use the oceanographic convention: the z -axis pointing vertically upwards with $z = 0$ at the equilibrium surface. The actual surface is located at

$$z = \zeta(x, y, t)$$

The kinematic surface condition is

$$\frac{\partial \zeta}{\partial t} + \nabla \varphi \cdot \nabla \zeta = \frac{\partial \varphi}{\partial z} \quad \text{at} \quad z = \zeta(x, y, t) \quad (2)$$

and the dynamical condition is

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}(\nabla \varphi)^2 + g\zeta = 0 \quad \text{at} \quad z = \zeta(x, y, t) \quad (3)$$

These equations are of course nonlinear, and therefore rather difficult to handle. Stokes (1847) found periodic solutions (Stokes waves) by expansion in the small steepness parameter ϵ . We shall use a generalized Stokes expansion

$$\varphi = \bar{\varphi} + \frac{1}{2}(Ae^{i\theta+ kz} + A_2e^{2i\theta+2kz} + A_3e^{3i\theta+3kz} + \dots + c.c.) \quad (4)$$

and

$$\zeta = \bar{\zeta} + \frac{1}{2}(Be^{i\theta} + B_2e^{2i\theta} + B_3e^{3i\theta} + \dots + c.c.) \quad (5)$$

where

$$\theta = \mathbf{k} \cdot \mathbf{x} - \omega t \quad \text{and} \quad \mathbf{x} = (x, y)$$

The complex amplitudes $A, A_2, A_3, \dots, B, B_2, B_3, \dots$ are slowly varying functions of space and time with the small parameter μ representing the slowness. Formally we assume that any space or time derivative is raising the order of these quantities in μ by one. Further $A, B = O(\epsilon)$ and $A_n, B_n = O(\epsilon^n)$. The quantities $\bar{\zeta}$ and $\bar{\varphi}$ are slowly varying real functions of space and time. They appear out of mathematical necessity for the expansion to be consistent. Physically they represent the slow response of the fluid to gradients in the wavestress, and as it turns out $\bar{\varphi} = O(\epsilon^2)$ and $\bar{\zeta} = O(\mu\epsilon^2)$.

The hierarchy

Inserting (4) and (5) into the basic equations (1), (2) and (3) and ordering in ϵ, μ and $\exp(i\theta)$ one obtains a hierarchy of equations. This hierarchy depends of course on the relative ordering of μ and ϵ . We will therefore in the following consider some of the most important cases.

- $O(\epsilon)$

To this lowest order in ϵ we have of course the linearized results. That is the *dispersion relation*

$$\mu^0 : \quad \omega^2 = gk$$

and

$$\mu^1 : \quad \frac{\partial B}{\partial t} + \mathbf{v}_g \cdot \nabla B = 0$$

which tell us that the wave envelope (and therefore the wave energy) is propagated by the *group velocity*

$$\mathbf{v}_g = \frac{\partial \omega}{\partial \mathbf{k}}.$$

To next order in μ *dispersion and diffraction* enters

$$\mu^2 : \quad i\left(\frac{\partial B}{\partial t} + v_g \frac{\partial B}{\partial x}\right) + \frac{1}{2} \left(\frac{dv_g}{dk} \frac{\partial^2 B}{\partial x^2} + \frac{v_g}{k} \frac{\partial^2 B}{\partial y^2} \right) = 0 \quad (6)$$

where for convenience we have put the x -axis along the $\mathbf{k}$ -direction.

- $O(\epsilon^2)$

Extending the equations to $O(\epsilon^2)$ has no impact on the evolution equation for the wave amplitude B . This is due to the fact that *resonant 3-wave interactions are not possible for gravity waves on deep water*. To this order, however, the *wavestress*¹ appears [1]. Gradients in the wavestress induces a slow motion given by the velocity potential $\bar{\varphi}$, and a corresponding elevation $\bar{\zeta}$ as

$$\mu^1 : \quad \frac{\partial \bar{\varphi}}{\partial t} + g\bar{\zeta} = 0 \quad \text{and} \quad \frac{\partial \bar{\varphi}}{\partial z} - \frac{\partial \bar{\zeta}}{\partial t} = \frac{\omega}{2} \frac{\partial}{\partial x} |B|^2 \quad (7)$$

Obviously $\bar{\varphi}$ must also satisfy Laplace equation

$$\nabla^2 \bar{\varphi} = 0 \quad (8)$$

This shows that $\bar{\varphi} = O(\epsilon^2)$ and $\bar{\zeta} = O(\mu\epsilon^2)$. Consequently the equations (7) can be simplified to the single equation

$$\frac{\partial \bar{\varphi}}{\partial z} = \frac{\omega}{2} \frac{\partial}{\partial x} |B|^2 \quad (9)$$

Consider the following simple example: A modulated wave train where the wave energy density $E = g/2 |B|^2$ varies like

$$E = E_0 + \Delta E \sin K(x - v_g t)$$

For a uniform wave train there is no mean Eulerian velocity, only a Lagrangian mass transport, the so-called Stokes drift. The potential $\bar{\varphi}$ is found from (8) and (9) to be $\bar{\varphi} = \frac{k}{\omega} \Delta E \exp(Kz) \cos K(x - v_g t)$.

¹Remark that for deep water waves the wavestress tensor $\mathbf{S}$ is given by

$$\mathbf{S} = \hat{\mathbf{k}}\hat{\mathbf{k}} \frac{E}{2}$$

where $E = \frac{g}{2} |B|^2$ is the wave energy density and $\hat{\mathbf{k}} = \mathbf{k}/k$

The corresponding Eulerian mean velocity at the surface is

$$\frac{\partial \bar{\varphi}(x, 0)}{\partial x} = -\frac{kK}{\omega} \Delta E \sin K(x - v_g t)$$

It is seen to be 180° out of phase with the energy modulation.

The feedback from this slow motion to the waves that caused it, does not appear to this order of approximation. This only occurs to the next order in ϵ .

- $O(\epsilon^3)$ and $\mu = O(\epsilon^2)$

Going to $O(\epsilon^3)$ gives the first nonlinear correction to the dispersion relation as shown already by Stokes (1847).

$$\mu^0 : \quad \omega^2 = gk(1 + k^2 |B|^2) \quad (10)$$

with the frequency shift $\delta\omega$ given by

$$\frac{\delta\omega}{\omega} = \frac{1}{2} k^2 |B|^2 \quad (11)$$

and further

$$\mu^1 : \quad i\left(\frac{\partial B}{\partial t} + v_g \frac{\partial B}{\partial x}\right) = \delta\omega B \quad (12)$$

The dynamics of the complex amplitude is limited to the phase as is appreciated by inserting $B = a \exp(i\psi)$ into the equation above which gives

$$a(x - v_g t, y) \quad \text{and} \quad \frac{\partial \psi}{\partial t} + v_g \frac{\partial \psi}{\partial x} = \frac{\omega}{2} (ka)^2$$

This is a modulation carried without change of form by the group velocity. The frequency is adjusted by $\delta\omega$ which is determined through the local amplitude (11). The uniform wave train solution of (12) (second order Stokes)

$$B = a \exp\left(-\frac{i}{2} (ka)^2 \omega t\right) \quad (13)$$

where a is the real amplitude, is stable to this approximation.

- $O(\epsilon^3)$ and $\mu = O(\epsilon)$

When $\mu = O(\epsilon)$, the dispersive/diffractive terms are of the same order of magnitude as the nonlinear term and must be added to (12) giving

$$\mu^2 : \quad i\left(\frac{\partial B}{\partial t} + v_g \frac{\partial B}{\partial x}\right) + \frac{1}{2} \left(\frac{dv_g}{dk} \frac{\partial^2 B}{\partial x^2} + \frac{v_g}{k} \frac{\partial^2 B}{\partial y^2} \right) - \delta\omega B = 0$$

This is the much celebrated and studied nonlinear Schrödinger (NLS) equation. It is a rather general model equation describing the evolution of modulated wave trains and is valid for dispersive and weakly nonlinear waves in isotropic media when 3-wave interaction is not possible. It has found applications in plasma physics, nonlinear optics as well as for gravity waves. Its success is due to the fact that it seems to contain at least qualitatively all the main physical effects that is found to be associated with such wave propagation. Which is quite a lot!

The uniform wave train solution (13) is no longer stable. Making small modulations in amplitude and phase $a(1+r) \exp(-\frac{1}{2}(ka)^2\omega t + i\psi)$ and assuming r and ψ to vary like $\exp i(K_x x + K_y y - \Omega t)$, one obtains upon linearization the dispersion relation

$$(\Omega - K_x v_g)^2 = D(2\delta\omega + D) \quad (14)$$

where

$$D = \frac{1}{2k}(v_g K_y^2 + \frac{dv_g}{dk} K_x^2)$$

Thus the uniform wave train is unstable provided

$$\delta\omega D < 0 \quad \text{and} \quad |\delta\omega| > \frac{1}{2}|D|$$

Depending on the properties of the medium through which the wave train is propagating one can have longitudinal or transversal instabilities.

- *Self-modulation* occurs when $\delta\omega dv_g/dk < 0$, (see [1]–[6])
- *Self-focusing* (transversal) is possible when $\delta\omega < 0$, (see [6]–[9])

For the case of gravity waves on deep water considered here, we see from (11) that $\delta\omega > 0$ so that self-focussing is not possible. Further we have

$$D = \frac{v_g}{2k}(K_y^2 - \frac{1}{2}K_x^2)$$

so that modulational instability is possible provided $K_y/K_x < \sqrt{2}$, meaning that the direction of the unstable modulation (K_x, K_y) is within the angular domain $\pm 35^\circ$ of $\mathbf{k}$. We note the interesting fact that this is exactly the angular domain of the transverse part of the so-called Kelvin wake pattern behind a ship.

The maximum growth rate for the modulational instability is equal to $\delta\omega$ and occurs when

$$K_x^2 - 2K_y^2 = 4k^2(ka)^2 \quad (15)$$

i.e. on a hyperbola in the (K_x, K_y) -plane.

Interest in the NLS equation took off after Zakharov and Shabat (1971) showed that the one-dimensional version of it could be solved by the inverse scattering method like the KdV equation before it. They used it to explain one dimensional self-modulation and two dimensional self-focusing where a central part is played by the envelope soliton solution

$$B = \frac{a \exp\left(-\frac{i}{2}\delta\omega t\right)}{\cosh[\delta k(x - v_g t)]} \quad (16)$$

Here a is the maximum amplitude and

$$\delta\omega = \frac{1}{2}\omega(ka)^2, \quad \delta k = \sqrt{2}ak^2 \quad (17)$$

A small shift of wavenumber $k \rightarrow k + K$ shifts the envelope velocity from v_g to $v_g + v'_g K$ and $B \rightarrow B \exp i [Kx - (v_g K + \frac{1}{2}v'_g K^2)t]$. Remark, however, that this velocity is independent of the maximum amplitude, in contrast to the KdV soliton. The width and amplitude are inversely proportional so that the “total mass” $\int_{-\infty}^{\infty} |B| dx$ is independent of a .²

The NLS equation has well known shortcomings as will be discussed later. It is useful mainly due to the fact that it is solvable by analytic techniques, and that a large number of interesting exact solutions are known.

Lately the research on extreme waves (freak- or rogue waves) has gained new momentum. From field measurements it seems clear that an extreme wave event on the open ocean consists of a short group of large waves standing out from the surrounding wave record [10], [11]. This has attracted renewed interest in some of the exact solutions of the NLS that represents short wave groups.

Simple solutions representing even shorter groups of nonlinear waves than the envelope soliton (16) are the so called breather solutions and the bound soliton solutions. Of the former we mention the simplest one [12] (see also [13])

$$B = a \exp(-i\delta\omega t) \left[1 - \frac{1 + 2i\delta\omega t}{1/4 + (\delta k x)^2 + (\delta\omega t)^2} \right] \quad (18)$$

where $\delta\omega$ and δk are given by (17). For convenience we have transformed to a frame of reference moving with the group velocity i.e. $x \rightarrow x + v_g t$. As $t \rightarrow \pm\infty$ the solution tends towards the uniform wave train solution (13) with amplitude a . When $t \rightarrow 0$ it forms an extremely short group of maximum amplitude $3a$. The number of waves in this group, N , is estimated by

$$N \sim \frac{0.2}{ka}$$

²This is of course connected to the fact that the NLS equation (in the frame moving with the group velocity v_g) is invariant to the transformation $t \rightarrow a^2 t, x \rightarrow ax, y \rightarrow ay$ and $B \rightarrow B/a$.

Another interesting solution that looks like the breather is the bound 2-soliton solution with two identical solitons [6], [12], [14]³

$$B = 4a \exp(-i \frac{\delta\omega}{2} t) \frac{\cosh(3\delta kx) + 3 \exp(-4i\delta\omega t) \cosh(\delta kx)}{\cosh(4\delta kx) + 4 \cosh(2\delta kx) + 3 \cos(4\delta\omega t)} \quad (19)$$

where again $\delta\omega$ and δk are given by (17). The envelope oscillates with a frequency $4\delta\omega$ between the form of two linearly superposed solitons of amplitude a , and a “breather-like” form of amplitude $4a$. The number of waves in the group in the latter phase is roughly

$$N \sim \frac{0.15}{ka}$$

The fact that the NLS equation predicts unstable growth for arbitrarily small wavelengths is of course a serious drawback. It means that growing perturbations are predicted outside the region of validity of the equation, which is a region of dimensions $O(\epsilon)$ around the origin of the (K_x, K_y) -plane. Numerical simulation using the NLS is working for one dimension, where the region of instability is limited. For two dimensions, however, energy leakage to higher and higher wavenumbers is unavoidable [15], [16].

It soon became clear that even for the one dimensional case some important features seen in laboratory experiments were not predicted by the NLS. It was demonstrated experimentally [17] that an initially symmetric wave packet evolves in an asymmetric manner. This asymmetric development is accelerated as the wave steepness is increased, and cannot be accounted for by the NLS which predicts a symmetric evolution. As an example consider the evolution of the bichromatic wave train shown in figures 1 and 2 (see [20]). Here the typical asymmetrical development seen in the experiment is not captured by the simulation using NLS.

This raised the question of whether the situation would be improved by going one order higher in the perturbation expansion. One physical effect that is not taken into account in the NLS is the back reaction of the wavestress induced streaming on the wave train causing it.

Before presenting the extension of the perturbation expansion (first done by one of the authors [18]) it is advantageous for esthetical reasons to bring the equations on a non-dimensional form by the transformation

$$t \rightarrow \omega t, \quad x \rightarrow kx, \quad y \rightarrow ky \quad \text{and} \quad B \rightarrow kB$$

The NLS equation then becomes

$$i\left(\frac{\partial B}{\partial t} + \frac{1}{2} \frac{\partial B}{\partial x}\right) + \frac{1}{4} \frac{\partial^2 B}{\partial y^2} - \frac{1}{8} \frac{\partial^2 B}{\partial x^2} = \frac{1}{2} |B|^2 B$$

³The form of the envelope of (19) is somewhat similar to that of the so-called Ma breather. They are both periodic in time.

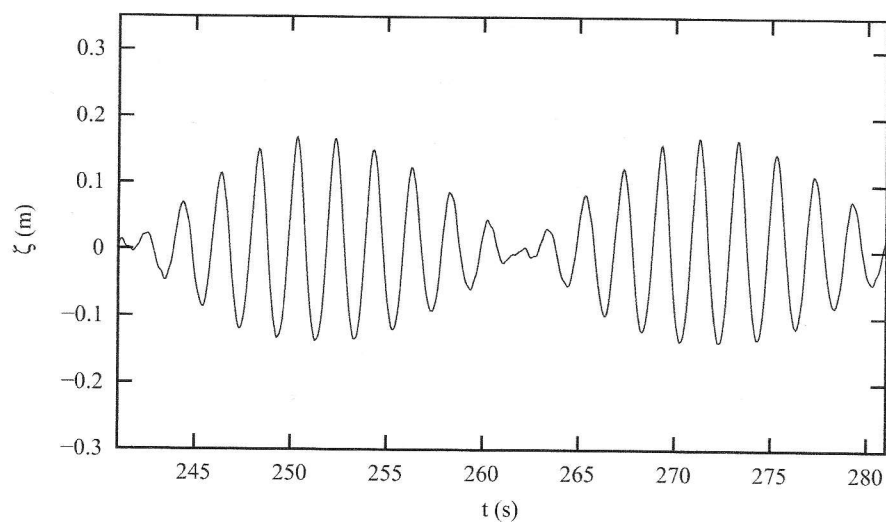


Figure 1: Bichromatic wave with central period 2 s, generated at one end of a long wavetank, measured by a wave staff at 9.3 m from the wave maker (experiment by Dr. Carl Trygve Stansberg).

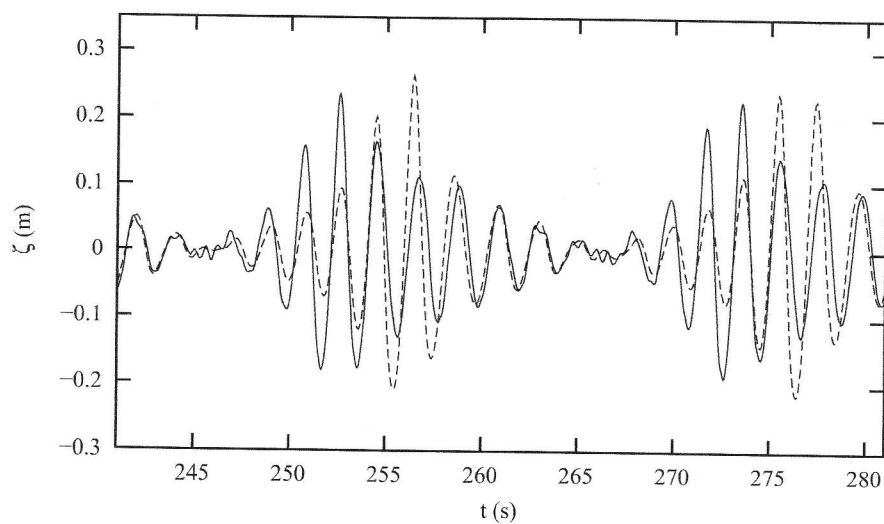


Figure 2: Comparison between experiment (—) and simulation with the NLS equation (---) at a distance of 80 m from the wave maker.

- $O(\mu\epsilon^3)$ and $\mu = O(\epsilon)$

To this order the *wavestress-induced motion is acting on the waves that caused it*. The evolution equations can be written

$$i\frac{\partial B}{\partial t} + L_3 B = \frac{1}{2}|B|^2 B - \frac{3i}{2}|B|^2 \frac{\partial B}{\partial x} - \frac{i}{4}B^2 \frac{\partial B^*}{\partial x} + \frac{\partial \bar{\varphi}}{\partial x} B \quad (20)$$

and

$$\begin{aligned} \frac{\partial \bar{\varphi}}{\partial z} &= \frac{1}{2} \frac{\partial}{\partial x} |B|^2 & z = 0 \\ \nabla^2 \bar{\varphi} &= 0 & z < 0 \end{aligned} \quad (21)$$

L_3 is the spatial part of the linear operator of the equation. It is given by an expansion to the third order in the derivatives of the operator

$$L \equiv \left\{ [(1 - \partial_x)^2 - \partial_y^2]^{1/4} - 1 \right\} \quad (22)$$

Further B^* is the complex conjugate of B . Remark that for the linearized case the equation

$$i\frac{\partial B}{\partial t} + LB = 0$$

give the *exact* solution for gravity waves on deep water.

Lo & Mei (1985) demonstrated that the asymmetric development seen in the experiments was reproduced by numerical simulations using the higher order equation (20). This can be seen in figure (3) (see [20]).

- $O(\mu\epsilon^3)$ and $\mu = O(\epsilon^{1/2})$

An evolution equation of the type (20) is well suited for 3D numerical simulations [25], [26]. The energy leakage to higher wave numbers turns out to be much reduced due to the higher order corrections in the equation. In simulating realistic ocean waves, however, the narrow band limitation $\mu = O(\epsilon)$ is too restrictive even when one takes into account only the most energetic part of an ocean wave spectrum⁴. If instead the less restrictive ordering $\mu = O(\epsilon^{1/2})$ is used we have a fairly realistic ordering. The evolution equation is almost identical to (20) except that the linear operator L_5 is substituted for L_3 [21] where L_5 is the expansion to the fifth order in the derivatives of the operator L given by (22). However, both the requirement for *broader bandwidth* and esthetical reasons have led us [22] to use L instead of L_5 in the evolution equation.

$$i\frac{\partial B}{\partial t} + LB = \frac{1}{2}|B|^2 B - \frac{3i}{2}|B|^2 \frac{\partial B}{\partial x} - \frac{i}{4}B^2 \frac{\partial B^*}{\partial x} + \frac{\partial \bar{\varphi}}{\partial x} B \quad (23)$$

⁴Using the so-called peak enhancement factor of the JONSWAP spectrum one can estimate the spectral width of the energy carrying waves to be in the range $\Delta k/k \sim 0.3 - 0.4$.

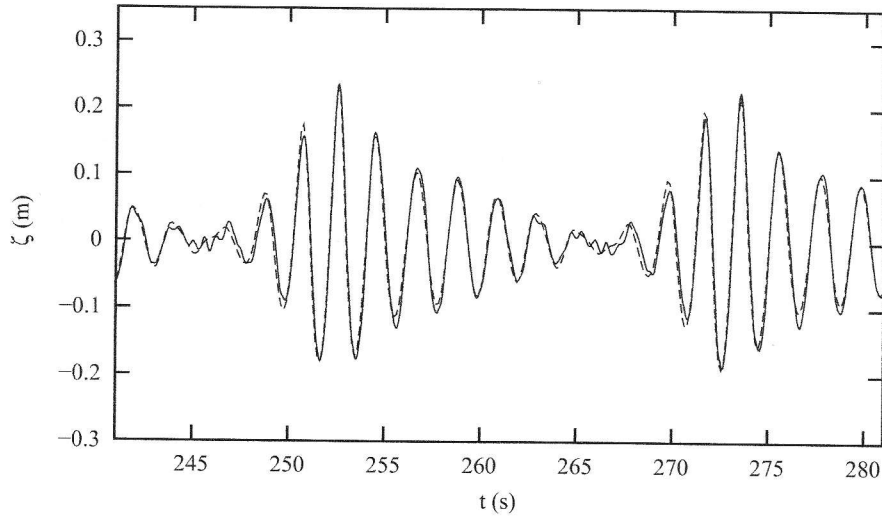


Figure 3: Comparison between experiment (—) and simulation with the higher order equation (---) at a distance of 80 m from the wave maker (same location as figure 2).

Numerical simulations based on this equation seems to have no energy leakage whatsoever, and is computationally as fast as the previous ones.

What happens to the solitons and breathers?

As mentioned above the NLS equation has serious shortcomings in describing the long term evolution of a modulated wave train. It is natural also to question whether the nice solutions of the type (16), (18) and (19) approximates corresponding solutions of the higher order equations.

This question has been considered for deep water waves by Akylas (1989, 1991). For the envelope solitons (16) he found similar solitary wave groups having a lower peak amplitude and moving slightly faster than the corresponding NLS soliton. The increase in speed is caused by a downshift in frequency. In figure 4 we take the NLS soliton as an initial condition for a simulation using the higher order equations in a frame following the linear group velocity. It is seen that a new solitary group forms, moving faster than the linear group velocity, and radiating excess energy in the form of a dispersive wave train with longer waves in front. Another solitary group one with shorter wavelengths forms behind, moving slower than the linear group velocity.

Corresponding simulations using breather solutions as initial conditions do not produce a similar phenomenon. Figure 5 shows the analytical Akhmediev

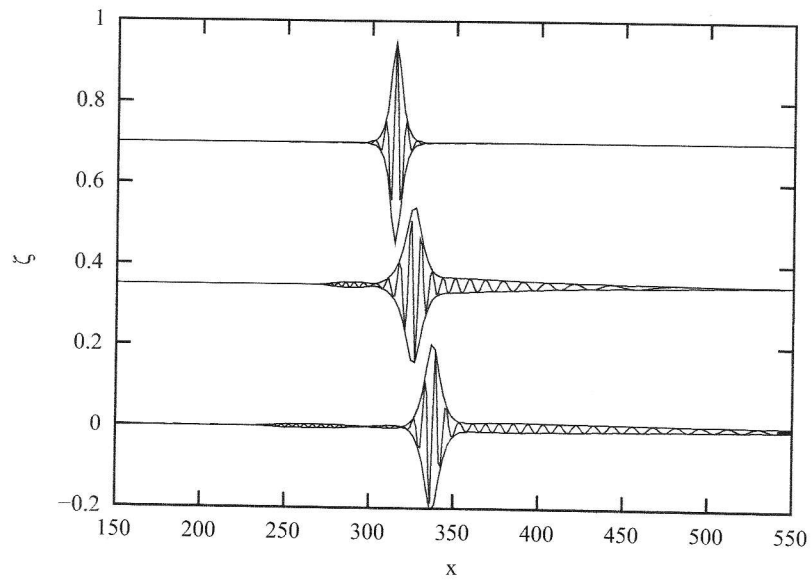


Figure 4: Analytical steady envelope soliton of the NLS equation (top) used as initial condition for the higher order equation. Propagation at two consecutively later times (middle and bottom). The envelope and the reconstructed first-harmonic wave are shown together.

breather solution of the NLS equation (see [13]). We see how a disturbance builds up in a symmetric manner, produces a short and large wave group, which is then attenuated in a symmetric manner. Using the higher order equation, figure 6 shows that the same initially symmetric disturbance develops in an asymmetric manner, first closing in front and then behind the short and large wave group. Later on, slight amplitude modulation emerges mainly in front of the short and large group. This may be due to the fact that the breather solutions are unstable to modulational instability on its uniform wave train wings.

The bound 2-soliton solution of Satsuma & Yajima for the NLS equation, figure 7, and with the higher order equation, figure 8.

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References

- [1] M. Longuet-Higgins and R. W. Stewart, *Deep Sea Res.*, 1964, **11**, 529–562.
- [2] M. J. Lighthill, *J. Inst. Math. Appl.*, 1965, **1**, 269–306.
- [3] M. J. Lighthill, *Proc. R. Soc. Lond. A*, 1967, **299**, 28–53.
- [4] T. B. Benjamin and J. E. Feir, *J. Fluid Mech.*, 1967, **27**, 417–430.
- [5] G. B. Whitham, *Proc. R. Soc. Lond. A*, 1967, **299**, 6–25.
- [6] V. E. Zakharov and A. B. Shabat, *Sov. Phys. JETP*, 1972, **34**, 62–69.
- [7] R. Y. Chiao, E. Garmire and C. H. Townes, *Phys. Rev. Lett.*, 1963, **13**, 479.
- [8] V. I. Talanov, *JETP Lett.*, 1964, **2**, 138.
- [9] A. Ostrovski, *JETP*, 1965, **24**, 797–800.
- [10] K. L. Henderson, D. H. Peregrine and J. W. Dold, *Wave Motion*, 2000, **29**, 341–361.
- [11] K. B. Dysthe, *Proc. Rogue Wave Workshop*, Brest, 2000.
- [12] D. H. Peregrine, *J. Austral. Math. Soc. Ser. B*, 1983, **25**, 16–43.
- [13] K. B. Dysthe and K. Trulsen, *Physica Scripta*, 1999, **T82**, 48–52.
- [14] J. Satsuma and S. Yajima, *Prog. Theor. Phys. Suppl.*, 1974, **55**, 284–306.

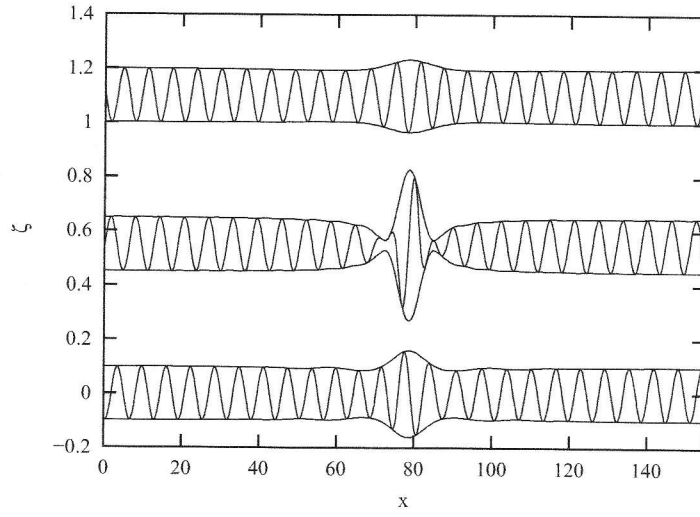


Figure 5: Analytical breather solution of the NLS equation at three consecutive times (top to bottom).

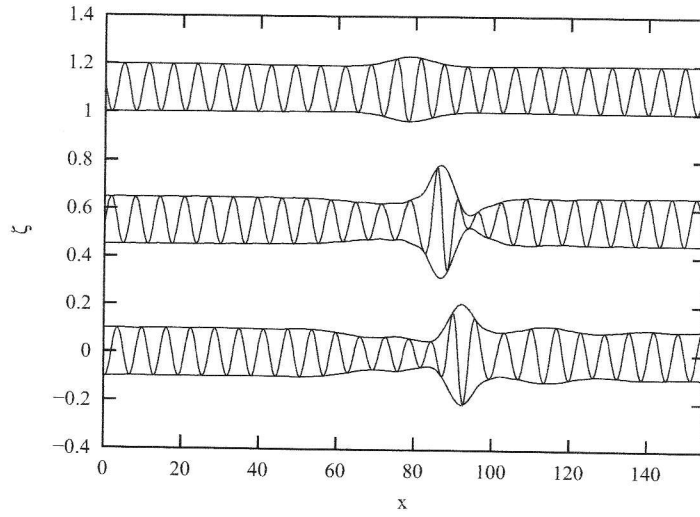


Figure 6: Using the analytical breather solution of the NLS equation used as initial condition for the higher order equation (top), we show the evolution at the same consecutive times (middle and bottom) as in figure 5.

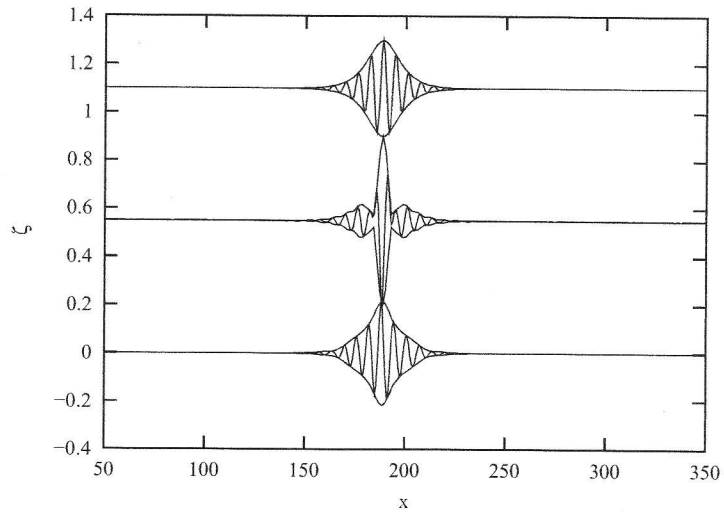


Figure 7: Analytical bound 2-soliton solution of the NLS equation at three consecutive times.

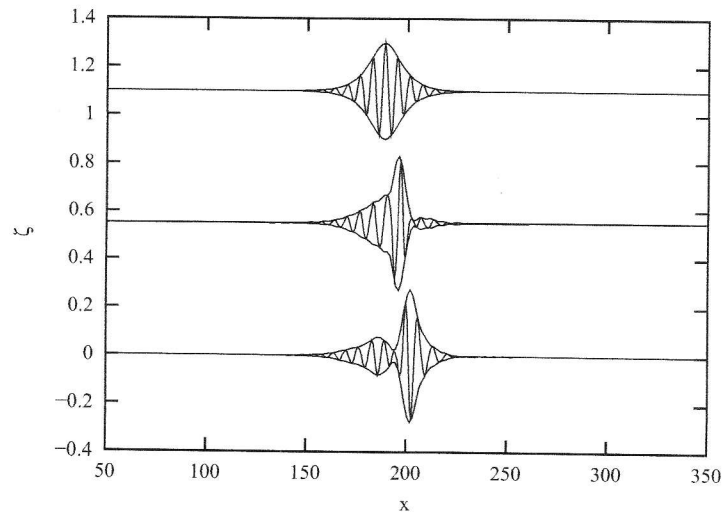


Figure 8: Analytical bound 2-soliton solution of the NLS equation used as initial condition for the higher order equation, at the same times as in figure 7.

- [15] H. C. Yuen and B. M. Lake, *Adv. Appl. Mech.*, **22**, 67–229.
- [16] D. U. Martin and H. C. Yuen, *Phys. Fluids*, **1980**, **23**, 881–883.
- [17] M. Y. Su, *Phys. Fluids*, **1982**, **25**, 2167–2174.
- [18] K. B. Dysthe, *Proc. R. Soc. Lond. A*, **1979**, **369**, 105–114.
- [19] E. Lo and C. C. Mei, *J. Fluid Mech.*, **1985**, **150**, 395–416.
- [20] K. Trulsen and C. T. Stansberg, *Proc. ISOPE*, Stavanger, **2001**, **3**, 71–77.
- [21] K. Trulsen and K. B. Dysthe, *Wave Motion*, **1996**, **24**, 281–289.
- [22] K. Trulsen, I. Klikaandler, K. B. Dysthe and M. G. Velarde, *Phys. Fluids*, **2000**, **12**, 2432–2437.
- [23] T. R. Akylas, *J. Fluid Mech.*, **1989**, **198**, 387–397.
- [24] T. R. Akylas, *J. Fluid Mech.*, **1991**, **224**, 417–428.
- [25] E. Lo and C. C. Mei, *Wave Motion*, **1987**, **9**, 245–259.
- [26] K. Trulsen and K. B. Dysthe, *J. Fluid Mech.*, **1997**, **352**, 359–373.