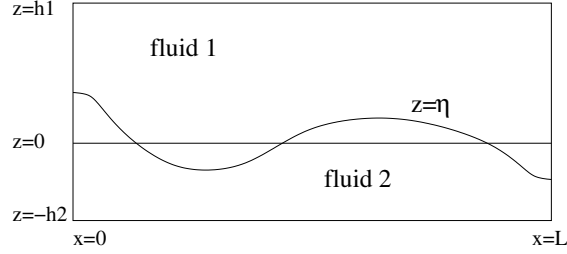


Voluntary oblig 1.5, MEK4320, spring 2021
suggested solution by Karsten Trulsen



Toy completely filled with two immiscible and incompressible liquids denoted “fluid 1” and “fluid 2” with densities $\rho_1 < \rho_2$ and viscosities ν_1 and ν_2 . It may be expected that $\nu_1 > \nu_2$. The equilibrium interface is at $z = 0$, the horizontal top is at $z = h_1$, the horizontal bottom is at $z = -h_2$. The width of the toy is L . Our attention is primarily directed toward the period of oscillation and damping rate of the lowest mode with wavelength $\lambda = 2L$. We assume the problem can be linearized.

Governing equations

The velocities in the two layers are

$$\mathbf{v}_j = u_j \mathbf{i}_x + w_j \mathbf{i}_z = (u_j, w_j) \quad \text{for } j = 1, 2 \quad (1)$$

The viscous stress tensors in the two layers are

$$\mathcal{P}_j = \rho_j \nu_j (\nabla \mathbf{v}_j + (\nabla \mathbf{v}_j)^T) = \rho_j \nu_j \begin{pmatrix} 2 \frac{\partial u_j}{\partial x} & \frac{\partial u_j}{\partial z} + \frac{\partial w_j}{\partial x} \\ \frac{\partial u_j}{\partial z} + \frac{\partial w_j}{\partial x} & 2 \frac{\partial w_j}{\partial z} \end{pmatrix} \quad (2)$$

The continuity equations in the two layers are

$$\nabla \cdot \mathbf{v}_j = 0 \quad \text{for } j = 1, 2 \quad (3)$$

The linearized Navier–Stokes equation is

$$\frac{\partial \mathbf{v}_j}{\partial t} = -\frac{1}{\rho_j} \nabla p_j - g \mathbf{i}_z + \nu_j \nabla^2 \mathbf{v}_j \quad \text{for } j = 1, 2 \quad (4)$$

The kinematic boundary conditions at the top and bottom are

$$\mathbf{v}_1 = 0 \quad \text{at } z = h_1 \quad \text{and} \quad \mathbf{v}_2 = 0 \quad \text{at } z = -h_2 \quad (5)$$

including both the normal condition that the fluid does not flow through the boundary and the tangential condition of no slip appropriate for viscous flow.

The Taylor-expanded and linearized kinematic boundary conditions and no-slip conditions at the interface are

$$\frac{\partial \eta}{\partial t} = w_1 = w_2 \quad \text{at } z = 0 \quad (6)$$

and

$$u_1 = u_2 \quad \text{at } z = 0 \quad (7)$$

The Taylor-expanded and linearized dynamic boundary conditions at the interface are

$$-p_1 + 2\rho_1\nu_1\frac{\partial w_1}{\partial z} = -p_2 + 2\rho_2\nu_2\frac{\partial w_2}{\partial z} \quad \text{at } z = 0 \quad (8)$$

and

$$\rho_1\nu_1\left(\frac{\partial u_1}{\partial z} + \frac{\partial w_1}{\partial x}\right) = \rho_2\nu_2\left(\frac{\partial u_2}{\partial z} + \frac{\partial w_2}{\partial x}\right) \quad \text{at } z = 0 \quad (9)$$

Thus we have 5 boundary conditions associated with the interface, and 4 boundary conditions associated with the ceiling and the floor. Let us neglect the boundary conditions at the ends $x = 0$ and $x = L$ since the toy is expected to be rather long in comparison to the height.

Scalar and vector potential

It is found useful to represent the velocity fields $\mathbf{v}_j$ as a superposition of curl-free and divergence-free parts by introducing scalar potentials ϕ_j and stream functions ψ_j such that $\mathbf{v}_j = \nabla\phi_j + \nabla \times (\psi_j \mathbf{i}_y)$ ¹

$$u_j = \frac{\partial \phi_j}{\partial x} - \frac{\partial \psi_j}{\partial z} \quad (10)$$

and

$$w_j = \frac{\partial \phi_j}{\partial z} + \frac{\partial \psi_j}{\partial x} \quad (11)$$

The continuity equations (3) now become the Laplace equations for the scalar potentials

$$\nabla^2 \phi_j = 0 \quad \text{for } j = 1, 2 \quad (12)$$

Taking the curl of the linearized Navier–Stokes equation (4), we find that the stream functions are governed by diffusion equations

$$\frac{\partial \psi_j}{\partial t} = \nu_j \nabla^2 \psi_j \quad \text{for } j = 1, 2 \quad (13)$$

¹In general, a vector field $\mathbf{v}$ can be represented by a combination of a scalar potential ϕ and a vector potential $\mathbf{A}$ by $\mathbf{v} = \nabla\phi + \nabla \times \mathbf{A}$. The divergence becomes $\nabla \cdot \mathbf{v} = \nabla^2\phi$ and the curl becomes $\nabla \times \mathbf{v} = \nabla\nabla \cdot \mathbf{A} - \nabla^2\mathbf{A}$. This decomposition is not unique, it thus allows us to impose an additional requirement known as “choosing a gauge”. The Coulomb gauge, $\nabla \cdot \mathbf{A} = 0$, is natural in our case since ψ can be independent of y .

Taking a second look at the linearized Navier–Stokes equation, we find that Euler’s pressure equation still holds within the linear approximation, this means that to linear order the pressure is determined only by the scalar potentials ϕ_j and not by the stream functions ψ_j . The pressure can then be given with reference to some reference pressure, e.g. the equilibrium pressure p_0 at the interface ($z = 0$) when the liquids are at rest, with the familiar expression

$$p_j = p_0 - \rho_j \left(\frac{\partial \phi_j}{\partial t} + gz \right) \quad \text{for } j = 1, 2 \quad (14)$$

The boundary conditions become

$$\frac{\partial \phi_1}{\partial x} - \frac{\partial \psi_1}{\partial z} = 0 \quad \text{and} \quad \frac{\partial \phi_1}{\partial z} + \frac{\partial \psi_1}{\partial x} = 0 \quad \text{at } z = h_1 \quad (15)$$

$$\frac{\partial \phi_2}{\partial x} - \frac{\partial \psi_2}{\partial z} = 0 \quad \text{and} \quad \frac{\partial \phi_2}{\partial z} + \frac{\partial \psi_2}{\partial x} = 0 \quad \text{at } z = -h_2 \quad (16)$$

$$\frac{\partial \phi_1}{\partial x} - \frac{\partial \psi_1}{\partial z} = \frac{\partial \phi_2}{\partial x} - \frac{\partial \psi_2}{\partial z} \quad \text{at } z = 0 \quad (17)$$

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi_1}{\partial z} + \frac{\partial \psi_1}{\partial x} = \frac{\partial \phi_2}{\partial z} + \frac{\partial \psi_2}{\partial x} \quad \text{at } z = 0 \quad (18)$$

$$\rho_1 \nu_1 \left(2 \frac{\partial^2 \phi_1}{\partial x \partial z} + \frac{\partial^2 \psi_1}{\partial x^2} - \frac{\partial^2 \psi_1}{\partial z^2} \right) = \rho_2 \nu_2 \left(2 \frac{\partial^2 \phi_2}{\partial x \partial z} + \frac{\partial^2 \psi_2}{\partial x^2} - \frac{\partial^2 \psi_2}{\partial z^2} \right) \quad \text{at } z = 0 \quad (19)$$

$$\rho_1 \left(\frac{\partial \phi_1}{\partial t} + g\eta + 2\nu_1 \frac{\partial^2 \phi_1}{\partial z^2} + 2\nu_1 \frac{\partial^2 \psi_1}{\partial x \partial z} \right) = \rho_2 \left(\frac{\partial \phi_2}{\partial t} + g\eta + 2\nu_2 \frac{\partial^2 \phi_2}{\partial z^2} + 2\nu_2 \frac{\partial^2 \psi_2}{\partial x \partial z} \right) \quad \text{at } z = 0 \quad (20)$$

Simple harmonic waves

Assume simple harmonic waves

$$\begin{pmatrix} \eta \\ \phi_1 \\ \phi_2 \\ \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} A \\ B_1 \cosh k(z - h_1) + C_1 \sinh k(z - h_1) \\ B_2 \cosh k(z + h_2) + C_2 \sinh k(z + h_2) \\ D_1 e^{m_1(z-h_1)} + E_1 e^{-m_1 z} \\ D_2 e^{-m_2(z+h_2)} + E_2 e^{m_2 z} \end{pmatrix} e^{i(kx - \omega t)} \quad (21)$$

This assumption expresses a lot of our experience obtained during previous work in the course. Limiting to the amplitudes A and B_j we should recover the solution of oblig 1. The amplitudes D_j should account for the viscous boundary layers near the floor and the ceiling, E_j should account for the viscous boundary layers near the interface, and C_j should be the corresponding adjustments necessary for the inviscid part of the flow field.

We are interested in waves attenuated in time and not in space, typically standing waves, primarily the longest standing wave with $k = \pi/L$. Thus k is real and the angular frequency is complex, $\omega = \omega_R + i\omega_I$.

The assumption (21) satisfies identically the Laplace equations (12), while the diffusion equations (13) become

$$m_j^2 - k^2 + \frac{i\omega}{\nu_j} = 0 \quad (22)$$

The assumptions (21) have been written with respect to $\text{Re}(m_j) > 0$, thus we have

$$m_j = \sqrt{k^2 - \frac{i\omega}{\nu_j}} = e^{-\frac{i\pi}{4}} \sqrt{\frac{\omega}{\nu_j}} \sqrt{1 + i\epsilon_j} \approx e^{-\frac{i\pi}{4}} \sqrt{\frac{\omega}{\nu_j}} \quad (23)$$

where

$$\epsilon_j = \sqrt{\frac{\nu_j k^2}{\omega}} \quad (24)$$

are small parameters $\epsilon_1 \approx 0.07$ and $\epsilon_2 \approx 0.01$, therefore we can safely neglect the contributions of D_j at the interface and E_j at the floor and the ceiling.

The boundary conditions become

$$ikB_1 - m_1 D_1 = 0 \quad (25)$$

$$kC_1 + ikD_1 = 0 \quad (26)$$

$$ikB_2 + m_2 D_2 = 0 \quad (27)$$

$$kC_2 + ikD_2 = 0 \quad (28)$$

$$\begin{aligned} ikB_1 \cosh(kh_1) - ikC_1 \sinh(kh_1) + m_1 E_1 \\ = ikB_2 \cosh(kh_2) + ikC_2 \sinh(kh_2) - m_2 E_2 \end{aligned} \quad (29)$$

$$-i\omega A = -kB_1 \sinh(kh_1) + kC_1 \cosh(kh_1) + ikE_1 \quad (30)$$

$$-i\omega A = kB_2 \sinh(kh_2) + kC_2 \cosh(kh_2) + ikE_2 \quad (31)$$

$$\begin{aligned} \rho_1 \nu_1 (-2ik^2 B_1 \sinh(kh_1) + 2ik^2 C_1 \cosh(kh_1) - k^2 E_1 - m_1^2 E_1) \\ = \rho_2 \nu_2 (2ik^2 B_2 \sinh(kh_2) + 2ik^2 C_2 \cosh(kh_2) - k^2 E_2 - m_2^2 E_2) \end{aligned} \quad (32)$$

$$\begin{aligned} \rho_1 (-i\omega B_1 \cosh(kh_1) + i\omega C_1 \sinh(kh_1) + gA \\ + 2\nu_1 k^2 B_1 \cosh(kh_1) + 2\nu_1 k^2 C_1 \sinh(kh_1) - 2i\nu_1 k m_1 E_1) \\ = \rho_2 (-i\omega B_2 \cosh(kh_2) - i\omega C_2 \sinh(kh_2) + gA \\ + 2\nu_2 k^2 B_2 \cosh(kh_2) + 2\nu_2 k^2 C_2 \sinh(kh_2) + 2i\nu_2 k m_2 E_2) \end{aligned} \quad (33)$$

This is a 9×9 system, which is a lot. We either need the help of a computer to do some algebra for us, or we need to employ a more efficient or clever approach. Here we resort to the first alternative, letting the computer do a perturbation expansion.

Let us assume a perturbation expansion of the form

$$\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \dots \quad (34)$$

with a generic small parameter that does not distinguish the layers,

$$\epsilon \sim \sqrt{\nu_j k^2 / \omega_0}.$$

First order $\mathcal{O}(\epsilon^0)$ — inviscid upper and lower layers

We get the familiar result

$$\omega^2 = \frac{(\rho_2 - \rho_1)gk}{\rho_2 \coth(kh_2) + \rho_1 \coth(kh_1)} \quad (35)$$

This same result is obtained by setting $\nu_j = 0$ and $C_j = 0$ and $D_j = 0$ and $E_j = 0$.

Second order $\mathcal{O}(\epsilon^1)$

$$\epsilon\omega_1 = \frac{-1 - i}{2} \sqrt{\frac{\omega_0}{2}} \frac{k(\nu_1 \rho_1^2 (C_1^2 - 1) + 2\sqrt{\nu_1 \nu_2} \rho_1 \rho_2 (C_2^2 + C_1 C_2 + C_1^2 - 1) + \nu_2 \rho_2^2 (C_2^2 - 1))}{(\rho_1 C_1 + \rho_2 C_2)(\sqrt{\nu_1} \rho_1 + \sqrt{\nu_2} \rho_2)} \quad (36)$$

where $C_j = \coth(kh_j)$.

Recognizing that $\coth(kh_j) > 1$ we notice that ω_1 is either zero or a complex number in the third quadrant. Provided ω_1 is nonzero, we have $\text{Re}(\omega) < \omega_0$ and $\text{Im}(\omega) < 0$, so the effect of viscosity is that the waves are damped in time, and the wave period is longer than in the inviscid case.

As equation (36) is rather complicated, it is interesting to consider three special limits:

1. The limit of inviscid lower layer. Typically the lower layer is water and the upper layer is oil, thus $\nu_2 \ll \nu_1$ while $\rho_2 \gtrsim \rho_1$, so let us here assume that $\sqrt{\nu_2} \rho_2 \ll \sqrt{\nu_1} \rho_1$. For simplicity, set $\nu_2 = 0$:

$$\epsilon\omega_1 = \frac{-1 - i}{2} \sqrt{\frac{\omega_0 \nu_1}{2}} \frac{k \rho_1 (C_1^2 - 1)}{(\rho_1 C_1 + \rho_2 C_2)} \quad (37)$$

2. The limit of distant floor and ceiling, set $C_j = 1$:

$$\epsilon\omega_1 = (-1 - i) \frac{k\rho_1\rho_2\sqrt{2\omega_0\nu_1\nu_2}}{(\rho_1 + \rho_2)(\sqrt{\nu_1}\rho_1 + \sqrt{\nu_2}\rho_2)} \quad (38)$$

3. The limit of inviscid lower layer and distant ceiling, set $\nu_2 = 0$ and $C_1 = 1$

$$\epsilon\omega_1 = 0 \quad (39)$$

so in this case we need to go up to higher order (at least $\epsilon^2\omega_2$) in order to find something of interest.

Third order $\mathcal{O}(\epsilon^2)$

We only need to consider the limit of inviscid lower layer and distant ceiling, so we limit to $\nu_2 = 0$ and $\coth kh_1 = 1$:

$$\epsilon\omega_2 = \frac{-2ik^2\rho_1\nu_1}{\rho_1 + \rho_2 \coth kh_2} \quad (40)$$

In this case, $\omega_1 = 0$, we have $\text{Re}(\omega) = \omega_0$ and $\text{Im}(\omega) < 0$, thus the waves are damped in time while their period is not affected.

Discussion

Even though we have not specifically analyzed the effect of each separate viscous boundary layer, we can still deduce that a viscous boundary layer next to a hard wall gives damping of order $\epsilon \sim k\sqrt{\nu/\omega}$, while a viscous boundary layer on just one side of a free surface gives a much smaller damping of order $\epsilon^2 \sim k^2\nu/\omega$. The combined viscous boundary layers on both sides of a free surface gives a damping of order $\epsilon \sim k\sqrt{\tilde{\nu}/\omega}$ where $\tilde{\nu} = \sqrt{\nu_1\nu_2}/(\sqrt{\nu_1} + \sqrt{\nu_2})$ (this approximation is valid for nearly equal densities).

There is an alternative approach, solving this problem in an iterative way: First solve the inviscid leading order problem with coefficients A and B_j . Then solve the viscous boundary layer corrections due to the leading order inviscid solution. Then solve a higher order inviscid solution with forcing at the boundaries due to the viscous corrections previously found. This will demand invoking the Fredholm alternative, conveniently expressed by Green's theorem. This can be found in chapter 8.4 in C.C.Mei (1989) "The applied dynamics of ocean surface waves".

A final concern: We have Taylor-expanded the boundary conditions at the interface around the equilibrium interface. This requires that the amplitude A does not exceed the viscous boundary layer thickness around the interface. That limits the analysis to depressingly small amplitudes. For larger amplitude internal waves one should either use curvilinear or Lagrangian coordinates.