

SPATIAL EVOLUTION OF WATER SURFACE WAVES

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Abstract: Weakly nonlinear phase-resolving modeling of the spatial evolution of water surface wave trains is capable of predicting the deterministic evolution of long-crested waves over a substantial number of wavelengths. The extension of this capability to directionally distributed waves is discussed. Application to long wave tanks are given.

INTRODUCTION

Conventional wave forecasting is of a statistical nature in two respects: Information about the initial phases of the waves is not used, and the computation of the evolving wave field does not resolve phase information. Indeed, one seldom has information available on the initial phases, and phase-resolving computations are extremely time consuming.

Deterministic forecasting of spatiotemporally evolving waves requires phase-resolved initial wave data for initialization, and models that account for the phase-resolved directional spatiotemporal evolution. Possible models include fully nonlinear and high-order spectral methods for the Euler equations, Zakharov integral equations, and nonlinear Schrödinger equations.

Here attention is directed toward “initial” wave data in the form of time series measured at fixed wave probes, and the propagation of waves from those probes. Application to long-crested waves was shown by Trulsen & Stansberg (2001) and Shemer *et al.* (2001, 2002), all with application to experiments with bichromatic waves.

We extend the consideration to irregular and weakly directional waves. The present consideration is limited to high-order nonlinear Schrödinger equations for spatial evolution, and in particular their initialization with measured data.

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MODIFIED NONLINEAR SCHRÖDINGER EQUATIONS

We assume inviscid, incompressible potential flow. The weakly nonlinear surface is described as a superposition of free waves and nonlinearly forced waves. The free waves are located mainly near the peak of the spectrum, and are designated as the first harmonic. This corresponds to an assumption of slow modulation of amplitude and phase. We assume that the energy outside the peak of the spectrum is important only if it is coherent with the energy near the peak. We account for nonlinearly forced contributions that are coherent with the free waves of the first harmonic, and distinguish between zeroth, second and third harmonics.

We adopt the harmonic series expansions for the surface displacement $\zeta(\mathbf{x}, t)$ and velocity potential $\phi(\mathbf{x}, z, t)$

$$\zeta = \bar{\zeta} + \frac{1}{2} (B e^{i\theta} + B_2 e^{2i\theta} + B_3 e^{3i\theta} + \dots + \text{c.c.}) \quad (1)$$

$$\phi = \bar{\phi} + \frac{1}{2} (A e^{i\theta} + A_2 e^{2i\theta} + A_3 e^{3i\theta} + \dots + \text{c.c.}) \quad (2)$$

The phase $\theta = k_0 x - \omega_0 t$ describes the principal direction of evolution of the wave field, where k_0 and ω_0 are characteristic values for the wavenumber and frequency. The first, second and third harmonic complex amplitudes of the surface displacement are $B(\mathbf{x}, t)$, $B_2(\mathbf{x}, t)$ and $B_3(\mathbf{x}, t)$, respectively. The first, second and third harmonic complex amplitudes of the potential, $A(\mathbf{x}, z, t)$, $A_2(\mathbf{x}, z, t)$ and $A_3(\mathbf{x}, z, t)$, will not be considered here, but can be computed if the fluid velocity and acceleration fields are desired. The zeroth harmonic of the surface elevation $\bar{\zeta}(\mathbf{x}, t)$ has an associated slow drift velocity potential $\bar{\phi}(\mathbf{x}, z, t)$. The horizontal position vector is $\mathbf{x} = (x, y)$, z is the vertical position with origin at the equilibrium surface, t is time, and c.c. denotes the complex conjugate.

In the wave vector plane $\mathbf{k} = (k_x, k_y)$, the contribution of the n th harmonic is imagined inside a circular disk centered at $\mathbf{k} = (nk_0, 0)$, with $(k_x - nk_0)^2 + k_y^2 \leq n^2 k_0^2$ for $n > 0$ and $k_x^2 + k_y^2 \leq 4k_0^2$ for $n = 0$, see Figure 1.

Depending on the assumptions of small steepness $\epsilon = k_0 \bar{a} \ll 1$, small bandwidth $|\Delta \mathbf{k}|/k_0 \ll 1$, large depth $(k_0 h)^{-1} \ll 1$, and the truncation level of the perturbation expansions, we obtain a hierarchy of models for wave evolution and reconstruction. Here $\bar{a} = \sqrt{2\eta^2}$ is a characteristic amplitude, $\Delta \mathbf{k}$ is a characteristic modulation wave vector relative to the characteristic wave vector, and h is the depth.

Space evolution of a temporal field

Assume $|\Delta \mathbf{k}|/k_0 = \mathcal{O}(\epsilon)$, $(k_0 h)^{-1} = \mathcal{O}(\epsilon)$. The cubic nonlinear Schrödinger (NLS) equation is

$$\frac{\partial B}{\partial x} + 2 \frac{\partial B}{\partial t} + i \frac{\partial^2 B}{\partial t^2} - \frac{i}{2} \frac{\partial^2 B}{\partial y^2} + i |B|^2 B = 0. \quad (3)$$

Assume $|\Delta \mathbf{k}|/k_0 = \mathcal{O}(\epsilon)$, $(k_0 h)^{-1} = \mathcal{O}(\epsilon)$ and include third-order reconstruction. The modified nonlinear Schrödinger (MNLS) equation, also known as the Dysthe equa-

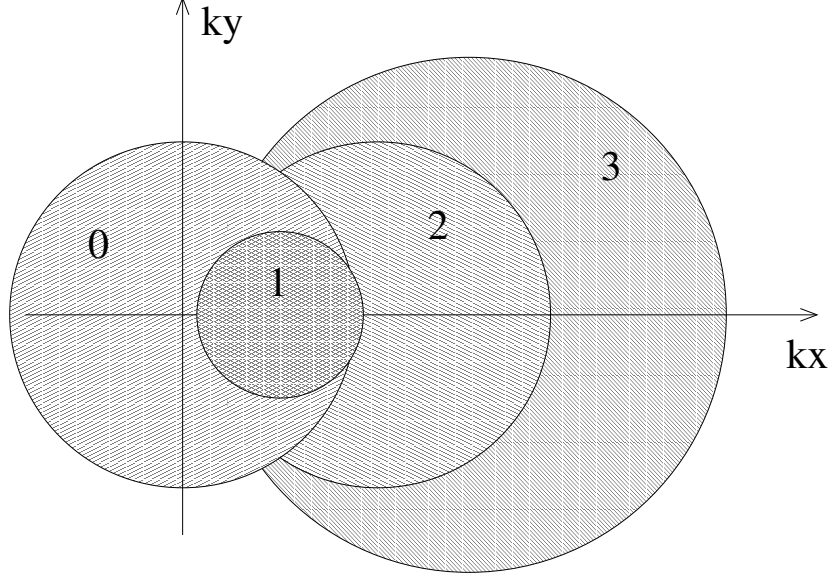


Fig. 1. Locations of the harmonics in the k wave vector plane.

tion is

$$\frac{\partial B}{\partial x} + 2\frac{\partial B}{\partial t} + i\frac{\partial^2 B}{\partial t^2} - \frac{i}{2}\frac{\partial^2 B}{\partial y^2} + i|B|^2 B - \frac{\partial^3 B}{\partial t \partial y^2} - 8|B|^2 \frac{\partial B}{\partial t} - 2B^2 \frac{\partial B^*}{\partial t} - 4iB \frac{\partial \bar{\phi}}{\partial t} = 0, \quad (4)$$

$$\frac{\partial \bar{\phi}}{\partial z} = -\frac{\partial |B|^2}{\partial t} \quad \text{at} \quad z = 0, \quad (5)$$

$$4\frac{\partial^2 \bar{\phi}}{\partial t^2} + \frac{\partial^2 \bar{\phi}}{\partial y^2} + \frac{\partial^2 \bar{\phi}}{\partial z^2} = 0 \quad \text{for} \quad -h < z < 0 \quad (6)$$

and

$$\frac{\partial \bar{\phi}}{\partial z} = 0 \quad \text{at} \quad z = -h. \quad (7)$$

The reconstruction formulas are

$$\bar{\zeta} = -\frac{\partial \bar{\phi}}{\partial t}, \quad B_2 = \frac{1}{2}B^2 + iB\frac{\partial B}{\partial t} \quad \text{and} \quad B_3 = \frac{3}{8}B^3. \quad (8)$$

The Dysthe equation for time evolution was published by Dysthe (1979) for infinite depth. Extension to great depth $(k_0 h)^{-1} = \mathcal{O}(\epsilon)$ and the reconstruction of the surface displacement to third order was published by Lo & Mei (1985) for long-crested waves. The Dysthe equation for spatial evolution was presented by Lo & Mei (1985). Kit & Shemer (2002) gave a derivation of these equations starting from the Zakharov integral equation.

Motivated from the desire to capture more rapidly modulated waves typically observed in the field (Trulsen & Dysthe 1997), we have modified the Dysthe equation further by including exact linear dispersion, corresponding to eliminating the constraint on $|\Delta \mathbf{k}|/k_0$ in the linear limit. This can be achieved if the linear part of the evolution equation is represented by a pseudo-differential operator (Trulsen, Kliakhandler,

Dysthe & Velarde 2000). The evolution equation (4) becomes

$$\frac{\partial B}{\partial x} - i \left\{ \left[\left(1 + i \frac{\partial}{\partial t} \right)^4 + \frac{\partial^2}{\partial y^2} \right]^{1/2} - 1 \right\} B + i|B|^2 B - 8|B|^2 \frac{\partial B}{\partial t} - 2B^2 \frac{\partial B^*}{\partial t} - 4i \frac{\partial \bar{\phi}}{\partial t} B = 0 \quad (9)$$

By series expansion of the pseudo-differential operator all the linear dispersive terms of lower-order theories are recovered. For long-crested waves, the pseudo-differential operator in (9) reduces to the linear terms already contained in the the MNLS equation (4). Hence equation (9) for space evolution only represents something new in the case of short-crested waves.

INITIALIZATION FROM MEASURED DATA

Our numerical implementation heavily depends on representation with Fourier transforms in the longitudinal (*i.e.* time) and transversal directions, and therefore some assumptions of periodic boundary conditions are necessary. We set

$$B(x, y_n, t_m) = \sum_{-\frac{M}{2} < i \leq \frac{M}{2}} \sum_{-\frac{N}{2} < j \leq \frac{N}{2}} \hat{B}(x, k_{yj}, \omega_i) e^{i(k_{yj}y_n - \omega_i t_m)} \quad (10)$$

for N collocation points in the transversal y -direction and M collocation points in time.

We stress that only the modulation is required to be periodic in time direction, not the wave itself. In the transversal direction we can either require that the wave itself is periodic, or we can impose transversal boundary conditions for a tank with vertical side walls.

The process of initialization implies two important steps: The determination of an optimal characteristic wave k_0 and ω_0 (tuning) and refining the extraction of the complex amplitude B . Practical experience suggests that the first step is more important.

Suppose we have given measured time series $\zeta(x, y_n, t_m)$ at $\tilde{M}$ times $t_m = mT/\tilde{M}$ for $m = 0, 1, \dots, \tilde{M} - 1$ and at $\tilde{N}$ transversal locations y_n for $n = 0, 1, \dots, \tilde{N} - 1$. Here we have constrained all transversal locations to have the same x -coordinate. Typically the number of observed samples $\tilde{M}$ is much greater than the number of collocation points M necessary for the representation of B . For a given characteristic wave (k_0, ω_0) , we rewrite the harmonic expansion (1) in the form

$$B = 2\zeta e^{-i\theta} - 2\bar{\zeta} e^{-i\theta} - B^* e^{-2i\theta} - B_2 e^{i\theta} - B_2^* e^{-3i\theta} - B_3 e^{2i\theta} - B_3^* e^{-4i\theta} - \dots \quad (11)$$

where $\theta = k_0 x - \omega_0 t$. A first approximation for B can be obtained by bandpass filtering the first term on the right hand side for $(\omega_i, k_{yj}) = (2\pi i/T, 2\pi j/b)$ close to the origin. Hence after taking the shifted Fourier transform

$$\hat{\zeta}'(x, k_{yj}, \omega_i) = \frac{1}{\tilde{M}\tilde{N}} \sum_{m=0}^{\tilde{M}-1} \sum_{n=0}^{\tilde{N}-1} \zeta(x, y_n, t_m) e^{-i[k_0 x + k_{yj} y_n - (\omega_0 + \omega_i) t_m]} \quad (12)$$

we set

$$\hat{B}(x, k_{yj}, \omega_i) = \begin{cases} 2\hat{\zeta}'(x, k_{yj}, \omega_i) & \text{for } |\omega_i| \text{ and } |k_{yj}| \text{ sufficiently small} \\ 0 & \text{elsewhere} \end{cases} \quad (13)$$

Let the Fourier transform of the measured time series be

$$\hat{\zeta}(x, k_{yj}, \omega_i) = \frac{1}{\tilde{M}\tilde{N}} \sum_{m=0}^{\tilde{M}-1} \sum_{n=0}^{\tilde{N}-1} \zeta(x, y_n, t_m) e^{-i(k_{yj}y_n - \omega_i t_m)}. \quad (14)$$

For application of a narrow-banded wave theory, the optimal choice for the characteristic wave (k_0, ω_0) is such that the width of the power spectrum of the first harmonic B is as narrow as possible. While typical choices for the characteristic frequency is either the mode or the mean of the distribution $\sum_j |\hat{\zeta}(x, k_{yj}, \omega_i)|^2$, better results are typically achieved by choosing the frequency that minimizes some parameterization of the width, e.g. the frequency that minimizes $\delta(\omega_0)$,

$$\delta(\omega_0) = \sum_i \sum_j \omega_i^2 |\hat{\zeta}'(x, k_{yj}, \omega_i)|^2. \quad (15)$$

In the subsequent validation, ω_0 without tuning is equal to the mean frequency. With *tuning* ω_0 is the frequency that minimizes the bandwidth $\delta(\omega_0)$. The tuning algorithm is fast, and the tuned ω_0 improves validation results slightly.

VALIDATION WITH LABORATORY MEASUREMENTS

An optimal criterion for validation depends on the application. Assuming that prediction of amplitude and phase are equally important, we introduce the error criterion

$$e(x) = \sqrt{\frac{\sum_i (\zeta_{\text{sim}}(x, t_i) - \zeta_{\text{exp}}(x, t_i))^2}{\sum_i (\zeta_{\text{exp}}(x, t_i))^2}} \quad (16)$$

where $\zeta_{\text{sim}}(x, t_i)$ is the simulated surface elevation and $\zeta_{\text{exp}}(x, t_i)$ is the measured surface elevation. Two sample applications using experimental data from Marintek are reported in the following:

Application to bichromatic waves

This experiment was carried out in the 270 m long and 10.5 m wide towing tank at Marintek (Stansberg 1993, 1995, 1998). Nine probes were located along the centerline of the tank, while one probe was located off the centerline. A sketch of the arrangement of wave probes is shown in Figure 2.

Measurements at probe 1 are used for initialization and detailed comparisons between simulated and measured waves are made at probes 2, 4, 5, 7 and 10. The probes are located at distances 9.3 m, 40 m, 80 m, 120 m, 160 m and 200 m from the wave maker, respectively. The error $e(x)$ is computed at all probes along the centerline. The sampling period is 0.05 s. In order to get rid of seiching in the experimental data, the measured time series were high-pass filtered at about 0.08 Hz.

The depth of the wave tank is 10 m for the first 80 m closest to the wave maker, and 5 m elsewhere. However, the wave periods employed in these experiments are so short that the waves are essentially on deep water. We have set the depth to 10 m in

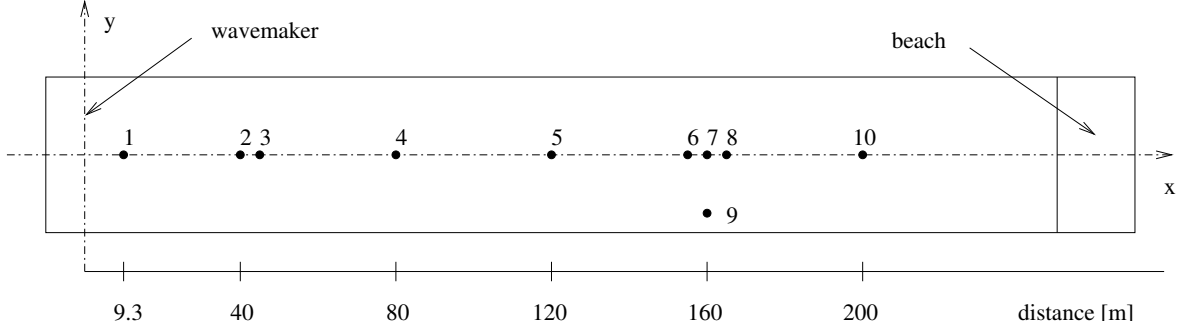


Fig. 2. Sketch of towing tank with wave probes.

all the simulations. While the non-uniform depth is practically invisible at the scale of individual waves, it may be visible for the induced current $\bar{\phi}$ and set-down $\bar{\zeta}$, the zeroth harmonic contributions to expansions (1) and (2). The effect of non-uniform depth becomes more important as the peak period increases, however our extended validation tests suggest that the best agreement between simulation and experiment is obtained for longer peak periods. Hence there is reason to believe that the neglect of non-uniform depth is not a principal source of error in the present case.

The bichromatic waves are generated with nominal periods 1.9 and 2.1 s, both with nominal wave heights 0.16 m. The steepness of the portion of the time series at probe 1 used for initialization is estimated to be 0.11. We use a time domain of length 204.8 s, starting at 90 s of the original time series, see Figure 3. The simulations are initialized by the relevant part of the experimental time series at probe 1. Comparisons for validation are made with a time interval of length 40 s, starting at 130 s at the initial probe, moving with the linear group velocity.

The normalized errors of various simulations are shown in Figure 4. here we compare the errors of a linear simulation (L), simulations using NLS without (N) and with (Nt) tuning, and simulations using MNLS without (M) and with (Mt) tuning. Detailed comparisons of experiment with simulation are given in Figure 5 for MNLS with tuning (Mt).

Application to irregular waves

This experiment was carried out in the ocean basin at Marintek. The arrangement of probes is outlined in Figure 6.

The wave maker was programmed with an irregular wave spectrum, with peak period 14.0 s and significant wave height 14.0 m (after Froude scaling 1:200). We use a time domain of length 2896.3 s, starting at 350.018 s of the original time series, see Figure 7. The simulations are initialized by the relevant part of the experimental time series at probe 5. Comparisons for validation are made with a time interval of length 200 s, starting at 1400 s at the initial probe, moving with the linear group velocity.

The normalized errors of various simulations are shown in Figure 8. It is not easy to distinguish between the quality of linear theory or NLS theory, but MNLS theory is clearly better than the other two.

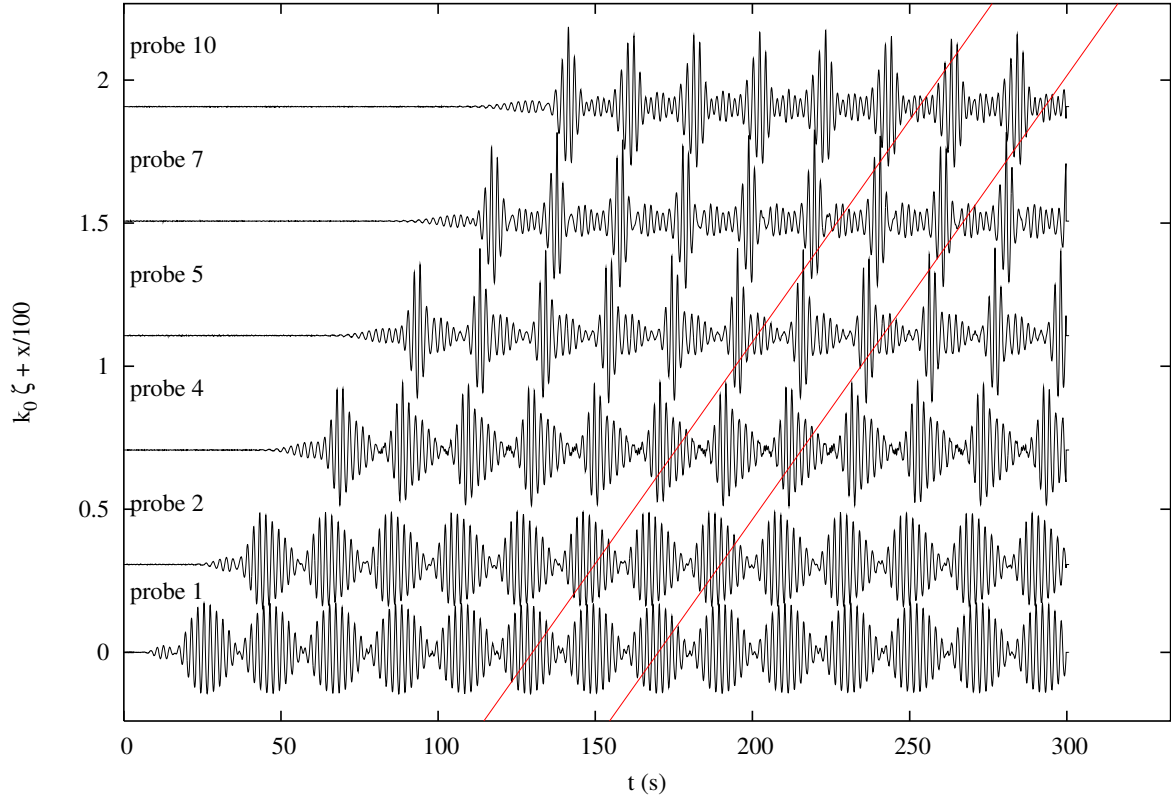


Fig. 3. Bichromatic waves. Time series from experiment. The two slanted lines show the evolution of the wave groups used for validation of the simulations.

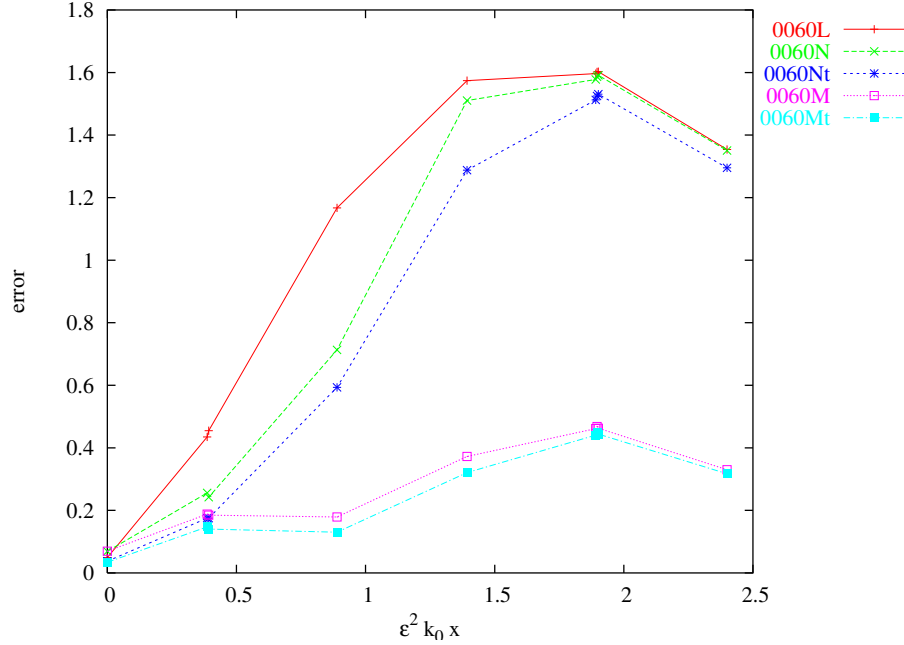


Fig. 4. Bichromatic waves: Normalized error $e(x)$ of simulations as a function of normalized propagation distance at subsequent probes. The linear simulation (L) is worst and the MNLS simulations (M, Mt) give the best results, while the NLS simulations (N, Nt) fall between. Tuning the characteristic frequency for minimum bandwidth (Nt, Mt) improves the nonlinear simulations slightly.

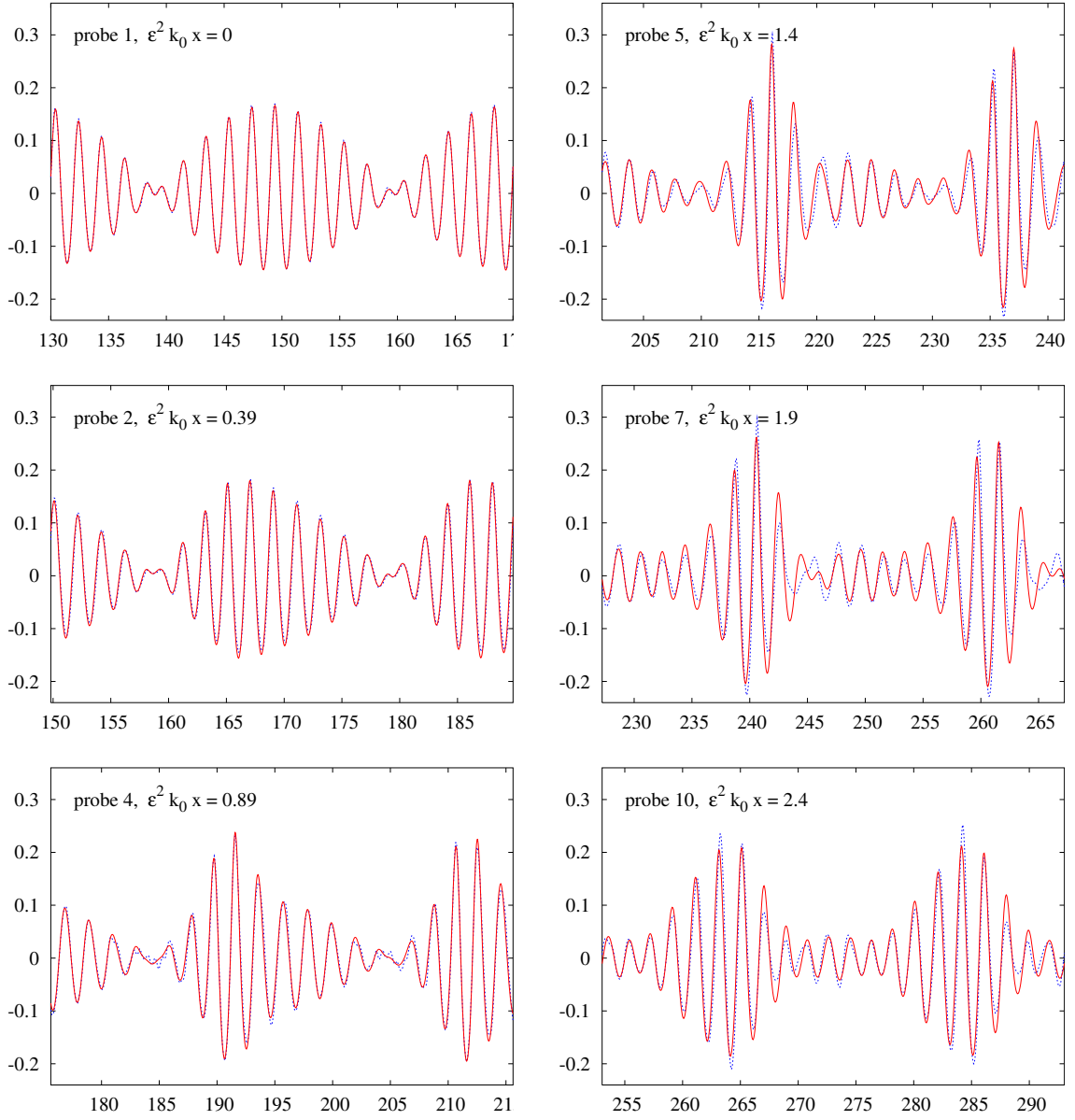


Fig. 5. Bichromatic waves: MNLS simulation with tuning. Comparison between simulation (solid curves) and experiment (dotted curves) at six probes. First axis is time in seconds. Second axis is normalized surface displacement $k_0 \zeta$.

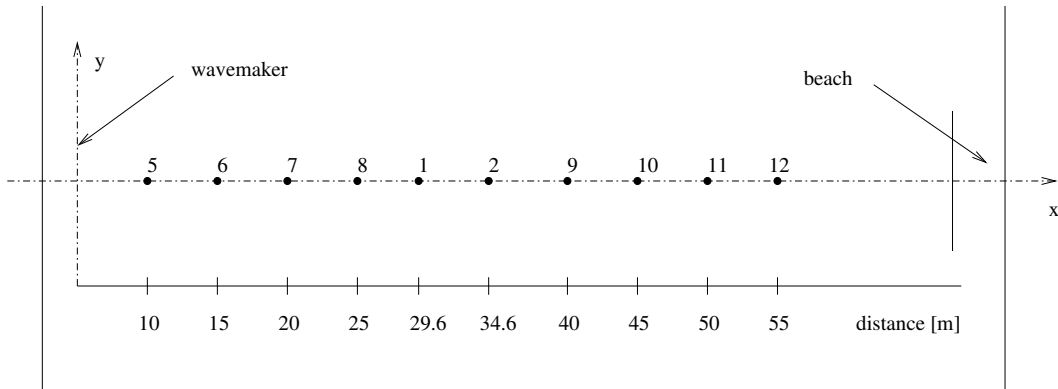


Fig. 6. Sketch of ocean basin with wave probes.

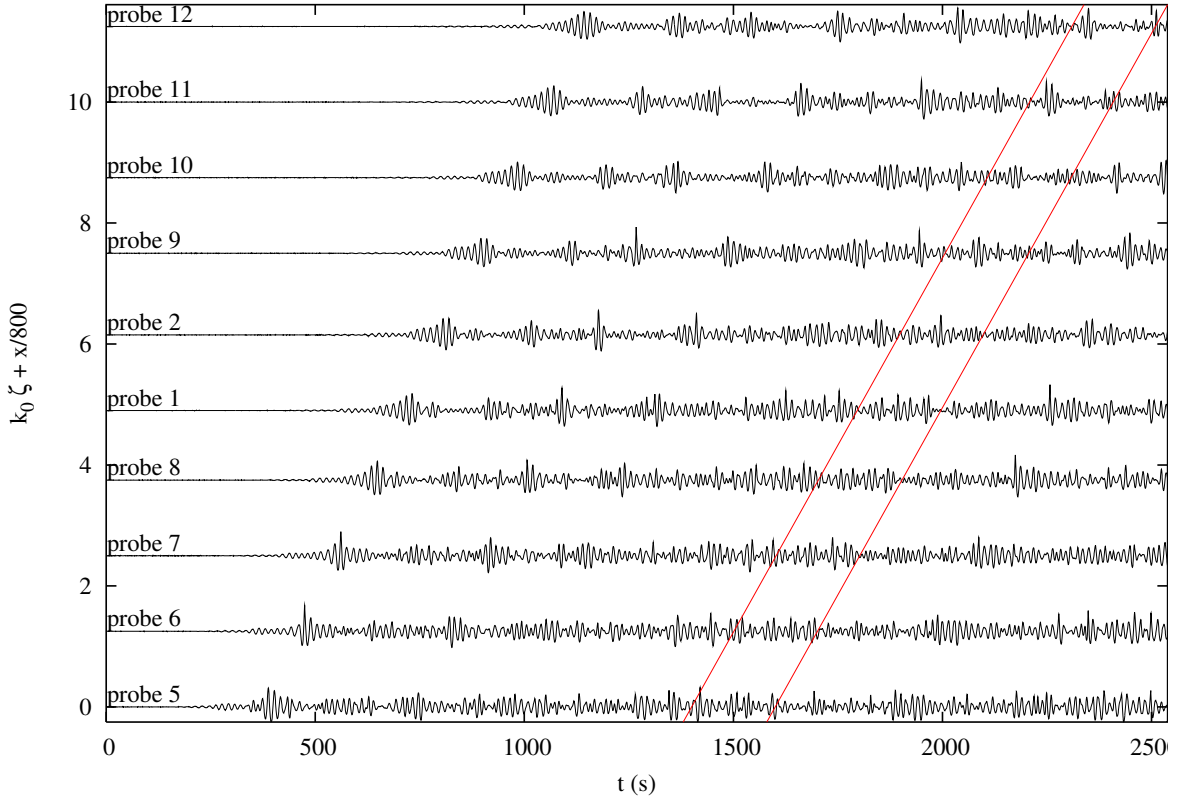


Fig. 7. Irregular waves: Time series from experiment. The two slanted lines show the evolution of the wave groups used for validation of the simulations.

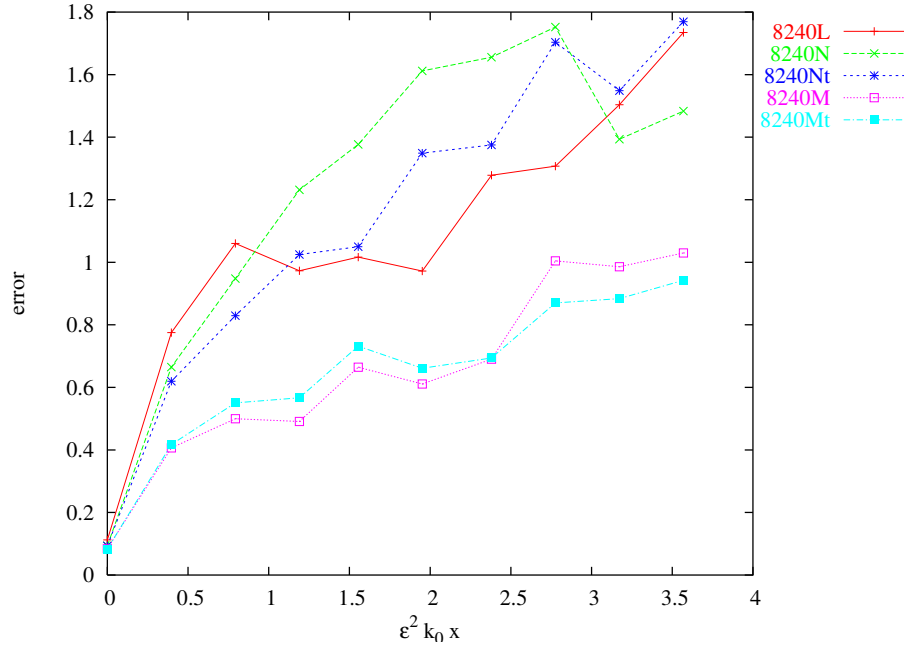


Fig. 8. Irregular waves: Normalized error $e(x)$ of simulations as a function of normalized propagation distance at subsequent probes. The linear simulation (L) and the NLS simulations (N, Nt) give the worst results while the MNLS simulations (M, Mt) give the best results.

CONCLUSIONS

We have demonstrated that deterministic forecasting of waves may be a realistic possibility for propagation distances corresponding to the nonlinear evolution scale $(k_0 \bar{a})^2 k_0 x$ being of order one, where k_0 is a characteristic wavenumber, $\bar{a}$ is a characteristic amplitude, and x is propagation distance. The cubic nonlinear Schrödinger equation hardly gives any improvement compared with linear theory, while a higher-order nonlinear Schrödinger equation does extend the forecasting horizon. Optimal tuning of the characteristic frequency/wavenumber may yield a slight improvement in performance of the nonlinear Schrödinger equations.

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